On the Additive Bandwidth of Graphs

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ABSTRACT. A *numbering* of graph $G = (V, E)$ is a bijection $f : V \to \{1, 2, \ldots, p\}$ where $|V| = p$. The *additive bandwidth of numbering* f is $B^+(G, f) = \max\{|f(u) + f(v) - (p+1)| : uv \in E\}$, and the *additive bandwidth* of G is $B^+(G) = \min\{B^+(G, f) : f \text{ a numbering of G}\}$. Labeling V by a numbering which yields $B^+(G)$ has the effect of causing the 1's in the adjacency matrix of G to be placed as near as possible to the main contradiagonal, a fact which offers potential storage savings for some classes of graphs. Properties of additive bandwidth are discussed, including relationships with other graphical invariants, its value for cycles, and bounds on its value for extensions of full k-ary trees.

1. Introduction

Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_p\}$. A *numbering* is a bijection $f : V \to \{1, 2, \ldots, p\}$. The *bandwidth of numbering* f is $B(G, f) = \max\{|f(u) - f(v)| : uv \in E\}$, and the *bandwidth* of G is

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$B(G) = \min\{B(G,f) : f$ is a numbering of G$\}$. Bandwidth has been studied by several investigators [1, 3, 4, 5, 6, 7, 8], in the main so as to provide a compact representation of sparse $n \times n$ matrices. A *bandwidth numbering* or *B-numbering* of G is a numbering whose bandwidth is the bandwidth of G. Using a bandwidth numbering, the 1's in the associated adjacency matrix of G (where the vertices are ordered by the numbering) lie within $B(G)$ diagonals above and below the main diagonal. Thus all of the information about G can be stored in a collection of bits corresponding to those diagonals.

Bascuñán, Ruiz, and Slater [2] defined the *additive bandwidth* $B^+(G)$ by $B^+(G) = \min\{\max\{|f(u) + f(v) - (p+1)| : uv \in E\} : f$ is a numbering of $G\}$. The quantity $f(u) + f(v)$ is called the *edge sum* of edge uv and $|f(u) + f(v) - (p+1)|$ is its *edge weight*. The number $p+1$ will be called the *target* and $B^+(G)$ is a measure of how far edge sums are from the target. An *additive bandwidth numbering* or B^+*-numbering* is a numbering whose additive bandwidth is $B^+(G)$. In an *additive bandwidth numbering* all 1's in the corresponding adjacency matrix lie within $B^+(G)$ contradiagonals above and below the main contradiagonal. Again, all the information about G can be stored in a collection of bits corresponding to those diagonals. The number of storage bits required for some graphs is less for a bandwidth numbering than for an additive bandwidth numbering, while the opposite holds for other graphs. Thus an additive bandwidth numbering offers the potential benefit of significant storage savings (in fact, close to one half) for some classes of graphs. The authors of the seminal paper [2] derive properties of $B^+(G)$ and compute its value for some classes of graphs, including grids and complete bipartite graphs. In particular, the n by m grid graph $P_n \times P_m$ with $n \le m$ has bandwidth n and additive bandwidth $\lceil n/2 \rceil$. Thus storage requirements for large n improve by a factor of essentially two (as described in [2]).

Section 2 of this paper describes maximum graphs for a given value of $B^+(G)$, Section 3 presents relationships between $B^+(G)$ and other graphical invariants, Section 4 is devoted to the additive bandwidth of cycles, Section 5 presents partial results on the additive bandwidth of trees, and Section 6 computes $B^+(G + K_n)$ in terms of $B^+(G)$.

2. Maximum Graphs

Let d_r be the contradiagonal of the adjacency matrix $A = (a_{ij})$ of graph G corresponding to the elements a_{ij} having $i + j = r$, $2 \le r \le 2p$. The main contradiagonal is d_{p+1}. Then any graph G having p vertices and $B^+(G) \le k$ must be a subgraph of $G^+_{p,k}$, the graph with adjacency matrix having 1's in all positions of $d_{p+1-k}, \ldots, d_{p+1+k}$, except the positions also residing on the main diagonal. Figure 1 shows $G^+_{p.k}$ for small values of p and k.

It does not follow that every subgraph G of $G^+_{p,k}$ has $B^+(G) \leq k$. Consider the graph $G^+_{5,1}$ of Figure 2. For this G, $B^+(G) = 1$, and a B^+-numbering is shown. Suppose graph H is obtained from G by deleting the vertex labeled 5 and further suppose $B^+(H) \leq 1$. Then we can produce a numbering $f : V \to \{1,2,3,4\}$ such that $f(u) + f(v) \in \{4,5,6\}$ for all $uv \in E$. But the triangle's vertices must include either labels 1 and 2 or labels 3 and 4, causing an edge sum of 3 or 7, a contradiction. Thus $B^+(H) \geq 2$. The problem here is that a subgraph G of $G^+_{p,k}$ is not necessarily a subgraph of $G^+_{p-1,k}$. For our example, H contains a $K_{1,3}$ but Figure 1 shows $G^+_{4,1}$ does not. However, the following straightforward result is true.

Proposition 1. *If H is a spanning subgraph of G, then $B^+(H) \leq B^+(G)$.*

The ability of B^+ to increase when a vertex is removed is indicative of the difficulties which arise when studying this parameter, difficulties which do not occur with normal bandwidth. It corresponds to shifting the target $p+1$ in $|f(u) + f(v) - (p+1)|$ by one which can cause an increase of one in this computed quantity even when $f(u)$ and $f(v)$ remain the same. This interferes with most induction approaches.

3. Invariant Relationships

Let $\beta_0(G)$ be the *vertex independence number* of graph G, that is, the maximum number of vertices with the property that no two are adjacent.

