Graphs with Dense Constant Neigbourhoods

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ABSTRACT. Let G be a finite graph and x be its vertex. The *neighbourhood* of x in G, denoted $N_G(x)$, is a subgraph of G induced by all vertices adjacent to x. G is a *graph with a constant neighbourhood* if there exists a graph H such that $N_G(x)$ is isomorphic to H for every vertex x of G.

We completely characterize graphs with constant neighbourhoods isomorphic to complements of regular disconnected graphs.

Let G be a finite connected graph without loops and multiple edges and x be its vertex. The *neighbourhood* of x in G (denoted $N_G(x)$) is the subgraph of G induced by all vertices adjacent to x. A graph G is a *graph with constant neighbourhood* if there exists a graph H such that $N_G(x) \cong H$ for any vertex x of G. We also say that G is a *locally* H graph. A graph H is *realizable* if there exists a locally H graph G. The *order* of a graph G is the number of its vertices while the *size* is the number of its edges.

R. Nedela [4] suggested that the structure of graphs with a "dense" constant neighbourhood might often (but not always) be quite restrictive. Here "dense" means that the number of edges in the neighbourhood is large.

Brouwer, Cohen and Neumaier [1, pp.12–13] proved that the structure of a graph with a dense neighbourhood is uniquely determined even under weaker assumptions. Namely, a regular graph with sufficiently dense regular neighbourhoods of the same degree must have a constant neighbourhood.

Theorem 1. (Brouwer, Cohen, Neumaier) *Let G be an r-regular graph and let the neighbourhood of any vertex of G be a t-regular graph. If $t \geq r - \sqrt{r} - 1/2$, then G is complete multipartite.*

A similar result (with a weaker bound) was independently proven by Šoltés [5].

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Another approach to the problem is to prescribe a structure of the "dense" neighbourhood in a locally H graph G. Results of this type were obtained by Zelinka [6], Bugata, Horňák and Jendrol' [2] and Nedela [4].

Zelinka [6] proved that a locally $\overline{P_n}$ graph exists for any $n \geq 4$, where $\overline{P_n}$ denotes the complement of a path with n vertices, while no locally $\overline{C_n}$ graph exists for a cycle C_n with more than 6 vertices. Bugata, Horňák and Jendrol' [2] gave a complete characterization of graphs with constant neighbourhoods isomorphic to complements of trees.

Theorem 2. (Bugata, Horňák, Jendrol') *Let T be a tree. Then a locally $\overline{T}$ graph exists if and only if T is a path or a star.*

Nedela [4] described a special class of non-realizable graphs.

Theorem 3. (Nedela) *Let H be a graph satisfying the following conditions:*

(i) there are two adjacent vertices in H whose degrees differ by at least 2,

(ii) each 4-vertex induced subgraph of H contains at least two adjacent edges.

Then there is no locally H graph.

In this article we examine a modification of the above mentioned structural approach. In our sense, a "dense" graph means that a neighbourhood of a vertex can be only "locally thin" in certain bounded areas. More precisely, we consider graphs with constant neighbourhoods isomorphic to complements of disjoint graphs.

A vertex x of a graph H is *universal* if it is adjacent to all other vertices of H. A *composition* $G[H]$ (also called a *lexicographic product*) of graphs G and H can be defined as follows: $V(G[H]) = V(G) \times V(H)$ and two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G[H]$ if and only if either (i) $x_1x_2 \in E(G)$ or (ii) $x_1 = x_2$ and $y_1y_2 \in E(H)$. In other words, take a graph G, put a copy of H to every vertex of G and replace each edge of G by $K_{n,n}$ ($n = |V(H)|$). A *join* (or a *sum*) $G+H$ of graphs G and H is a graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$.

P. Hell [3] presented a characterization of realizable graphs with universal vertices.

Theorem 4. (Hell) *If H has n universal vertices, then H is not realizable unless $H \cong K_n + H'[K_{n+1}]$ for a realizable graph H' without universal vertices.*

The graphs described above are (if H' is not an empty graph) complements of disconnected graphs with n isolated vertices and at least one nontrivial component. In contrast to Hell's result [3] we study graphs which are complements either of disconnected regular graphs or of disconnected graphs with all components of the same order or the same size. We denote a subgraph of G induced by a vertex set X by $\langle X\rangle_G$.

Lemma 5. *Let H be a disconnected graph whose all components are of the same order q and $\overline{H}$ be a realizable graph. Then every component of H is a regular graph.*

Proof: If $q = 1$ then H is indeed regular. Suppose that v is a vertex of a locally $\overline{H}$ graph G, H is a disjoint union of connected components $H_1, H_2, \dots, H_s$ of order $q \geq 2$ each and m is a minimum degree of H. We first prove that if a component H_i contains a vertex of degree m, then all other components are regular of degree m.

