Clique Decompositions of Strong Short Ladders

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ABSTRACT. An explicit recurrence is obtained for the clique polynomial of a short ladder in which the two diagonals are drawn in each cell. From this result, an explicit formula for the number of decompositions of the ladder into triangles and 4-cliques is obtained. The recurrence is then used to obtain results for the matching polynomial of the ladder. Finally, an association is made with a particular tiling problem.

1. Introduction

The graphs considered here are finite and contain no loops nor multiple edges. Let G be such a graph. We define an *n-clique* in G to be a subgraph of G which is isomorphic to the complete graph with n nodes. For $n > 2$, the clique will be called *proper*. A *clique cover* (or *vertex-clique covering*) of G, is spanning subgraph of G in which all components are cliques. We will use the word 'cover' to mean clique cover. A *proper clique cover* is a cover in which all the components are proper cliques, whereas a matching ls a cover in which none of the components is proper.

Let us associate with each clique α in G, an indeterminate or *weight* w_α, and with every clique cover C, the weight $w(C) = \prod_\alpha w_\alpha$, where the product is taken over all the components of C. Then the *clique polynomial* of G is

$$K(G;\underline{w}) = \sum w(C),$$

where the summation is taken over all the (clique) covers in G, and $\underline{w}$ is a vector of indeterminates. In this paper, we will assign the same weight w_r to each r-clique in G. Therefore we will have $\underline{w} = (w_1, w_2, w_3, \dots)$. Some basic results on clique polynomials have been given in the introductory paper [1]. The *proper clique polynomial* of G is the clique polynomial in which only proper covers are considered. The *matching polynomial* of G is analogously defined.

The *short ladder* s_n is the graph obtained by joining the corresponding nodes of two equal paths with $n+1$ nodes. The n squares so formed will be called *cells*. If in each cell we now draw the two diagonals, then the resulting graph will be called the *strong short ladder* A_n. We will take A_0 to be an edge. It can be easily confirmed that A_n will contain $2n+2$ nodes and $5n+1$ edges. We note that A_n is a simple version of the graph associated with a two-dimensional lattice graph in the Ising problem with magnetic field (see Harary [5]).

In this paper, we obtain an explicit recurrence for the clique polynomial of A_n. From this, we deduce the parallel result for the proper clique polynomial of A_n. We then obtain an explicit formula for the number of decompositions of A_n into triangles and 4-cliques. Results for the matching polynomial of A_n are then obtained. This yields results for the number of decompositions of A_n into matchings. Finally, an association is made with the clique covers of A_n and a particular tiling problem.

2. Some Basic Result

Let G be a graph containing an edge e. Then either a clique cover contains e or it does not. We can therefore partition the covers into two classes: (i) those containing e and (ii) those not containing e. The covers which contain e will be covers of the graph G^* obtained from G by incorporating e, i.e. e is required to belong to every cover of G^*. The covers in class (ii) will be covers of the graph G' obtained from G by deleting e. Hence we obtain the following result.

Theorem 1. (The Fundamental Edge Theorem) *Let G be a graph containing an (unincorporated) edge e. Then*

$$K(G;\underline{w}) = K(G';\underline{w}) + K(G^*;\underline{w}).$$

Let us denote the node set of G by $V(G)$. Let $S \subseteq V(G)$. Then $G - S$ will denote the graph obtained from G by removing the nodes in S.

Theorem 2. (The Fundamental Node Theorem) *Let G be a graph with node set $V(G) = \{v_1, v_2, \ldots, v_p\}$. Then*

$$K(G;\underline{w}) = w_1 K(G - \{v_j\};\underline{w}) + \sum_{r=2}^{p} w_r \sum K(G - \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\};\underline{w}),$$

where $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ is the node set of an r-clique ($r > 1$) containing node v_j; and the second summation is taken over all such r-cliques in G. Also, if no such r-clique exists, then $K(G - \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\};\underline{w}) = 0$.

Proof: We can partition the covers of G into two classes: (i) those in which node v_j is isolated and (ii) those in which it is not. If v_j is isolated, then it will have a weight of w_1 and the remaining components of the cover will be a cover of the graph $G - \{v_j\}$. If v_j belongs to an r-clique, for $r > 1$, then the clique will have weight w_r and the remaining elements of the cover will be a cover of the graph obtained from G by removing the nodes of the r-clique to which v_j belong. Hence the result follows. □

It is clear that the above theorems can be used recursively to obtain clique polynomials of graphs. These algorithms are called the *fundamental edge algorithm* and *fundamental node algorithm* respectively. Both algorithms are suitable for computer implementation, and have been programmed in PASCAL. The 'built in' recursive feature of PASCAL seems ideal for algorithms of this type.