Proposition 2. *For any graph G, $p - 2\beta_0 \leq B^+(G) \leq p - \lfloor \beta_0/2 \rfloor - 1$, and these bounds are sharp.*

Proof: The lower bound is established in [2]. Label the vertices of a maximum independent set by $1, 2, \ldots \lfloor \beta_0/2 \rfloor, p - \lceil \beta_0/2 \rceil + 1, p - \lceil \beta_0/2 \rceil + 2, \ldots, p$ and label the rest of the vertices arbitrarily with $\lfloor \beta_0/2 \rfloor + 1$ through $p - \lceil \beta_0/2 \rceil$. Then the smallest possible edge sum is $\lfloor \beta_0/2 \rfloor + 2$ and the largest is $2p - \lceil \beta_0/2 \rceil$. Now $p + 1 - (\lfloor \beta_0/2 \rfloor + 2) = p - \lfloor \beta_0/2 \rfloor - 1$ and $2p - \lceil \beta_0/2 \rceil - (p+1) = p - \lceil \beta_0/2 \rceil - 1 \leq p - \lfloor \beta_0/2 \rfloor - 1$. The upper bound is achieved by $\overline{K}_{\beta_0} + K_{p-\beta_0}$ where $\overline{K}_{\beta_0}$ is the empty graph on β_0 vertices and the sum $G + H$ of graphs G and H is formed by adjoining every vertex of G to every vertex of H (see Lemma 17). The lower bound is achieved by K_p. □

A lower bound in terms of maximum degree Δ is obtained easily.

Proposition 3. *For any graph G, $B^+(G) \geq \lceil \frac{\Delta - 1}{2} \rceil$ and this bound is sharp.*

Proof: The Δ edge sums associated with the edges incident to a vertex of maximum degree may include $p+1$, but always leave at least $\Delta - 1$ others to be distributed on either side of $p+1$. The bound is achieved by $K_{1,n}$ for n even [2]. □

In [2] it was mentioned without proof that $B(G) \leq 2B^+(G)$. We now justify that.

Theorem 4. *For any graph G with p vertices and $B^+(G) \geq 1$, $B(G) \leq 2B^+(G)$. Furthermore, this bound is sharp.*

Proof: We first show the result is true for $G = G^+_{p,k}$ and p even. In this case $G^+_{p,0}$ is a collection of $p/2$ independent edges, which we shall call *rungs*, labeled as in Figure 3a. Call the two vertices on a single rung *mates*. Then $G^+_{p,k}$ is obtained from $G^+_{p,0}$ by connecting each vertex to all vertices whose label differs from the label of its mate by at most k. It is easy to see that these added edges always connect rungs which are at most k apart, and, when the rungs are exactly k apart, the edge connects a vertex on the left side to one on the right. Figures 3b, 3c, and 3d show $G^+_{p,1}, G^+_{p,2}$, and $G^+_{p,3}$, respectively. Define a new numbering f of $G^+_{p,k}$ as follows. Label the vertices of the rungs from the top rung down with successive integers beginning with 1. The top k rungs are labeled left to right, the next k right to left, and so on until all rungs have been labeled. This procedure is illustrated for $k = 1, 2$, and 3 in Figures 4a, 4b, and 4c, respectively. Consider vertices on all rungs between rung i and rung $i + k$, inclusive, when such a difference exists. These are labeled with $2i-1$ through $2i+2k$. However, the vertices labeled $2i - 1$ and $2k$ are on the same side, left or right, and, since the corresponding rungs are exactly k apart, there is no edge between them. Thus the largest value of $|f(u) - f(v)|$ for any edge uv is $2k$. This numbering then shows $B(G^+_{p,k}) \leq 2B^+(G^+_{p,k})$ if p is even. Thus, if $B^+(G) = k$, G is a subgraph of $G^+_{p,k}$ so $B(G) \leq B(G^+_{p,k}) \leq 2B^+(G^+_{p,k}) = 2k = 2B^+(G)$, and we are done.

Notice that, if p is odd, $G^+_{p,k}$ can be obtained from $G^+_{p+1,k}$ by contracting the bottom rung. The labeling f of $G^+_{p+1,k}$ described above induces a labeling of $G^+_{p,k}$ by labeling all vertices the same as in $G^+_{p+1,k}$ except for the new combined vertex which is labeled p. Once again assuming $B^+(G) = k$, it follows that $B(G) \leq B(G^+_{p,k}) \leq B(G^+_{p+1,k}) \leq 2B^+(G^+_{p+1,k}) = 2k = 2B^+(G)$.

Sharpness is provided by the grid graph $P_{2t} \times P_{2t}$ for which it is known that $B(P_{2t} \times P_{2t}) = 2t$ [7] and $B^+(P_{2t} \times P_{2t}) = t$ [2]. □

The next section demonstrates that there are graphs G for which $B^+(G)$ is arbitrarily larger than $B(G)$. In particular, if G is the disjoint union of n odd cycles, $B(G) = 2$ but $B^+(G) = n$.

4. Graphs with Odd Cycles

For any numbering f of graph G, define the (additive) *diameter* $d(G, f)$ by $d(G,f) = \max\{f(u) + f(v) : uv \in E\} - \min\{f(u) + f(v) : uv \in E\}$ and extend the notion to a subgraph H (labeled with the restriction of f to H) by simply taking the maximum and minimum over edges in $E(H)$. Note that for a B^+-numbering $f, B^+(G) \leq d(G,f) \leq 2B^+(G)$. In the following C_p is the cycle on $p \geq 3$ vertices.