Let $N_G(v) \cong \overline{H}$, $H_i = \langle X\rangle_H = \langle x_1, x_2, \dots, x_q\rangle_H, H - H_i = \langle Y\rangle_H = \langle y_1, y_2, \dots, y_{r-q}\rangle_H$ (where $r = sq$) and $deg_H x_1 = m$. Suppose, to the contrary, that acomponent H_j contains a vertex, say y_1, such that $deg_H y_1 > m$. Since $deg_H x_1 = m$, and x_1 is in G adjacent to all $r - q$ vertices of Y, $q-m-1$ vertices of X and to v, x_1 must be adjacent to m vertices of G not belonging to the set $X \cup Y \cup v$, say $z_1, z_2, \dots, z_m$. Because $deg_H y_1 > m$, y_1 is adjacent to at most $r - m - 1$ vertices of the set $X \cup Y \cup v$. On the other hand, G is regular of degree r and then y_1 must have at least one other neighbour different from $z_1, z_2, \dots, z_m$. Let this vertex be u. Since the only vertices in $V(G) - \{v \cup X \cup Y\}$ adjacent to x_1 are $z_1, z_2, \dots, z_m$, the vertex u is in G not adjacent to x_1. Then H contains the edge ux_1 and $\langle u, x_1, x_2, \dots, x_q\rangle_H$ is a connected component of $\overline{N_G(y_1)}$ with $q + 1$ vertices. Therefore $N_G(y_1)$ is not isomorphic to $\overline{H}$, which is the desired contradiction.

Thus all the components of $H - H_i$ are regular of degree m and we can apply our assertion once more to show that H_i is also regular of degree m, which completes the proof. □

The converse of Lemma 5, i.e., the regularity of H implies that all components of H have the same order, is also true. However, before proving it (even in a stronger form) we examine disconnected graphs with all components of the same size. The methods are quite similar to those used in the proof of the previous lemma.

Suppose that a disconnected graph H has order r and all its components have size $k \geq 1$ (if $k = 0$ then H is clearly regular). G is again a locally $\overline{H}$ graph, v is a vertex of G and x_1 is a vertex of minimum degree, m, in $\overline{N_G(v)} \cong H$. Let x_1 belong to a component $H_1 = \langle X\rangle_H = \langle x_1, x_2, \dots, x_t\rangle_H$ of H and $Y = \{y_1, y_2, \dots, y_{r-t}\}$ be a set of all vertices of $N_G(v)$ not belonging to X. Since $deg_H x_1 = m$, there must be again m neighbours of x_1 not belonging to $X \cup Y \cup v$, say $z_1, z_2, \dots z_m$. Suppose that there is a vertex $y_j \in Y$ such that $deg_H y_j > m$. Then there is a vertex $u \notin X \cup Y \cup v$ which is a neighbour of y_j in G. Clearly, u is not adjacent to x_1 in G.

Then $\overline{N_G(y_j)}$ contains a connected component $\langle x_1, x_2, \ldots, x_t, u\rangle_{\overline{G}}$, because y_j is in G adjacent to all vertices of X. But this component contains at least $k+1$ edges: k edges induced by the set X and the edge ux_1, which is a contradiction. Hence all vertices $y_1, y_2, \ldots, y_{r-t}$ are of degree m in H. Repeating the arguments once more, we can see that the component H_1 is also regular of degree m and therefore the following lemma holds.

Lemma 6. *Let H be a disconnected graph with all components of the same size k and $\overline{H}$ be a realizable graph. Then H is regular.*

Applying the arguments once again, we now prove the converse of Lemma 5 and determine a structure of a realizable complement of a disconnected regular graph.

Lemma 7. *Let H be a disconnected, p-regular graph of order r. Let G be a locally $\overline{H}$ graph. Then H is a disjoint union of $r/(p+1)$ complete graphs K_{p+1} and $G \cong K_{p+1,p+1,\ldots,p+1}$ with $r/(p+1)+1 \geq 3$ parts.*