3. The Clique Polynomial of A_n

For brevity, we will write G for $K(G;\underline{w})$, when it would lead to no confusion and especially in recurrences. $G(t)$ will denote the generating function of $K(G;\underline{w})$ with the indicator function t.

The graph A_n and one of its subgraphs B_{n-1} are shown below in Figure 1. B_{n-1} is obtained from A_n by removing a node of valence 3.

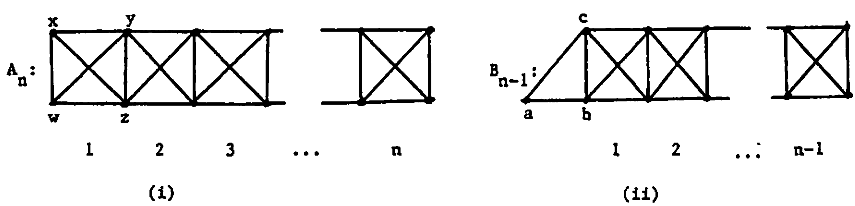

Figure 1

Let us apply the node algorithm to A_n, using node x (i.e. $v_i = x$). Then from Theorem 2 we get

$$K(A_n;\underline{w}) = w_1 K(A_n - \{x\};\underline{w}) + \sum_{r=2} w_r \sum K(G - \{x, v_{i_2}, \ldots, v_{i_r}\};\underline{w}) \quad (1)$$

Now $A_n - \{x\}$ is the graph B_{n-1}. For $r = 2$, we obtain the following graphs: $A_n - \{x, y\}$, $A_n - \{x, w\}$ and $A_n - \{x, z\}$ (see Figure 1(i)). $A_n - \{x, y\}$ consists of B_{n-2} with the twig wz attached to it. By applying the edge algorithm to this graph, using wz, we get

$$A - \{x, y\} = w_1 B_{n-2} + w_2 A_{n-2}. \quad (2)$$

Similarly, we get

$$A_n - \{x, z\} = w_1 B_{n-2} + w_2 A_{n-2} \tag{3}$$

Clearly

$$A_n - \{x, w\} = A_{n-1}. \tag{4}$$

For $r = 3$, x must belong to a triangle. In this case we obtain the graphs $A_n - \{x, y, z\}$, $A_n - \{x, w, z\}$ and $A_n - \{x, y, w\}$. Hence we obtain the following relationships between the polynomials;

$$A_n - \{x, y, z\} = w_1 A_{n-2}, \tag{5}$$

$$A_n - \{x, w, z\} = B_{n-2}, \tag{6}$$

$$\text{and } A_n - \{x, y, w\} = B_{n-2}. \tag{7}$$

For $r = 4$, x belongs to a 4-clique. The only possible 4-clique is the cell containing x. Hence we get

$$A_n - \{w, x, y, z\} = A_{n-2}. \tag{8}$$

By substituting the results of Equations (2), (3), (4), (5), (6) and (7) into Equation (1), we obtain the following lemma.

Lemma 1.

$$A_n = w_1 B_{n-1} + (2w_1 w_2 + 2w_3) B_{n-2} + (2w_2^2 + w_1 w_3 + w_4) A_{n-2} + w_2 A_{n-1} (n > 1).$$

Let us apply the node algorithm to B_n, using node a (see Figure 1 (ii)). This yields

$$B_n = w_1 K(B_n - \{a\}; \underline{w}) + \sum_r w_r \sum K(B_n - \{a, v_{i_2}, \ldots, v_{i_r}\}; \underline{w}).$$

$B_n - \{a\}$ is the graph A_n. For $r = 2$, the graphs are $B_n - \{a, b\}$ and $B_n - \{a, c\}$. These graphs are isomorphic to B_{n-1}. For $r = 3$, the only graph is $B - \{a, b, c\}$; which is A_{n-1}. Hence we obtain the following lemma.

