On the Linear Vertex-Arboricity of a Surface

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ABSTRACT. The linear vertex-arboricity of a surface S is the maximum of the linear vertex-arboricities of all graphs embeddable into S. Poh showed that the linear vertex-arboricity of a sphere is three. We show that the linear vertex-arboricities of a projective plane and a torus are three and four respectively. Moreover we show that the linear vertex-arboricity of a Klein bottle is three or four.

1 Introduction

In this paper we assume that all graphs are finite, undirected graphs without loops or multiple edges. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. We denote the number of elements of a set A by $|A|$. For any subset V of $V(G)$, the subgraph of G induced by V is denoted by $G[V]$. A forest is called a *linear forest* if each component is a path. An n-partition $(V_1, \cdots, V_n)$ of $V(G)$ is said to be *linear arborescent* if $G[V_i]$ is a linear forest for $1 \leq i \leq n$. The *linear vertex-arboricity of a*

JCMCC 18 (1995), pp. 3-10

graph G, denoted by $la(G)$, is the smallest number n such that $V(G)$ has a linear arborescent n-partition.

A compact, connected, 2-dimensional manifold is called a *surface*. By ∂S we denote the boundary of a surface S. If $\partial S = \emptyset$, then S is called a *closed surface*. A closed orientable surface with g handles is denoted by S_g and a closed nonorientable surface with k crosscaps is denoted by N_k. Then S_0, S_1, N_1 and N_2 mean a sphere, a torus, a projective plane and a Klein bottle respectively. The *linear vertex-arboricity of a surface* S, denoted by $la(S)$, is defined to be the maximum of the linear vertex arboricities of all graphs embeddable into S. In our notation Poh [3] showed that $la(S_0) = 3$.

In Section 2 we prove the following theorem.

Theorem 1.1. $la(N_1) = 3$.

In Section 3 we show Theorem 1.2.

Theorem 1.2. $la(S_1) = 4$.

Moreover in Section 4 we give the upper bound of the linear vertex-arboricity of a closed surface S by using the Euler characteristic $\chi(S)$ of S.

Theorem 1.3. *Let S be a closed surface. If $\chi(S) \leq -3$, then*

$$la(S) \leq \left\lfloor \frac{15 + \sqrt{169 - 48\chi(S)}}{4} \right\rfloor .$$

If $-2 \leq \chi(S) \leq 1$, then $la(S) \leq 4 - \chi(S)$.

Since the complete graph with at most 6 vertices is embeddable into N_2, we have $la(N_2) \geq 3$. By Theorem 1.3, we have the following corollary.

Corollary 1.4. $la(N_2) = 3$ *or* 4.

2 The Linear Vertex-Arboricity of a Projective Plane

Let G be a graph and H a subgraph of G. For $u, v \in V(H)$, the edge $uv \in E(G)$ is called a *chord of* H if $uv \notin E(H)$. If H has no chord, then H is said to be *chordless in* G.

A graph G on a surface S is called a *triangulation of* S if G is the 1-skelton of a triangulation of S. Poh [3] showed the following lemma for a triangulation of a disk.

Lemma 2.1 [Poh]. *Let G be a triangulation of a disk D and $\partial D = v_1 v_2 \cdots v_k v_1$ ($k \geq 3$). If two paths $P_1 = v_1 \cdots v_r$ and $P_2 = v_{r+1} \cdots v_k$ are chordless in G for some integer r ($1 \leq r < k$), then there exists a linear arborescent 3-partition (V_1, V_2, V_3) of $V(G)$ such that P_i is a component of $G[V_i]$ for $i = 1, 2$.*

For any vertex v of a graph G, the degree of v in G is denoted by $\deg_G(v)$. At first we show the following lemma.

Lemma 2.2. *Let G be a triangulation of a disk D and $\partial D = v_0v_1\cdots v_kv_0$ ($k \geq 2$). If two paths $P_1 = v_1\cdots v_r$ and $P_2 = v_{r+1}\cdots v_k$ are chordless in G for some integer r ($1 \leq r < k$), then there exists a linear arborescent 3-partition (V_1, V_2, V_3) of $V(G)$ satisfying the following properties:*

(1) P_i is a component of $G[V_i]$ for $i = 1, 2$ and

(2) $v_0 \in V_3$ and $\deg_{G[V_3]}(v_0) \leq 1$.