Proof: Let $H = H_1 \cup H_2 \cup \ldots H_s$ with $|H_1| \leq |H_2| \leq \cdots \leq |H_s|$. Let v be any vertex of a locally $\overline{H}$ graph G. Then $\overline{N_G(v)} = H_1 \cup \cdots \cup H_s$ where $H_1 = \langle X\rangle_H = \langle x_1, x_2, \ldots, x_q\rangle_H$, $H_2 \cup \cdots \cup H_s = \langle Y\rangle_H = \langle y_1, y_2, \ldots, y_{r-q}\rangle_H$. Consider a neighbourhood of x_1 in G. Since x_1 is adjacent to v, all vertices $y_1, \ldots, y_{r-q}$ and $q-p-1$ vertices of X, say $x_{p+2}, \ldots, x_q$, it must be adjacent to p other vertices, say $z_1, \ldots, z_p$. Every vertex $y_i \in H_j, 2 \leq j \leq s$, is adjacent in $N_G(x_1)$ to $r-q-p-1$ vertices of the set Y, $q-p-1$ vertices $x_{p+2}, \ldots, x_q$, the vertex v and to no other vertex of $\{v \cup X \cup Y\}$. Because $N_G(x_1)$ is $(r-p-1)$-regular, it follows that y_i has to be adjacent, in addition to the $r-2p-1$ vertices of $\{v \cup X \cup Y\}$, to p other vertices. Since the only vertices of $N_G(x_1) - \{v \cup X \cup Y\}$ are $z_1, z_2, \ldots, z_p$, clearly y_i is adjacent to all vertices of $Z = \{z_1, z_2, \ldots, z_p\}$.

Let $x_i \neq x_1$ be another vertex of X and suppose that $N_G(x_i)$ contains, besides the vertices $y_1, \ldots, y_{r-q}$ and $q-p-1$ vertices of X, a vertex $u \notin Z$. This vertex is then adjacent to all vertices of Y, but in this case any $y_j \in Y$ is adjacent to $r-p-1$ vertices of $X \cup Y$, p vertices of Z and the vertices u and v. Hence $deg_G y_j \geq r+1$, which contradicts the regularity of G. Thus any vertex $x_i \in X$ is adjacent to all vertices $z_1, \ldots, z_p$. Because now every vertex of Z is adjacent to all vertices $x_1, \ldots, x_q$ and $y_1, \ldots, y_{r-q}$, there is no edge in $\langle Z\rangle_G$ and $N_G(x_i)$ contains $\overline{K_{p+1}} = \langle z_1, \ldots,, z_p, v\rangle_G$. Therefore H contains K_{p+1}, and the minimality of H_1 immediatelly yields $H_1 \cong K_{p+1}$.

Now suppose that there is at least one component of H which is not isomorphic to K_{p+1}. Let $m+1$ be the smallest number such that $H_{m+1} \ncong K_{p+1}$. Then $\overline{N_G(v)} \cong H$ contains exactly m copies of K_{p+1}, since $p+1 < |H_{m+1}| \leq \cdots \leq |H_s|$. Obviously, if y_j is a vertex of H_{m+1}, it is adjacent in G to all vertices of $H_1 \cup \cdots \cup H_m$. Furthermore, as we have seen above, it is adjacent to all vertices $z_1, \ldots, z_p$ and, of course, to v.

But $\langle z_1, \ldots, z_p \rangle_H \cong K_p$ and no vertex of Z is adjacent to v in G. Then $\langle v, z_1, \ldots, , z_p \rangle_{\overline{G}} \cong K_{p+1}$ and $\overline{N_G(y_j)}$ contains $m+1$ disjoint copies of K_{p+1}, namely $H_1, \ldots, H_m, \langle v \cup Z \rangle_{\overline{G}}$, which contradicts the fact that H contains exactly m disjoint copies of K_{p+1}. Hence $m = s$ and G is a complete $(s+1)$-partite graph $K_{p+1,\ldots,p+1}$. □

Combining Lemmas 5, 6 and 7, we immediatelly have the following.

Theorem 8. *Let H be a disconnected graph and $\overline{H}$ be a realizable graph. Then the following are equivalent:*

(i) all components of H have the same size,

(ii) H is regular,

(iii) all components of H have the same order.

Proof:

(i) $\Rightarrow$ (ii) follows from Lemma 6.

(ii) $\Rightarrow$ (iii) follows from Lemma 7.

(iii) $\Rightarrow$ (i). From Lemmas 5 and 7 it follows that (ii) and (iii) are equivalent, thus, H is regular and has all components of the same order. Then, obviously, all components of H are of the same size. □

Our main result now easily follows from Lemma 7 and Theorem 8 and from the well-known fact (see, e.g., [3]) that every regular complete s-partite graph is uniquely realizable by a complete $(s+1)$-partite graph.

Theorem 9. *The complement $\overline{H}$ of a disconnected graph H which satisfies one of the following equivalent conditions:*

(i) all components of H have the same size, or

(ii) H is regular, or

(iii) all components of H have the same order,
is realizable if and only if $\overline{H}$ is regular complete multipartite graph.

An equivalent rephrasing of Theorem 9 may be of some interest:

Theorem 9'. *Let H be a disconnected graph satisfying one of the conditions (i)–(iii) above and let G be a locally $\overline{H}$ graph. Then G is a regular complete multipartite graph with at least three parts.*

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