Lemma 2.

$$B_n = w_1 A_n + 2w_2 B_{n-1} + w_3 A_{n-1} (n > 0).$$

The following relations between $A(t)$ and $B(t)$, the generating functions for A_n and B_n respectively, can be immediately obtained from the above lemmas.

$$\begin{aligned}(1 - w_2 t - 2w_2^2 t^2 - w_1 w_3 t^2 - w_4 t^2) A(t) &= (A_0 + A_1 t - w_1 B_0 t - w_2 A_0 t) \\ &+ (w_1 t + 2w_1 w_3 t^2 + 2w_3 t^2) B(t)\end{aligned} \tag{9}$$

and

$$(1 - 2w_2 t)B(t) = (w_1 + w_3 t)A(t) + B_0 - w_1 A_0, \tag{10}$$

where $A_0 = w_1^2 + w_2$ and $B + 0 = w_1^3 + 3w_1 w_2 + w_3$.

Equations (8) and (9) can be solved simultaneously to obtain an explicit generating function for A_n. This is given in the following lemma.

Lemma 3.

$$A(t) = \frac{w_1^2 + w_2 + (2w_1^2 + 4w_1 w_3 + w_4)t - (4w_2^3 + 2w_2 w_4 - 2w_3^2)t^2}{[1 - (w_1^2 + 3w_2)t - (2w_1^2 w_2 + 4w_1 w_3 + w_4)t^2 + (2w_2 w_4 + 4w_2^3 - 2w_3^2)t^3]}.$$

From this lemma, we obtain the explicit recurrence for A_n given in the following theorem.

Theorem 3.

$$A_n = (w_1^2 + 3w_2)A_{n-1} + (2w_1^2 + 4w_1 w_3 + w_4)A_{n-2} - (2w_2 w_4 + 4w_2^3 - 2w_3^2)A_{n-3} \quad (n > 2),$$

where

$$A_0 = w_1^2 + w_2, \; A_1 = w_1^4 + 6w_1^2 w_2 + 4w_1 w_3 + 3w_2^2 + w_4$$

and

$$A_2 = w_1^6 + 11w_1^4 w_2 + 8w_1^3 w_3 + 23w_1^2 w_2^2 + 2w_1^2 w_4 + 16w_1 w_2 w_3 + 5w_2^3 + 2w_2 w_4 + 2w_3^2.$$

The following table gives values of $K(A_n; \underline{w})$ for $n = 0$, 1, 2, 3 and 4.

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n	$K(A_n; \underline{w})$
0	$w_1^2 + w_2$
1	$w_1^4 + 6w_1^2 w_2 + 4w_1 w_3 + 3w_2^2 + w_4$
2	$w_1^6 + 11w_1^4 w_2 + 8w_1^3 w_3 + 23w_1^2 w_2^2 + 2w_1^2 w^4 + 16w_1 w_2 w_3 + 5w_2^3$ $+2w_2 w_4 + 2w_3^2$
3	$w_1^8 + 16w_1^6 w_2 + 12w_1^5 w_3 + 68w_1^4 w_2^2 + 3w_1^4 w_4 + 72w_1^3 w_2 w_3 + 76w_1^2 w_2^3$ $+14w_1^2 w_2 w_4 + 20w_1^2 w_3^2 + 60w_1 w_2^2 w_3 + 8w_1 w_3 w_4 + 11w_2^4 + 11w_2^4$ $+7w_2^2 w_4 + 8w_2 w_3^2 + w_4^2$
4	$w_1^{10} + 21w_1^8 w_2 + 16w_1^7 w_3 + 138w_1^6 w_2^2 + 4w_1^6 w_4 + 168w_1^5 w_2 w_3$ $+322w_1^4 w_2^3 + 36w_1^4 w_2 w_4 + 54w_1^4 w_3^2 + 400w_1^3 w_2^2 w_3 + 24w_1^3 w_3 w_4$ $+225w_1^2 w_2^4 + 64w_1^2 w_2^2 w_4 + 148w_1^2 w_2 w_3^2 + 3w_1^2 w_4^2 + 184w_1 w_2^3 w_3$ $+40w_1 w_2 w_3 w_4 + 16w_1 w_3^3 + +21w_2^5 + 16w_2^3 w_4 + 30w_2^2 w_3^2$ $+3w_2 w_4^2 + 4w_3^2 w_4$