Proof: We show Lemma 2.2 by induction on $|V(G)|$. If $|V(G)| = 3$, then Lemma 2.2 holds clearly. If $|V(G)| > 3$, then the following two cases occur:

Case 1: $P_1 \cup P_2$ has a chord and

Case 2: $P_1 \cup P_2$ is chordless in G.

Case 1. Let v_sv_t be a chord of $P_1 \cup P_2$ for $1 \leq s \leq r$ and $r+1 \leq t \leq k$. We divide D into two disks D_1 and D_2 such that $D_1 \cup D_2 = D$, $D_1 \cap D_2 = v_sv_t$, $\partial D_1 = v_0v_1\cdots v_sv_t\cdots v_kv_0$ and $\partial D_2 = v_s\cdots v_rv_{r+1}\cdots v_tv_s$. Let $G_j = G \cap D_j$ for $j = 1, 2$. We note that G_j is a triangulation of D_j for $j = 1, 2$. We set $P_i^j = P_i \cap \partial D_j$ for $1 \leq i, j \leq 2$. Since $|V(G_1)| < |V(G)|$, by induction hypothesis, there exists a linear arborescent 3-partition (V_1^1, V_2^1, V_3^1) of $V(G_1)$ satisfying the following properties:

(i) P_i^1 is a component of $G_1[V_i^1]$ for $i = 1, 2$ and

(ii) $v_0 \in V_3^1$ and $\deg_{G_1[V_3^1]}(v_0) \leq 1$.

By Lemma 2.1, there exists a linear arborescent 3-partition (V_1^2, V_2^2, V_3^2) of $V(G_2)$ such that P_i^2 is a component of $G_2[V_i^2]$ for $i = 1, 2$. Let $V_i = V_i^1 \cup V_i^2$ for $1 \leq i \leq 3$. Then (V_1, V_2, V_3) is a linear arborescent 3-partition of $V(G)$ that satisfies the properties (1) and (2) of Lemma 2.2.

Case 2. There exists a vertex u of $G - \partial D$ such that a cycle $uv_rv_{r+1}u$ is the boundary circuit of a triangular face Δ. Since $P_1 \cup P_2$ is chordless in G, there exists a shortest path $P = u_1\cdots u_l$ of $G - P_1 \cup P_2$ such that $u_1 = v_0$ and $u_l = u$. There exist two disks D_1 and D_2 in D such that $D_1 \cup D_2 = D - \Delta \cup \{v_rv_{r+1}\}$, $D_1 \cap D_2 = P$, $\partial D_1 = v_0v_1\cdots v_ru_l\cdots u_1$ and $\partial D_2 = u_1\cdots u_lv_{r+1}\cdots v_kv_0$. Let $G_j = G \cap D_j$ for $j = 1, 2$. Since P is chordless in G_j for $j = 1, 2$, by Lemma 2.1, there exists a linear arborescent 3-partition (V_1^j, V_2^j, V_3^j) of $V(G_j)$ such that P_j and P are components of $G_j[V_1^j]$ and $G_j[V_2^j]$ respectively. Let $V_1 = V_1^1 \cup V_3^2$, $V_2 = V_3^1 \cup V_1^2$ and $V_3 = V_2^1 \cup V_2^2$. Then (V_1, V_2, V_3) is a linear arborescent 3-partition satisfying the properties (1) and (2) of Lemma 2.2.

In order to determine $la(N_1)$, we prove Lemma 2.3 by using Lemma 2.2.