Lemma 5. *If G contains an odd cycle C and f is a numbering such that $d(C,f) \leq 2$, then the label set $f[V(C)]$ is composed of consecutive integers.*

Proof: Suppose the vertices of C are labeled successively by $a_1, a_2, \ldots, a_{2k-1}$. We may assume $a_1 = 1$. Then $|(a_2+a_3)-(a_1+a_2)| \leq 2$, or $1 \leq a_3 - 1 \leq 2$. Similarly, $1 \leq a_{2k-2}-1 \leq 2$. Thus $a_3 = 2$ or 3 and $a_{2k-2} = 2$ or 3. Without loss of generality let $a_3 = 3$ and $a_{2k-2} = 2$. With these choices, the following values are forced, using similar reasoning: $a_{2j+1} = 2j+1$ and $a_{2k-2j} = 2j$ for $j = 1, 2, \ldots, k-1$. Thus the label set of C is $\{1, 2, \ldots, 2k-1\}$. □

It follows that, if the actual minimum label of such a C is "a", its label set is $\{a, a+1, \ldots, a+2k-2\}$ and the three edge sums generated by C are $2k+2a-3$, $2k+2a-2$, and $2k+2a-1$. Also, note that the "middle" label of such a C is $a+k-1$.

Corollary 6. *If a graph G with an even number of vertices contains an odd cycle, then $B^+(G) \geq 2$.*

Proof: If $B^+(G) \leq 1$, then $d(G,f) \leq 2$ for some B^+-numbering f. Thus the odd cycle must be labeled with consecutive integers, producing edge sums as indicated above. Since the middle sum must be $p+1$, we have $2k-2a-2 = p+1$, where $p = |V(G)|$. Thus p is odd, a contradiction. □

Proposition 7. *If G contains two odd cycles, each of which contains a vertex not in the other, then $B^+(G) \geq 2$.*

Proof: Let C and C' be two such odd cycles where C has $2k-1$ vertices and minimum label a, and C' has $2k'-1$ vertices and minimum label a'. Now assume $B^+(G) \leq 1$ and let f be a B^+-numbering of G. Then $d(C,f) \leq 2$ and $d(C',f) \leq 2$. By Lemma 5 each label set must consist of consecutive integers. Since C and C' produce edge sums $2k+2a-3, 2k+2a-2, 2k+2a-1$ and $2k'+2a'-3, 2k'+2a'-2, 2k'+2a'-1$, respectively, we have $2k+2a-2 = p+1 = 2k'+2a'-2$, or $k+a-1 = k'+a'-1$. Thus the "middle label" is the same for both C and C' and so the label set for the smaller cycle is a subset of the label set for the larger cycle. But then every vertex of the smaller cycle is in the larger cycle which contradicts the assumed vertex properties of C and C'. □

Note that the above result does not extend directly to three cycles, as is seen by the graph G of Figure 5 for which $B^+(G) = 2 < 3$.

Lemma 8. *$B^+(C_p) = 1$ for all p.*

Proof: Clearly $B^+(C_p) \geq 1$. Thus it suffices to display a numbering f such that $B^+(C_p, f) = 1$. Such a numbering depends on the parity of

p. If $p = 2k-1$, denote the labels $1, 2, \ldots, p$ in order as follows: $1 = \ell_{k-1}, \ldots, \ell_1, k, u_1, \ldots, u_{k-1} = p$, that is, $\ell_i = k - i$ and $u_i = k + i$, $1 \leq i \leq k-1$. Label one vertex of the cycle with k and continue alternately left and right according to the following pattern:

$$\ldots u_4 \ell_3 u_2 \ell_1 k u_1 \ell_2 u_3 \ell_4 \ldots \tag{1}$$

Identifying vertices by their labels, we have edges $\ell_1 k$ with edge sum $(k-1)+k = 2k-1$, ku_1 with edge sum $2k+1$, $\ell_i u_{i-1}$ with edge sum $(k-i)+(k+i-1) = 2k-1$ for $2 \leq i \leq k-1$, $\ell_i u_{i+1}$ with edge sum $(k-i)+(k+i+1) = 2k+1$ for $1 \leq i \leq k-2$, and $\ell_{k-1}u_{k-1}$ with edge sum $2k$. Thus all edge sums are one of p, $p+1$ or $p+2$, so $B^+(C_p, f) = 1$. If p is even, an analogous construction works with the set $1 = \ell_{k-1}, \ldots, \ell_1, u_1, \ldots, u_{k-1} = p$. □

Lemma 9. *If G contains a disjoint union of n odd cycles, then $B^+(G) \geq n$.*

Proof: Let $p = |V(G)|$; $C_1, C_2, \ldots, C_n$ be n disjoint odd cycles contained in G with C_i having length $2j_i + 1$; and $j = j_1 + j_2 + \cdots + j_n$. Let C be the subgraph induced by $\bigcup_{i=1}^{n} C_i$ and $s = |V(C)| = 2j + n$. Note, $\beta_0(C) \leq j$. For any numbering $f : V \to \{1, 2, \ldots, p\}$ let $f[V(C)] = \{a_1, a_2, \ldots, a_s\}$ with $a_i < a_{i+1}$. Because $\beta_0(C) \leq j$, at least two vertices with labels in $\{a_1, a_2, \ldots, a_j, a_{j+1}\}$ are adjacent, as are two vertices in $\{a_{s-j}, a_{s-j+1}, \ldots, a_s\}$. Thus there are edges uv and wx with $f(u)+f(v) \leq a_j + a_{j+1}$ and $f(w) + f(x) \geq a_{s-j} + a_{s-j+1} = a_{j+n} + a_{j+n+1}$. Now $a_{j+n} - a_j \geq n$ and $a_{j+n+1} - a_{j+1} \geq n$ so $(a_{j+n} + a_{j+n+1}) - (a_j + a_{j+1}) \geq 2n$ which implies $\beta^+(C) \geq n$. □