4. The Proper Polynomial of A_n

It is interesting to consider only the proper covers of A_n. These will be covers in which each component is either a triangle or a 4-clique. A recurrence for the proper polynomial of A_n can be immediately obtained from Theorem 3, by putting $w_1 = w_2 = 0$.. i.e. $\underline{w} = (0, 0, w_3, w_4, \dots) = \underline{w}'$. We will denote $K(A_n; \underline{w}')$ by A'_n. Hence we get

Corollary 3.1.

$$A'_n = w_4 A'_{n-2} + 2w_3^2 A'_{n-3} (n > 2),$$

where

$$A'_0 = 0, A'_1 = w_4 \text{ and } A'_2 = 2w_3^2.$$

By using standard techniques, we can obtain the generating function $A'(t)$ for A'_n, given below.

Lemma 4.

$$A'(t) = \frac{w_4 t + 2w_3^2 t^2}{1 - w_4 t^2 - 2w_3^2 t^3}.$$

After some calculations, we obtain the following theorem.

Theorem 4.

$$A'_n = \sum_{a=0}^{\lfloor (n-1)/2 \rfloor} \binom{(n-a+1)/2}{a} 2^a w_3^{2a} w_4^{(n-3a+1)/2},$$

where $\binom{k}{r} = 0$, *when* $k \leq 0$.

From this theorem, we can deduce the number of ways of covering A_n with a given number of triangles and 4-cliques. The result is given in the following theorem.

Theorem 5. *A_n can be covered with r triangles and s 4-cliques if and only if r is even and $3r + 4s = 2n + 2$. In this case, the number of ways of covering A_n is $\binom{(2n-r+2)/4}{r/2} 2^{r/2}$.*

The following corollary is immediate.

Corollary 5.1.

(i) *A_n can be covered with r triangles if and only if $r = 2k$ and $n = 3k-1$, for some positive integer k; and in this case, the number of ways of covering A_n is 2^k.*

(ii) *A_n can be covered with s 4-cliques if and only if $s = k + 1$ and $n = 2k + 1$, for some non-negative even integer k; and in this case, there is only one way of covering A_n.*

We note that Theorem 5 (and its corollary) can be obtained by direct combinatorial arguments.

5. The Matching Polynomial of A_n

In this section we will consider decompositions of A_n into nodes and edges only. As mentioned above, such a decomposition is called a matching, and the resulting polynomial—a matching polynomial. The basic properties of matching polynomials are given in Farrell [2] and Godsil and Gutman [4]. Formally, the matching polynomial of G; denoted by $M(G;\underline{w})$, and the clique polynomial of G are related as given in the following lemma. (Note: Some authors define 'matching' differently.)

Lemma 5.

$$M(G;\underline{w}) = K(G;(w_1, w_2, 0, 0, \dots)).$$

By applying the result of this lemma to Theorem 3, we obtain the following analogous result for matching polynomials in which $M(G)$ is written for $M(G;\underline{w})$.

Theorem 6.

$$M(A_n) = (w_1^2 + 3w_2)M(A_{n-1}) + 2w_1^2 w_2 M(A_{n-2}) - 4w_2^3 M(A_{n-3})(n > 2)$$

with

$$M(A_0) = w_1^2 + w_2, M(A_1) = w_1^4 + 6w_1^2 w_2 + 3w_2^2 \text{ and}$$
$$M(A_2) = w_1^6 + 11w_1^4 w_2 + 23w_1^2 w_2^2 + 5w_2^3.$$

Definition: A *defect-d* matching is a matching with d isolated nodes.

We shall denote the number of defect-d matchings in G by $N_d(G)$. We can obtain a recurrence for $N_d(A_n)$ directly from Theorem 6. It is given in the following corollary.

Corollary 6.1. *A_n has a defect-d matching if and only if d is even, and in this case,*

$$N_d(A_n) = N_{d-2}(A_{n-1}) + 3N_d(A_{n-1}) + 2N_{d-2}(A_{n-2}) - 4N_d(A_{n-3})(n > 2).$$

Corollary 6.1 is a useful result. It can be used to obtain explicit formulae for the coefficients of $M(A_n)$, by using appropriate values of d (see Farrell and Wahid [3]). For example, by putting $d = 0$, we obtain the following recurrence for the number of perfect matchings in A_n.