Lemma 2.3. *Let G be a triangulation of a disk D and $\partial D = v_0^1 v_1^1 \cdots v_k^1 v_0^2 v_1^2 \cdots v_k^2 v_0^1$ $(k \geq 1)$. We set $P_j = v_1^j \cdots v_k^j$ for $j = 1, 2$. If ∂D is chordless in G, then there exists a linear arborescent 3-partition (V_1, V_2, V_3) of $V(G)$ satisfying the following properties:*

(1) P_1 and P_2 are components of $G[V_1]$,

(2) $v_0^j \in V_3$ and $\deg_{G[V_3]}(v_0^j) \leq 1$ for $j = 1, 2$ and

(3) v_0^1 and v_0^2 are not contained in the same component of $G[V_3]$.

Proof: Since ∂D is chordless in G, there exist two vertices u' and u'' of $G - \partial D$ such that two cycles $u' v_0^1 v_k^2 u'$ and $u'' v_k^1 v_0^2 u''$ are the boundary circuits of triangular faces Δ' and Δ'' respectively. There exists a shortest path $P = u_1 \cdots u_l$ $(l \geq 1)$ of $G - \partial D$ such that $u_1 = u'$ and $u_l = u''$. We divide D into two disks D_1 and D_2 such that $D_1 \cup D_2 = D - \Delta' \cup \Delta'' \cup \{v_0^1 v_k^2, v_k^1 v_0^2\}$, $D_1 \cap D_2 = P$, $\partial D_1 = v_0^1 v_1^1 \cdots v_k^1 u_l \cdots u_1 v_0^1$ and $\partial D_2 = v_0^2 v_1^2 \cdots v_k^2 u_1 \cdots u_l v_0^2$. Let $G_j = G \cap D_j$ for $j = 1, 2$. Since P_j and P are chordless in G_j for $j = 1, 2$, by Lemma 2.2, there exists a linear arborescent 3-partition (V_1^j, V_2^j, V_3^j) of $V(G_j)$ satisfying the following properties:

(i) P_1 and P are components of $G_1[V_1^1]$ and $G_1[V_2^1]$ respectively,

(ii) P_2 and P are components of $G_2[V_1^2]$ and $G_2[V_2^2]$ respectively and

(iii) $v_0^j \in V_3^j$ and $\deg_{G_j[V_3^j]}(v_0^j) \leq 1$ for $j = 1, 2$.

Let $V_i = V_i^1 \cup V_i^2$ for $1 \leq i \leq 3$. Then (V_1, V_2, V_3) is a linear arborescent 3-partition of $V(G)$ satisfying the properties (1), (2) and (3) of Lemma 2.3.

Proof of Theorem 1.1: By the definition of the linear arboricity, it holds that $la(K_n) = \lceil \frac{n}{2} \rceil$, where K_n is the complete graph with n vertices. Since K_6 is embeddable into N_1, we have $la(N_1) \geq la(K_6) = 3$. Therefore it is sufficient to show that $la(N_1) \leq 3$. Let G be a graph embedded into N_1. Every graph on a surface S is a subgraph of some triangulation of S. We may assume that G is a triangulation of N_1. There exists a shortest cycle $C = v_0 v_1 \cdots v_k v_0$ $(k \geq 2)$ of G such that C is non-separating in N_1, that is, $N_1 - C$ is connected. We obtain a disk D by cutting N_1 along C. Let $\tilde{G}$ be the resultant graph embedded into D and θ the graph map from $\tilde{G}$ to G. There exist two paths $C_1 = v_0^1 v_1^1 \cdots v_k^1$ and $C_2 = v_0^2 v_1^2 \cdots v_k^2$ of $\tilde{G}$ such that $\theta(v_i^1) = \theta(v_i^2) = v_i$ for $0 \leq i \leq k$. Then $\partial D = v_0^1 v_1^1 \cdots v_k^1 v_0^2 v_1^2 \cdots v_k^2 v_0^1$. Since C is chordless in G, $C_1 \cup C_2$ is chordless in $\tilde{G}$. Let $P_j = C_j - v_0^j$ for $j = 1, 2$. By Lemma 2.3, there exists a linear arborescent 3-partition $(\tilde{V}_1, \tilde{V}_2, \tilde{V}_3)$ of $V(\tilde{G})$ satisfying the following properties:

(i) P_1 and P_2 are components of $\tilde{G}[\tilde{V}_1]$,

(ii) $v_0^j \in \tilde{V}_3$ and $\deg_{\tilde{G}[\tilde{V}_3]}(v_0^j) \leq 1$ for $j = 1, 2$ and

(iii) v_0^1 and v_0^2 are not contained in the same component of $\tilde{G}[\tilde{V}_3]$.

