A Note on the Coloring of a Certain Class of Graphs

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Abstract. Let Γ_ℓ be the union of n complete graphs $A_1, A_2, \ldots, A_n$ of size n each such that $|A_i \cap A_j| \leq \ell$ whenever $i \neq j$, we prove that the chromatic number of Γ_ℓ is bounded above by $(2n-4)\ell + 1$.

1. Introduction

Let $A_1, A_2, \ldots, A_n \in \binom{X}{n}$, called *blocks*, such that $|A_i \cap A_j| \leq \ell$ whenever $i \neq j$, without loss of generality we may assume that $X = \cup_{1 \leq i \leq n} A_i$. Consider the graph Γ_ℓ defined over X such that $x, y \in X$ being adjacent if and only if $x, y \in A_i$ for some i. In 1972, Erdös, Faber and Lovász conjectured that the chromatic number of the graph Γ_1 is n. Erdös [1] also suggested that some bounds for the chromatic numbers of Γ_ℓ can be obtained. Although the bounds of the chromatic numbers of Γ_1 have been improving during the last twenty years, see for examples: [2], [3], [4], [5], [6], [7], [8], this problem remains open. A stronger result than that of [7] is given in [8]. For fixed ℓ and large n, the method of [6] gives a bound of about $n\ell$. No bounds for the chromatic numbers of Γ_ℓ ($\ell \geq 2$) were given yet. In this paper, we shall prove:

Theorem. *Let Γ_ℓ be the union of n complete graphs $A_1, A_2, \ldots, A_n$ of size n each such that $|A_i \cap A_j| \leq \ell$ whenever $i \neq j$, then the chromatic number of Γ_ℓ is bounded above by $(2n-4)\ell + 1$.*

In order to prove this theorem, a hypergraph $\mathcal{H}$ is associated with Γ_ℓ. For a fixed edge B of $\mathcal{H}$ of size k, following a counting argument used by Chang and Lawler [2], an upper bound for the number of edges of $\mathcal{H}$ with

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non-empty inetrsections with B is derived, and then an upper bound for the chromatic number $\chi(\Gamma_\ell)$ of Γ_ℓ is obtained.

2. Proof of the Theorem

Let n and ℓ be fixed. For each $x \in X$, let $\beta(x) = \{i \mid 1 \leq i \leq n$ with $x \in A_i\}$. Clearly, the chromatic number of Γ_ℓ is equal to that of the induced subgraph of Γ_ℓ over $\{x \mid x \in X$ with $|\beta(x)| \geq 2\}$. A hypergraph $\mathcal{H}$ is associated with the above system naturally with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{\beta(x) \mid x \in X$ with $|\beta(x)| \geq 2\}$. For $\ell \geq 2$, the hypergraph $\mathcal{H}$ is not necessarily simple since different blocks might have more then two common points. However, the conditions that $|A_i \cap A_j| \leq \ell$ whenever $i \neq j$ insure that $\mathcal{H}$ is loopless and with edge multiplicity at most ℓ (i.e., any distinct i, j are contained in at most ℓ edges). A coloring of Γ_ℓ corresponds to an edge coloring of $\mathcal{H}$ and vice versa, i.e., a color can be assinged to $x \in X$ as well as to the edge $\beta(x)$ of $\mathcal{H}$ simultaneously.

For $i, j \in \{1, 2, \ldots, n\}$, the vertex set of $\mathcal{H}$, let $m(i, j)$ be the number of edges of $\mathcal{H}$ containing i and j. Since $i, j \in \beta(x)$ if and only if $x \in A_i \cap A_j$, and $|A_i \cap A_j| \leq \ell$ for distinct i and j, it follows that $m(i, j) \leq \ell$. Let B be an edge of $\mathcal{H}$ with $|B| = k$. In order to estimate the number of edges of $\mathcal{H}$ incident with B and of size at least k, counting the number of two elements subsets of such edges shows that it is bounded above by

$$\sum_{i \in B} \sum_{j \notin B} m(i, j) \leq k\ell(n - k).$$

Since each such edge contributes at least $k - 1$ two elements subsets, the number of such edges is bounded above by $k\ell(n - k)$. This observation is summarized in the following lemma.

Lemma. *If B is an edge in $\mathcal{H}$ with $|B| = k$, then there are at most $k\ell(n - k)/(k - 1)$ edges of $\mathcal{H}$ distinct from B which are of sizes at least k and have non-empty intersections with B.*

To prove this theorem, arrange the edges of $\mathcal{H}$ in a nonincreasing order of sizes. The first edge can be colored arbitrarily, suppose that all edges with sizes at least $k + 1$ and some edges with sizes k have been colored properly, and now an edge B in $\mathcal{H}$ of size k is to be colored. As shown in the previous lemma, B has non-empty intersections with at most $k\ell(n-k)/(k-1)$ edges of $\mathcal{H}$ with sizes at least k, it follows that at most $k\ell(n - k)/(k - 1)$ colors would be enough to color those edges in the preceding of B and incident with B. As a function of k, $f(k) = k\ell(n - k)/(k - 1)$ is decreasing, so

$$k\ell(n - k)/(k - 1) \leq f(2) = (2n - 4)\ell.$$

it follows that there is one more color remained for B. This completes the proof.

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