Lemma 10. *If G is a disjoint union of n odd cycles, then $B^+(G) = n$.*

Proof: By Lemma 9 we need only show a numbering f such that $B^+(G, f) = n$. Let $C_1, C_2, \ldots, C_n$ be the n odd cycles arranged so that $|V(C_1)| \geq |V(C_2)| \geq \cdots \geq |V(C_n)|$. We will represent f as an $n \times q$ array where $q = |V(C_1)|$ and where the nonzero entries in row i of the array will give the consecutive labels around the cycle C_i. Figure 6 presents an example of the technique described in this proof. The columns of the array (K, L_i, and U_i), described below, will be arranged as in (1) of the proof of Lemma 8: $\ldots L_3 U_2 L_1 K U_1 L_2 U_3 \ldots$. The columns are constructed as follows, with the understanding that any undefined entry will be 0 and thus not used as a label. Let m_i be the number of cycles of length at least $2i+1$, $i = 1, 2, \ldots, \frac{q-1}{2}$, and $p = |V(G)|$. Let the set $\{1, 2, \ldots, p\}$ be partitioned in order as $B_{(q-1)/2} \cup \cdots \cup B_1 \cup S \cup A_1 \cup \cdots \cup A_{(q-1)/2}$, where $|S| = n$ and, for each i, $|A_i| = |B_i| = m_i$. Observe that $2\sum_{i=1}^{(q-1)/2} m_i + n = p$, by definition of the m_i. The columns are now created with values entered from top to bottom:

(i) K-elements of S in decreasing order

(ii) U_i-elements of A_i in decreasing order if i is even, increasing order if i is odd

(iii) L_i-elements of B_i in decreasing order if i is even, increasing order if i is odd

Verification that $B^+(G,f) = n$ involves a straightforward examination of four types of edge sums:

(i) $b+s$, where $b \in B_1$, $s \in S$

(ii) $s+a$, where $s \in S$, $a \in A_1$

(iii) $a+b$, where $a \in A_i$, $b \in B_{i\pm1}$

(iv) $a+b$, where $a \in A_i$, $b \in B_i$. □

Lemma 11. *If $B^+(G) > 0$, then $B^+(G \cup C_{2k}) \leq B^+(G)$.*

Proof: Suppose $|V(G)| = p$ and let f be a B^+-numbering of G and g be the B^+-numbering on C_{2k} from Lemma 8. Define a labeling h on $G \cup C_{2k}$ as follows:

$$h(v) = \begin{cases} g(v) & \text{if } v \in V(C_{2k}) \text{ and } g(v) \leq k \\ g(v)+p & \text{if } v \in V(C_{2k}) \text{ and } g(v) > k \\ f(v)+k & \text{if } v \in V(G) \end{cases}$$

It is easily verified that h is a numbering and $B^+(G \cup C_{2k}, h) \leq B^+(G)$. □

The next theorem combines the previous results.

Theorem 12. *If G is a disjoint union of cycles, exactly n of which are odd, then $B^+(G) = \max\{1, n\}$.*

5. Trees

We have seen that determining $B^+(G)$ even for simple graphs G can be difficult, a fact which remains true for trees. Here we confine ourselves to determining bounds for the additive bandwidth of a special but important class of trees. Define $T_{r,m,d}$ to be the rooted tree of depth $d \geq 2$ where all leaf vertices are on level d (the root is level 0), the root has degree $r \geq 2$, and all other non-leaf vertices have degree $m \geq 3$.

Observe that $|V(T_{r,m,d})| = p = 1 + r\sum_{i=0}^{d-1}(m-1)^i = 1 + r\frac{(m-1)^d-1}{m-2}$ so the target is $t = 2 + r\frac{(m-1)^d-1}{m-2}$. This class of trees includes full binary trees by setting $r = 2$ and $m = 3$. For integers x and y with $x \leq y$ let $[x, y]$ represent the set $\{x, x+1, \ldots, y-1, y\}$.

Lemma 13. *In any B^+-numbering of $T_{r,m,d}$ where the root is labeled a, the vertices at level ℓ must be labeled with members of $[t - \ell B^+ - a, t + \ell B^+ - a] \cap [1,p]$ if ℓ is odd and $[a - \ell B^+, a + \ell B^+] \cap [1,p]$ if ℓ is even.*

Proof: Employ induction on ℓ. All labels z on level $\ell = 1$ must obey $z + a \in [t - B^+, t + B^+]$. When $\ell \geq 2$ is even any label x at level $\ell - 1$ is in $[t - (\ell - 1)B^+ - a, t + (\ell - 1)B^+ - a]$ by the inductive hypothesis and any label z at level ℓ must obey $z + x \in [t - B^+, t + B^+]$. It follows that $z \in [t - B^+ - (t + (\ell - 1)B^+ - a), t + B^+ - (t - (\ell - 1)B^+ - a)]$. A similar argument establishes the case for $\ell \geq 3$ odd. □

Note that the root label a can be taken to be $a \leq t/2$ since, if it is not, the *complement numbering* defined by replacing each label x by $p + 1 - x$ can be employed. This revised labeling produces the same edge weights as the original. We first determine a lower bound on the additive bandwidth of $T_{r,m,d}$. With normal bandwidth a lower bound can be obtained using the relation $B(G) \geq \lceil \frac{p-1}{\text{diameter}} \rceil$ [4], a relation which is invalid in general for additive bandwidth, as the cycle shows. Surprisingly, though, the value produced by this formula does yield a valid lower bound for the additive bandwidth of $T_{r,m,d}$.

Lemma 14. $B^+(T_{r,m,d}) \geq \lceil r\frac{(m-1)^d - 1}{2d(m-2)} \rceil$.

Proof: In light of the paragraph preceding this theorem, we will assume throughout this proof that $a \leq t/2$.

