On Certain Combinatorial Arrays

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ABSTRACT. In this paper we derive and present some necessary conditions for the existence of certain combinatorial arrays (called balanced arrays (B-arrays)) with two elements by making use of some classical inequalities. We discuss briefly the usefulness of these arrays in combinatorics, and statistical design of experiments.

1 Introduction and Preliminaries

The concept of balanced arrays (B-arrays) has served to unify various combinatorial areas of experimental designs. For example, orthogonal arrays and incidence matrices of incomplete block designs are B-arrays with suitable values of the parameters. Furthermore, B-arrays play a very important role in the construction and existence of balanced fractional factorial designs (BFF designs) for symmetrical as well as asymmetrical situations.

For ease of reference, we recall the definition of a B-array with two levels (say, 0 and 1). A B-array T of strength t with two symbols (or levels), N columns (runs, or treatment combinations), m rows (constraints) and index set $\{\mu_0, \mu_1, \mu_2, \ldots, \mu_t\}$ is merely a matrix T of Size $(m \times N)$ whose elements are the two levels (or symbols 0 and 1) such that in every $(t \times N)$ submatrix T_0 (clearly, $t \leq m$) of T, each t-vector (i.e., a vector with t symbols in it) $\underline{\alpha}$ of weight i (the weight of $\underline{\alpha}$ is the number of 1's in it; $0 \leq i \leq t$) appears as a column of T_0 precisely μ_i times. The constants $\mu_i (i = 0, 1, 2, \ldots, t)$, m, N, and t are called the parameters of the B-array T, and sometimes we denote T by $BA\{m, N, 2, t; \mu_0, \mu_1, \ldots, \mu_t\}$. This definition can be easily generalized to a B-array with s symbols. Given the index set

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$\underline{\mu}' = \{\mu_0, \mu_1, \ldots, \mu_t\}$ of a B-array, we can easily find N by using:

$$N = \sum_{i=0}^{t} \binom{t}{i} \mu_i.$$

The existence and construction of B-arrays for an arbitrary index set $\underline{\mu}' = \{\mu_0, \mu_1, \ldots, \mu_t\}$, and m is clearly nontrivial for $m > t$. To construct such arrays, for a given $\underline{\mu}'$ (and hence N), with arbitrary maximum possible value of m, is an important problem both in combinatorial mathematics and statistical design of experiments. Such a problem for Orthogonal arrays (O-arrays) and B-arrays has been investigated, among others, by Rao [13], Bose and Bush [3], Seiden and Zemach [18], Rafter and Seiden [12], Saha and Mukerjee and Kageyama [15], Chopra and Dioś [8], etc. The necessary conditions presented in the form of inequalities in this paper should prove useful in the existence of B-arrays for arbitrary values of m and $\underline{\mu}'$, and also in obtaining an upper bound on m for a given $\underline{\mu}'$. The readers interested to gain further insight into B-arrays and their importance to statistical design of experiments and combinatorics may consult the list of references given at the end, and also consult further references given therein.

2 Main Results and Applications

In this paper, for the sake of simplicity, we confine ourselves to B-arrays with $t = 4$, but the results presented can be extended to general 't' with notations and symbols becoming messy and cumbersome. Next, we state some results which are easy to prove.

Lemma 2.1. *Consider a B-array T with $t = 4$ and $\underline{\mu}' = \mu_0, \mu_1, \mu_2, \mu_3 \mu_4)$. Then T is also of strength t' where $0 \leq t' \leq 4$.*

Remark: Considered as an array of strength $t' = 3, 2$, and 1, its index sets are (A_0, A_1, A_2, A_3), (B_0, B_1, B_2), and (C_0, C_1) respectively, where $A_i = \mu_1 + \mu_{i+1} (i = 0, 1, 2, 3)$; $B_j = A_j + A_{j+1} (i = 0, 1, 2)$; and $C_k = B_k + B_{k+1} (k = 0, 1)$.

Lemma 2.2. *Let X_j denote the number of $(m \times 1)$ columns of weight j $(j = 0, 1, 2, \ldots, m)$ in a B-array $T(m, N, 4, \underline{\mu}')$, then the following results* *must hold:*

$$\sum_{j=0}^{m} X_j = N \tag{2.1}$$

$$\sum jX_j = m_1C_1 \tag{2.2}$$

$$\sum j^2X_j = M_2\ B_2 + m_1C_1 \tag{2.3}$$

$$\sum J^3X_j = m_3A_3 + 3m_2B_2 + m_1C_1 \tag{2.4}$$

$$\sum j^4X_j = m_4\mu_4 + 6m_3A_3 + 7m_2B_2 + m_1C_1 \tag{2.5}$$

where m_r is defined to be $m(m-1)(m-2)\dots(m-r+1)$.

Note: Clearly results (2.1)-(2.5) express the moments of the weights of the columns of T in terms of the parameters m and $\underline{\mu'}$ of T.