Let $V_i = \theta(\tilde{V}_i)$ for $1 \leq i \leq 3$. Then (V_1, V_2, V_3) is a linear arborescent 3-partition of $V(G)$ such that $C - v_0$ is a component of $G[V_1]$ and $v_0 \in V_3$. Therefore $la(G) \leq 3$.

3 The Linear Vertex-Arboricity of a Torus

In this section we prove Theorem 1.2 by the argument similar to that in the previous section.

Proof of Theorem 1.2: Since K_7 is embeddable into S_1, we have $la(S_1) \geq la(K_7) = 4$. It is sufficient to show that $la(S_1) \leq 4$. Let G be a graph embedded into S_1. We may assume that G is a triangulation of S_1. There exists a shortest cycle $C = v_0 v_1 \cdots v_k v_0$ $(k \geq 2)$ of G such that C is non-separating in S_1. We obtain an annulus A by cutting S_1 along C. Let $\tilde{G}$ be the resultant graph on A and θ the graph map from $\tilde{G}$ onto G. There exist two cycles $C_1 = v_0^1 v_1^1 \cdots v_k^1 v_0^1$ and $C_2 = v_0^2 v_1^2 \cdots v_k^2 v_0^2$ of $\tilde{G}$ such that $\theta(v_i^1) = \theta(v_i^2) = v_i$ for $0 \leq i \leq k$. We obtain a sphere $\hat{A}$ from A by capping off C_1 and C_2 with 2-cells Δ_1 and Δ_2 respectively. Let $\hat{G}$ be a graph on $\hat{A}$ obtained from $\tilde{G}$ by adding the edges $v_0^j v_i^j$ on Δ_j for $0 \leq i \leq k$ and $j = 1, 2$. Then $\hat{G}$ is a triangulation of $\hat{A}$. There exists a shortest path $P = v_0^1 u_1 \cdots u_l v_0^2$ $(l \geq 1)$ of $\hat{G}$. We obtain the disk D by cutting $\hat{A}$ along P. Let G' be the resultant graph on D and $\hat{\theta}$ the graph map from G' onto $\hat{G}$. There exist two paths $P_1 = u_1^1 \cdots u_l^1$ and $P_2 = u_1^2 \cdots u_l^2$ of G' such that $\hat{\theta}(u_i^1) = \hat{\theta}(u_i^2) = u_i$ for $1 \leq i \leq l$. Since $\partial D = \hat{\theta}^{-1}(P)$ is chordless in G', by Lemma 2.3, there exists a linear arborescent 3-partition (V_1', V_2', V_3') of $V(G')$ satisfying the following properties:

(i) P_1 and P_2 are components of $G'[V_1']$,

(ii) $v_0^j \in V_3'$ and $\deg_{G'[V_3']}(v_0^j) \leq 1$ for $j = 1, 2$ and

(iii) v_0^1 and v_0^2 are not contained in the same component of $G'[V_3']$.

Let $\hat{V}_i = \hat{\theta}(V_i')$ for $1 \leq i \leq 3$. Then $(\hat{V}_1, \hat{V}_2, \hat{V}_3)$ is a linear arborescent 3-partition of $V(\hat{G})$ such that $v_0^j \in \hat{V}_3$, $\deg_{\hat{G}[\hat{V}_3]}(v_0^j) \leq 1$ for $j = 1, 2$ and v_0^1 and v_0^2 are not contained in the same component of $\hat{G}[\hat{V}_3]$. Let $\tilde{V}_i = \hat{V}_i - (V(C_1 \cup C_2) - \{v_0^1, v_0^2\})$ for $1 \leq i \leq 3$ and $\tilde{V}_4 = V(C_1 \cup C_2) - \{v_0^1, v_0^2\}$. Then $(\tilde{V}_1, \tilde{V}_2, \tilde{V}_3, \tilde{V}_4)$ is a linear arborescent 4-partition of $V(\tilde{G})$ satisfying the following properties:

(i) $v_0^j \in \tilde{V}_3$ and $\deg_{\tilde{G}[\tilde{V}_3]}(v_0^j) \le 1$ for $j = 1, 2$,

(ii) v_0^1 and v_0^2 are not contained in the same component of $\tilde{G}[\tilde{V}_3]$ and

(iii) $C_j - v_0^j$ is a component of $\tilde{G}[\tilde{V}_4]$ for $j = 1, 2$.

Let $V_i = \theta(\tilde{V}_i)$ for $1 \le i \le 4$. Then (V_1, V_2, V_3, V_4) is a linear arborescent 4-partition of $V(G)$. Therefore $la(S_1) \le 4$.

4 The Upper Bound of the Linear Vertex-Arboricity

A graph G is said to be *critical* if $la(G - v) < la(G)$ for every vertex v of G. Matsumoto [2] showed the following lemmas.

Lemma 4.1. *If a graph G is critical, then $\delta(G) \ge la(G) - 1$, where $\delta(G)$ is the minimum degree of G.*

Lemma 4.2. $la(G) \le 1 + \left\lfloor \frac{\Delta(G)}{2} \right\rfloor$ *for any graph G, where $\Delta(G)$ is the maximum degree of G.*

Let f be a 2-cell embedding of a graph G into a surface S. Then $\chi(S) = |V(G)| - |E(G)| + |R(f)|$, where $R(f)$ is the set of all faces of f. Since $2|E(G)| \ge 3|R(f)|$, we have the following lemma by easy calculation.

Lemma 4.3. *If a graph G is embeddable into a surface S, then $\bar{d}(G) \le 6 - \frac{6\chi(S)}{|V(G)|}$, where $\bar{d}(G)$ is the average degree of G.*

The following lemma is obtained from Lemma 4.1, 4.2 and 4.3.

Lemma 4.4. *For any closed surface S,*

$$la(S) \le \left\lfloor \frac{15 + \sqrt{169 - 48\chi(S)}}{4} \right\rfloor.$$

Proof: If $\chi(S) > 0$, then $S = S_0$ or N_1, and $la(S) = 3$. Therefore Lemma 4.4 holds. We may assume that $\chi(S) \le 0$. Let G be a critical graph embeddable into S with $la(G) = la(S)$. From Lemmas 4.1 and 4.2, it follows that

$$la(G) - 1 \le \delta(G) \le \bar{d}(G) \le 6 - \frac{6\chi(S)}{|V(G)|}.$$

By Lemma 4.2, we have $|V(G)| \ge \Delta(G) + 1 \ge 2la(G) - 1$. Since $\chi(S) \le 0$, we have $-\frac{6\chi(S)}{|V(G)|} \le -\frac{6\chi(S)}{2la(G) - 1}$. Hence $la(G) - 1 \le 6 - \frac{6\chi(S)}{2la(G) - 1}$. Therefore $la(S) = la(G) \le \left\lfloor \frac{15 + \sqrt{169 - 48\chi(S)}}{4} \right\rfloor$.

For a closed surface S of the high Euler characteristic, we estimate the upper bound of $la(S)$ better than that in Lemma 4.4.

Lemma 4.5. *Let S be a closed surface. If $\chi(S) \leq 1$, then $la(S) \leq 4-\chi(S)$.*

Proof: We prove Lemma 4.5 by induction on $\chi(S)$. If $\chi(S) = 1$, then $la(S) = 3 = 4 - \chi(S)$ by Theorem 1.1. If $\chi(S) \leq 0$, then we show that $la(G) \leq 4 - \chi(S)$ for every graph G embedded into S. There exists a shortest cycle $C = v_0v_1 \cdots v_kv_0$ ($k \geq 2$) of G such that C is non-separating in S.