Case 1: d is odd. It follows from Lemma 13 that even level labels are selected from $[a - (d-1)B^+, a + (d-1)B^+] \cap [1,p]$ and odd level labels from $[t - dB^+ - a, t + dB^+ - a] \cap [1,p]$. If $t/2 - dB^+ \leq 1$ we have $B^+ \geq \frac{t-2}{2d} = r\frac{(m-1)^d - 1}{2d(m-2)}$. Suppose then that $t/2 - dB^+ > 1$. Now $t - dB^+ - a \geq t/2 - dB^+ > 1$ so the smallest possible label on an odd level is greater than 1, that is, the label "1" must appear at an even level. This can happen only if $a - (d-1)B^+ \leq 1$. Thus the smallest possible label that can appear on an odd level is greater than or equal to $t - dB^+ - [(d-1)B^+ + 1] = t - 2dB^+ + B^+ - 1$. It follows that labels 1 through $t - 2dB^+ + B^+ - 2$ must occur on even levels. Assuming none of these labels is assigned to the root, each vertex so labeled has $m - 1$ children on an odd level. Thus by the pigeon hole principle at least one of these children has a label no larger than $t - (m-1)(t - 2dB^+ + B^+ - 2)$. The child with this label is adjacent to a vertex having label at most $t - 2dB^+ + B^+ - 2$ to give an edge sum of at most $2t - (m-1)t + 2(m-2)dB^+ - (m-2)B^+ + 2(m-2)$. Since this sum must be at least $t - B^+$, we can solve the resultant inequality to see $B^+ \geq r\frac{(m-1)^d - 1}{2d(m-2) - (m-3)}$ which is never smaller than the bound found when $t/2 - dB^+ \leq 1$. If one of the vertices labeled 1 through $t - 2dB^+ + B^+ - 2$ is the root, the number of children on odd levels is $r + (m-1)(t - 2dB^+ +$

$B^+-3)$. By modifying the preceding analysis only slightly, it can be shown that this forces a value for B^+ which is also always at least as much as the bound of the theorem.

Case 2: d is even. Now odd level labels are in $[t-(d-1)B^+-a, t+(d-1)B^+-a]\cap[1,p]$ and even level labels are in $[a-dB^+, a+dB^+]\cap[1,p]$. If $t/2+dB^+\geq t-1$, $B^+\geq r\frac{(m-1)^d-1}{2d(m-2)}$. If $t/2+dB^+<t-1$, label $t-1$ must appear on an odd level, which is possible only if $t+(d-1)B^+-a\geq t-1$, that is, $a\leq(d-1)B^++1$. Thus even levels can have labels up to at most $(d-1)B^++1+dB^+=2dB^+-B^++1$. This means odd levels must include all labels from $2dB^+-B^++2$ to $t-1$ of which there are $t-2dB^++B^+-2$. Each vertex so labeled has $m-1$ children on even levels and the ones on the smallest level must also have at least one even level father, so the entire collection of even level neighbors has at least $(m-1)(t-2dB^++B^+-2)+1$ members. This means there is an edge sum of at least $(m-1)t-2(m-1)dB^++(m-1)B^+-2(m-1)+1+2dB^+-B^++2$ which can be at most $t+B^+$. Solving for B^+ yields $B^+\geq\frac{r[(m-1)^d-1]+1}{2d(m-2)-(m-3)}$ which exceeds previous bounds. □

Lemma 15. $B^+(T_{r,m,d})\leq\lfloor\frac{r}{2}(m-1)^{d-1}\rfloor$.

Proof: *Case 1:* r is even. Figure 7(a) illustrates the numbering scheme which shows the bound. The scheme is:

(i) Label half the leaves left to right with 1 to $\frac{r}{2}(m-1)^{d-1}$. Label the other half right to left with $t-1$ down to $t-\frac{r}{2}(m-1)^{d-1}$.

(ii) On level $d-1$ label half the vertices left to right with $t-\frac{r}{2}(m-1)^{d-1}-1$ down to $t-\frac{r}{2}(m-1)^{d-1}-\frac{r}{2}(m-1)^{d-2}$. Label the other half from right to left with $\frac{r}{2}(m-1)^{d-1}+1$ to $\frac{r}{2}(m-1)^{d-1}+\frac{r}{2}(m-1)^{d-2}$.

(iii) Continue in this manner to successively smaller numbered levels, alternating which half gets increasing labels and decreasing labels. The labels used at level ℓ are those in $\left[\frac{r}{2}\sum_{i=1}^{d-\ell}(m-1)^{d-i}+1, \frac{r}{2}\sum_{i=1}^{d-\ell+1}(m-1)^{d-i}\right]\cup\left[t-\frac{r}{2}\sum_{i=1}^{d-\ell+1}(m-1)^{d-i}, t-\frac{r}{2}\sum_{i=1}^{d-\ell}(m-1)^{d-i}-1\right]$.

(iv) Label the root with $t/2$.

It is easily seen that the largest deviation from t occurs for edges between levels $d-1$ and d, and its value is $\frac{r}{2}(m-1)^{d-1}$.

Case 2: r is odd.

Case 2a: m is odd. Then every subtree rooted at a level 1 vertex has an even number of subtrees rooted at level 2. This allows us to use the same scheme as in Case 1 for level ℓ, $2\leq\ell\leq d$. Figure 7(b) illustrates the entire scheme, which for levels 0 and 1 is:

(i) Level 1 uses labels $\left[\frac{r}{2}\sum_{i=1}^{d-1}(m-1)^{d-i}+1, \frac{r}{2}\sum_{i=1}^{d-1}(m-1)^{d-i}+\frac{r+1}{2}\right] \cup \left[t-\frac{r}{2}\sum_{i=1}^{d-1}(m-1)^{d-i}-\frac{r-1}{2}, t-\frac{r}{2}\sum_{i=1}^{d-1}(m-1)^{d-i}-1\right]$ with the left-right direction opposite to that used on level 2.

(ii) The root is labeled $\frac{t+1}{2}$.

Again the largest deviation from t occurs for edges between levels $d-1$ and d, and its value is $\frac{r}{2}(m-1)^{d-1}$.