Corollary 6.2.

$$N_0(A_n) = 3N_0(A_{n-1}) - 4N_0(A_{n-3})(n > 2),$$

with

$$N_0(A_0) = 1, N_0(A_1) = 3 \text{ and } N_0(A_2) = 5.$$

Theorem 7.

$$N_0(A_n) = \frac{1}{3}\left[2^{n+2} - (-1)^{n+2}\right] (n \geq 0).$$

Proof: Multiply the recurrence by t^n and sum from $n = 3$ to infinity. This yields

$$\sum_{n=3} N_0(A_n)t^n = 3\sum_{n=3} N_0(A_{n-1})t^n - 4\sum_{n=3} N_0(A_{n-3})t^n.$$

By simplifying, using the boundary conditions, and writing $N_0(t)$ for the generating function of $N_0(A_n)$, we get

$$\begin{aligned} N_0(t) &= \frac{1-4t^2}{1-3t+4t^3} = \frac{1+2t}{(1+t)(1-2t)} \\ &= \frac{-\frac{1}{3}}{1+t} + \frac{\frac{4}{3}}{1-2t}. \end{aligned}$$

The result follows by extracting the coefficient of t^n on the RHS. □

The following result was proved in [3] .

Lemma 6. *Let G be a graph with p nodes and q edges. Then*

(i) $N_p(G) = 1$

(ii) $N_{p-2}(G) = q$, *and*

(iii) $N_{p-4}(G) = \binom{q}{2} - \varepsilon$, *where ε is the number of paths of length 2 in G. That is,*

$$N_{p-4}(G) = \binom{q}{2} - \sum_{i=1}^{p}\binom{v_i}{2},$$

where v_i is the valency of node i in G.

Theorem 8.

(i) $N_{2n+2}(A_n) = 1$

(ii) $N_{2n}(A_n) = 5n + 1$

(iii) $N_{2n-2}(A_n) = \frac{5n}{2}(5n+1) - 20n + 8$ $(n > 0)$.

Proof: A_n has $2n + 2$ nodes and $5n + 1$ edges. Also for $n > 0$, A_n has 4 nodes of valencey 3 and $2n - 2$ nodes of valence 5. Hence the result follows from Lemma 6. □

Corollary 6.1 can be used to extend the results given in Theorem 8, by putting $d = 2n - 4$, $2n - 6$, etc.

6. An Equivalent Tiling Problem

Consider a tiling of the $2 \times n$ rectangle $R_{2,n}$ shown below in Figure 2,

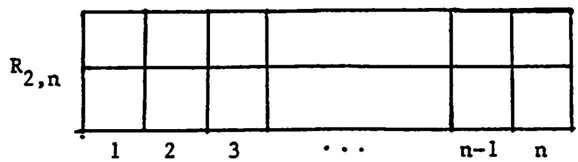

Figure 2

with tiles of the form shown below in Figure 3.

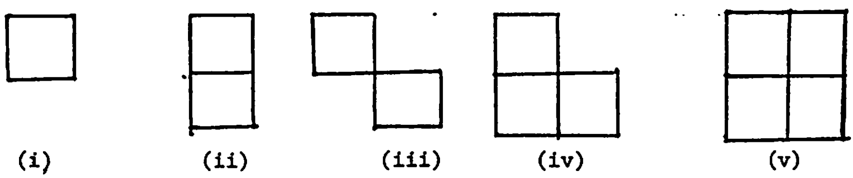

Figure 3

We can associate with $R_{2,n}$ a graph G constructed as follows: The nodes of G will represent the $2n$ squares in $R_{2,n}$. Two nodes u and v will be joined by an edge if and only if the corresponding squares have a common corner. It can be seen that G will be the graph A_n. Also, the graphs associated (in the same way) with the tiles in Figure 3 will be a node, an edge, an edge, a triangle and a 4-clique respectively.

It is easy to see that the problem of tiling $R_{2,n}$ with the tiles in Figure 3 is equivalent to that of covering A_n with cliques. Thus our earlier results give the answers to questions about the number of ways of tiling $R_{2,n}$ with different numbers of tiles of the given type. Notice that the tiles represent all the possible tiles that can be formed by removing cells from $R_{2,2}$ (Figure 3(v)).

We note that the tiles in Figure 3 have also been called *animals* (see Harary and Palmer [6]). Also, a connection between $R_{2,n}$ and A_n was established by Harary ([5] page 28), in a discussion of the Ising problem.

References

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