If C is an orientation preserving curve in S, then we obtain a closed surface $\tilde{S}$ by cutting S along C and capping off each component of the resultant boundary with a 2-cell. Then $\chi(\tilde{S}) = \chi(S) + 2$. Let $\tilde{G}$ be the resultant graph on $\tilde{S}$ and $\theta : \tilde{G} \to G$ the natural graph map from $\tilde{G}$ onto G. There exist two cycles $C_1 = v_0^1v_1^1 \cdots v_k^1v_0^1$ and $C_2 = v_0^2v_1^2 \cdots v_k^2v_0^2$ of $\tilde{G}$ such that $\theta(v_i^1) = \theta(v_i^2) = v_i$ for $1 \leq i \leq k$. Then $C_1 \cup C_2$ is chordless in $\tilde{G}$ because C is chordless in G. Since $\chi(\tilde{S}) = \chi(S)+2$, by induction hypothesis, there exists a linear arborescent n-partition $(\tilde{V}_1, \ldots, \tilde{V}_n)$ of $V(\tilde{G})$ for some $n \leq 4 - \chi(\tilde{S}) = 2 - \chi(S)$. Let $\hat{V}_i = \tilde{V}_i - V(C_1 \cup C_2)$ for $1 \leq i \leq n$, $\hat{V}_{n+1} = V(C_1 \cup C_2) - \{v_0^1, v_0^2\}$ and $\hat{V}_{n+2} = \{v_0^1, v_0^2\}$. Since $C_1 \cup C_2$ is chordless in $\tilde{G}$, $(\hat{V}_1, \ldots, \hat{V}_{n+2})$ is a linear arborescent $(n+2)$-partition of $V(\tilde{G})$. Let $V_i = \theta(\hat{V}_i)$ for $1 \leq i \leq n+2$. Then $(V_1, \ldots, V_{n+2})$ is a linear arborescent $(n+2)$-partition of $V(G)$. Therefore $la(G) \leq n+2 \leq 4-\chi(S)$.

If C is an orientation reversing curve in S, then a regular neighborhood of C in S is a Möbius band. We obtain a closed surface $\tilde{S}$ by cutting S along C and capping off the resultant boundary with a 2-cell D. Then $\chi(\tilde{S}) = \chi(S) + 1$. Let $\tilde{G}$ be the resultant graph embedded into $\tilde{S}$, and let v_i^1 and v_i^2 be two vertices of $\tilde{G}$ obtained by separating v_i for $0 \leq i \leq k$. We obtain a graph $\hat{G}$ embedded into $\tilde{S}$ by identifying v_0^1 and v_0^2 on D. Let $\theta : \hat{G} \to G$ be the natural graph map from $\hat{G}$ onto G. We set $P_j = v_1^j \cdots v_k^j$ for $j = 1, 2$. Then $P_1 \cup P_2$ is chordless in $\hat{G}$ because C is chordless in G. Since $\chi(\tilde{S}) = \chi(S) + 1$, by induction hypothesis, there exists a linear arborescent n-partition $(\hat{V}_1, \ldots, \hat{V}_n)$ of $V(\hat{G})$ for some $n \leq 4 - \chi(\tilde{S}) = 3 - \chi(S)$. Let $\tilde{V}_i = \hat{V}_i - V(P_1 \cup P_2)$ for $1 \leq i \leq n$ and $\tilde{V}_{n+1} = V(P_1 \cup P_2)$. Then $(\tilde{V}_1, \ldots, \tilde{V}_{n+1})$ is a linear arborescent $(n+1)$-partition of $V(\hat{G})$ because $P_1 \cup P_2$ is chordless in $\hat{G}$. Let $V_i = \theta(\hat{V}_i)$ for $1 \leq i \leq n+1$. Then $(V_1, \ldots, V_{n+1})$ is a linear arborescent $(n+1)$-partition of $V(G)$. Therefore $la(G) \leq n+1 \leq 4-\chi(S)$.

Theorem 1.3 follows from Lemma 4.4 and 4.5.

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