Case 2b: m is even. Now every level has an odd number of vertices. The center vertices on each level are temporarily left unlabeled, while the other vertices on levels 1 and higher are labeled by a technique paralleling that of previous cases. Figure 7(c) illustrates the scheme, which involves the following:

(i) The non-center vertices of level ℓ, $1 \leq \ell \leq d$, use labels in

$$\left[\frac{r\sum_{i=1}^{d-\ell}(m-1)^{d-i}-(d-\ell)}{2}+1, \frac{r\sum_{i=1}^{d-\ell+1}(m-1)^{d-i}-(d-\ell+1)}{2}\right] \cup$$
$$\left[t-\frac{r\sum_{i=1}^{d-\ell+1}(m-1)^{d-i}-(d+1-\ell)}{2}, t-\frac{r\sum_{i=1}^{d-\ell}(m-1)^{d-i}-(d-\ell)}{2}-1\right]$$

(ii) The middle vertex on level ℓ, $0 \leq \ell \leq d$, is labeled with $\frac{t-d}{2}+\ell$.

Yet again the largest deviation from t occurs for edges between levels $d-1$ and d, and its value is $\frac{r(m-1)^{d-1}-1}{2}$. This maximum edge weight does not involve an edge incident to a center vertex. Checking such edge weights and showing they don't exceed the bound is a tedious but straightforward computation. □

Combining Lemmas 14 and 15 gives the following.

Theorem 16. $\left\lceil r\frac{(m-1)^d-1}{2d(m-2)}\right\rceil \leq B^+(T_{r,m,d}) \leq \left\lfloor\frac{r}{2}(m-1)^{d-1}\right\rfloor$.

For full binary trees the theorem becomes $\left\lceil\frac{2^d-1}{d}\right\rceil \leq B^+(T_{2,3,d}) \leq 2^{d-1}$. The numberings of Figure 8 show the lower bound is correct for $d \leq 5$. We believe the lower bound of Theorem 16 is closer to the true value. If so, it would parallel the recently reported fact [8] that the ordinary bandwidth of $T_{k,k+1,d}$ is given by $\left\lceil\frac{k(k^d-1)}{2d(k-1)}\right\rceil$.

6. $B^+(G+K_n)$

As a first step in determining the additive bandwidth of combinations of graphs we compute $B^+(G+K_n)$ in terms of $B^+(G)$, where G is a graph on p vertices and $G + K_n$ is obtained from disjoint copies of G and K_n by connecting every vertex of G to every vertex of K_n. A series of lemmas paves the way. The first gives the result when $G = \overline{K}_p$, the empty graph on p vertices.

Lemma 17. *For $p \geq 2$, $B^+(\overline{K}_p + K_n) = n + \lceil p/2 \rceil - 1$.*

Proof: Label the vertices of K_n by $\lfloor p/2 \rfloor + 1, \ldots, \lfloor p/2 \rfloor + n$ and the vertices of $\overline{K}_p$ with the remaining labels between 1 and $p + n$. Since no vertices of $\overline{K}_p$ are adjacent, the smallest edge sum is $\lfloor p/2 \rfloor + 2$ and the largest is $\lfloor p/2 \rfloor + 2n + p$ which implies $B^+(\overline{K}_p + K_n) \leq n + \lceil p/2 \rceil - 1$. For the reverse inequality consider any numbering where x_S is the smallest label assigned to any vertex of K_n and x_L is the largest. Since $x_L \geq x_S + n - 1$ it follows that one of $x_S \leq \lfloor p/2 \rfloor + 1$ and $x_L \geq n + \lfloor p/2 \rfloor + 1$ holds. Let $x_S \leq \lfloor p/2 \rfloor + 1$. If $x_S \neq 1$, $1 + x_S \leq \lfloor p/2 \rfloor + 2$ is an edge sum and $B^+(\overline{K}_p + K_n) \geq n + \lceil p/2 \rceil - 1$. If $x_S = 1$, the vertices labeled 1 and 2 are adjacent so $B^+(\overline{K}_p + K_n) \geq n + p + 1 - 3 \geq n + p - \lfloor p/2 \rfloor - 1$ for $p \geq 2$. A similar argument holds when $x_L \geq n + \lfloor p/2 \rfloor + 1$ where the case $x_L = p + n$ must be treated specially. □

The next two lemmas deal with the $n = 1$ case.

Lemma 18. *For any graph G, $B^+(G + K_1) > B^+(G)$.*

Proof: Let g be a B^+-numbering of $G + K_1$, G having p vertices and with u being the vertex corresponding to K_1. Define a (not necessarily B^+-) numbering f of G by

$$f(v) = \begin{cases} g(v) & \text{if } g(v) < g(u) \\ g(v) - 1 & \text{if } g(v) > g(u). \end{cases}$$

Let x and y be vertices such that $|f(x)+f(y)-(p+1)| \geq B^+(G)$ for edge xy. Without loss of generality assume $f(x) < f(y)$ so that $g(x) < g(y)$. Suppose $g(x) < g(y) < g(u)$. If $f(x)+f(y) > p+1$, then $f(x)+f(y)-p-1 \geq B^+(G)$. In $G + K_1$ we have for edge uy that $|g(u) + g(y) - (p+2)| \geq |g(x) + 2 + f(y) - p - 2| = |f(x) + f(y) - (p+1) + 1| \geq B^+(G) + 1$. On the other hand, if $f(x) + f(y) \leq p + 1, p + 1 - f(x) - f(y) \geq B^+(G)$. Thus, in $G + K_1$, $|(p+2) - g(x) - g(y)| = |(p+1) - f(x) - f(y) + 1| \geq B^+(G) + 1$. Similar arguments for the cases $g(x) < g(u) < g(y)$ and $g(u) < g(x) < g(y)$ always show the existence of an edge ab such that $|g(a)+g(b)-(p+2)| \geq B^+(G)+1$ so the result follows. □

Lemma 19. *Let G be a graph on p vertices. If $B^+(G) \geq \lceil p/2 \rceil - 1$, then $B^+(G + K_1) \leq B^+(G) + 1$.*

Proof: Let f be a B^+-numbering of G and define numbering g on $G + K_1$, where u is the vertex corresponding to K_1, by

$$g(v) = \begin{cases} f(v) & \text{if } f(v) < \lceil p/2 \rceil + 1 \\ \lceil p/2 \rceil + 1 & \text{if } v = u \\ f(v) + 1 & \text{if } f(v) \geq \lceil p/2 \rceil + 1 \end{cases}.$$

The largest edge weights involving u are at most $g(u) + (p+1) - (p+2) = \lceil p/2 \rceil$ and $(p+2) - [g(u)+1] \leq \lceil p/2 \rceil$. Furthermore, the edge weight of an edge in G increases by at most one in $G + K_1$. □

Theorem 20. *For any graph G on $p \geq 2$ vertices,* $B^+(G + K_n) = \begin{cases} n + \lceil p/2 \rceil - 1 & \text{if } B^+(G) < \lceil p/2 \rceil - 1 \\ n + B^+(G) & \text{if } B^+(G) \geq \lceil p/2 \rceil - 1 \end{cases}.$

Proof: Suppose first that $n = 1$ and $B^+(G) \geq \lceil p/2 \rceil - 1$. Then $B^+(G + K_1) = B^+(G) + 1$ by Lemmas 18 and 19. Observe now that $B^+(G+K_1) \geq \lceil p/2 \rceil \geq \lceil (p+1)/2 \rceil - 1$ so the second condition is satisfied for $G + K_1$. Repeating the argument for this graph gives $B^+[(G+K_1)+K_1] = B^+(G+K_2) = 1 + B^+(G+K_1) = 2 + B^+(G)$ and repeating $n-2$ additional times yields the second result. Now assume $B^+(G) < \lceil p/2 \rceil - 1$ and label $G+K_1$ as in the proof to Lemma 19. Then the edge weight in $G+K_1$ of any edge not involving u is at most $\lceil p/2 \rceil - 1$ and of any edge involving u is at most $\lceil p/2 \rceil$. Since B^+ can only increase as edges are added, it follows from Lemma 17, by adding edges to $\overline{K}_p$ to obtain G, that $B^+(G + K_1) \geq \lceil p/2 \rceil$, yielding the result when $n = 1$. As before $\lceil p/2 \rceil \geq \lceil (p+1)/2 \rceil - 1$ so the second condition holds for $G + K_1$ and $B^+(G + K_n) = B^+[(G + K_1) + K_{n-1}] = n - 1 + B^+(G + K_1) = n - 1 + \lceil p/2 \rceil$. □

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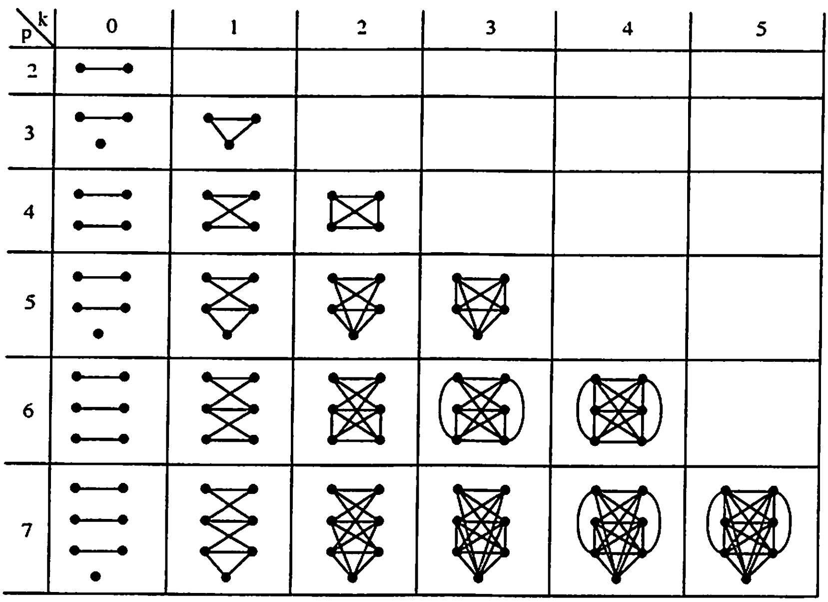

Figure 1: $G_{p,k}$ for small values of p and k

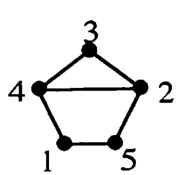

Figure 2: Graph whose additive bandwidth increases when a vertex is removed

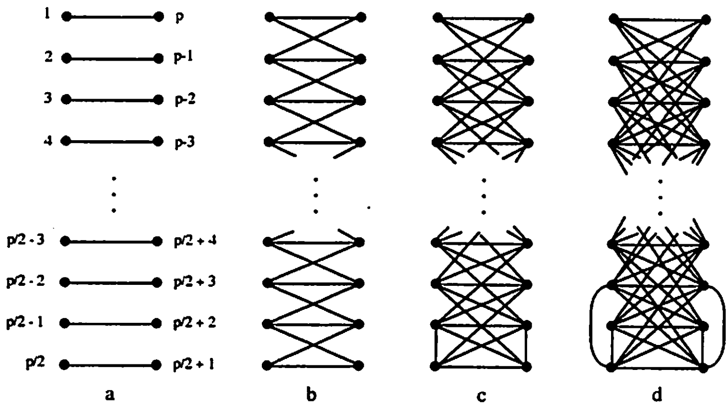

Figure 3: B^+-labeling of $G^+_{p,k}$ for even p and $k = 0, 1, 2, 3$

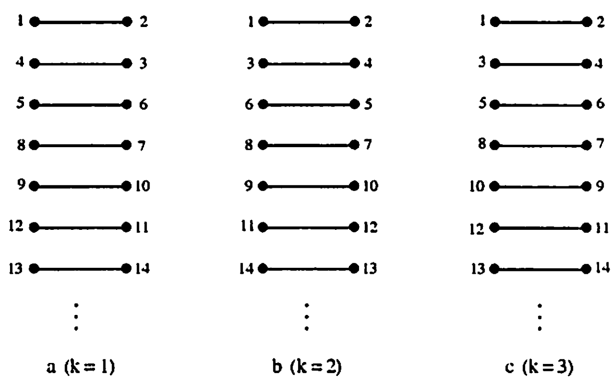

Figure 4: B-labelings of the graphs of Figure 3 for $k = 1, 2, 3$ (only the rungs are shown)

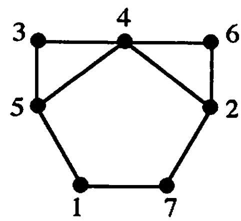

Figure 5: Graph with three odd cycles each of which contains a vertex not in the others and for which $B^+ = 2$

$$G = C_3 \cup C_3 \cup C_7 \cup C_9 \text{ and } p = 22$$

1,	2,3,	4,5,	6,7,8,9,	10, 11, 12, 13,	14, 15, 16, 17,	18, 19,	20, 21,	22
B_4	B_3	B_2	B_1	S	A_1	A_2	A_3	A_4

Array:

C_9	22	2	19	6	13	14	5	20	1
C_7		3	18	7	12	15	4	21	
C_3				8	11	16			
C_3				9	10	17			

(Undefined entries are zero)

Figure 6: Example of the labeling procedure used in the proof of Lemma 10

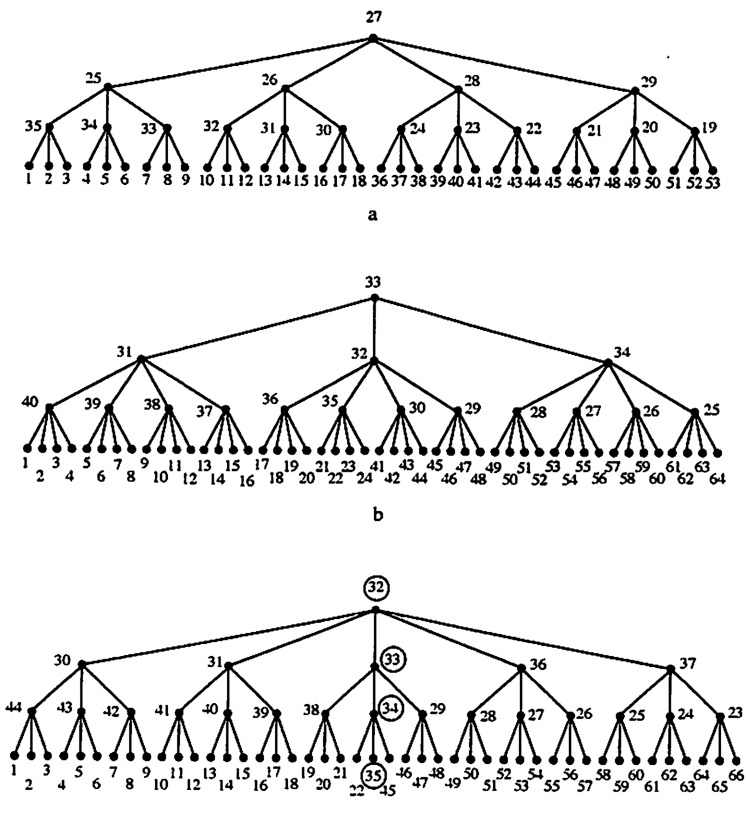

Figure 7: Examples of tree labelings which demonstrate upper bounds

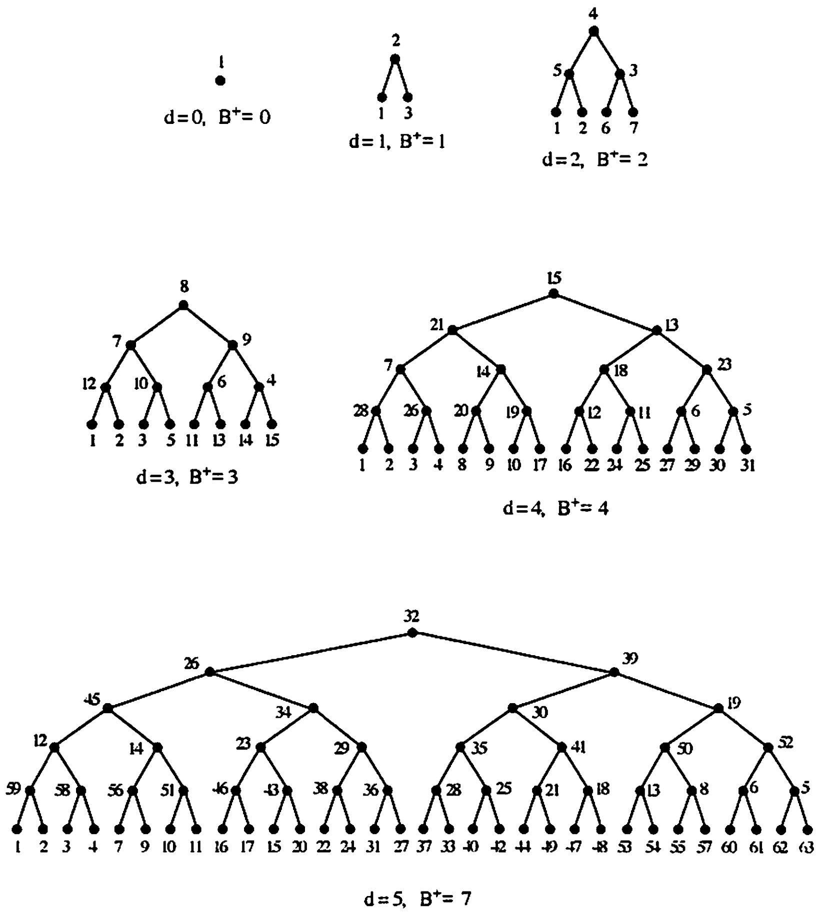

Figure 8: B^+ **labelings of full binary trees to depth 5**