Theorem 2.1. *Consider a B-array $T(m \times N)$ with $t = 4$ and $\underline{\mu'} = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$. Then the following holds:*

$$ln\left[\frac{\frac{m_1C_1(m_2B_2+m_1C_1)}{m_2B_2+2m_1C_1}+\frac{(m_3A_3+3m_2B_2+m_1C_1)(m_4\mu_4+6m_3A_3+7m_2B_2+m_1C_1)}{m_4\mu_4+7m_3A_3+10m_2B_2+2m_1C_1}}{}\right]$$

$$\leq ln(m_3A_3+3m_2B_2+2m_1C_1)+ln(m_4\mu_4+6m_3A_3+8m_2B_2+2m_1C_1)$$
$$- ln(m_4\mu_4+7m_3A_3+11m_2B_2+4m_1C_1) \tag{2.6}$$

Proof: In order to obtain (2.6), we consider the following inequality (See Mitrinović, [11]): $\frac{a_1a_2}{a_1+a_2} + \frac{a_3a_4}{a_3+a_4} \leq \frac{(a_1+a_3)(a_2+a_4)}{a_1+a_2+a_3+a_4}$, $a_k > 0$ for all k. We replace each $a_k (k = 1, 2, 3, 4)$ by $\sum j^kX_j$ from (2.2)-(2.5), and taking logs on both sides of the inequality we obtain (2.6).

Theorem 2.2. *Let T be a B-array $(m, N, t = 4, \underline{\mu'})$. Then we have*

$$ln[N + 4m_1C_1 + 11m_2B_2 + 7m_3A_3 + m_4\mu_4] - ln\ 5$$
$$\leq \left[\begin{array}{c} m_1C_1\, ln\, m_1C_1 + (m_1C_1 + m_2B_2)\, ln\,(m_1C_1 + m_2B_2)+ \\ (m_3A_3 + 3m_2B_2 + m_1C_1)\, ln\,(m_3A_3 + 3m_2B_2 + m_1C_1)+ \\ (m_4\mu_4 + 6m_3A_3 + 7m_2B_2 + m_1C_1)\, ln\,(m_4\mu_4 + 6m_3A_3+ \\ 7m_2B_2 + m_1C_1) \end{array}\right]$$
$$\div [N + 4m_1C_1 + 11m_2B + 7m_3A_3 + m_4\mu_4] \tag{2.7}$$

Proof: In order to obtain the above result, we make use of the following inequality (See Mitrinović, [11]):

$$\left[\sum_{i=1}^{n} a_i/n\right]^{\sum a_i} \leq \prod_{i=1}^{n} a_i^{a_i}$$

where each $a_i > 0$. We specialize above to $n = 5$, and set $a_k = \sum j^k X_j (k = 0, 1, 2, 3$, and $4)$. Next we make use of results (2.1)-(2.5), and then take logs on both sides to obtain (2.7).

Theorem 2.3. *For a B-array $T(m, N, t = 4, \underline{\mu'})$ to exist, we must have*

$$\begin{aligned}
&(m_3A_3+3m_2B_2+m_1C_1)\, ln\; m_1C_1+\\
&(m_4\mu_4+6m_3A_3+7m_2B_2+m_1C_1)\, ln\,(m_2B_2+m_1C_1)+\\
&(m_4\mu_4+7m_3A_3+10m_2B_2+2m_1C_1)\, ln\,(m_4\mu_4+7m)3A_3+10m_2B_2+2m_1C_1)\\
&\le (m_3A_3+6m_3A_3+7m_2B_2+m_1C_1)\, ln\,(m_3A_3+3m_2B_2+m_1C_1)+\\
&(m_4\mu_4+6m_3A_3+7m_2B_2+m_1C_1)\, ln\,(m_4\mu_4+6m_3A_3+7m_2B_2+m_1C_1)+\\
&(m_4\mu_4+7m_3A_3+10m_2B_2+2m_1C_1)\, ln\,(m_2B_2+2m_1C_1) \qquad (2.8)
\end{aligned}$$

Proof: The above result can be easily proved by using (from Mitrinović, [11]) the inequality: $a_1^{a_3}a_2^{a_4}(a_3+a_4)^{a_3+a_4} \le a_3^{a_3}a_4^{a_4}(a_1+a_2)^{a_3+a_4}$, $a_i > 0$, we set $a_k = \sum j^k X_j$ $(k = 1, 2, 3, 4)$. Next, we use (2.2)-(2.5), and take logs on both sides.

Theorem 2.4. *Consider a B-array T with $m \ge 5$, $t = 4$, and with index set $\underline{\mu'} = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$. Then we must have*

$$\begin{aligned}
&[m_4\mu_4+7m_3A_3+11m_2B_2+4m_1C_1]\bullet\\
&ln\begin{bmatrix} m_1C_1(m_2B_2+m_1C_1)(m_4\mu_4+7m_3A_3+10m_2B_2+2m_1C_1)+\\ (m_3A_3+3m_2B_2+m_1C_1)(m_4\mu_4+6m_3A_3+7m_2B_2+m_1C_1)\\ (m_2B_2+2m_1C_1)\end{bmatrix}\\
&\ge (m_4\mu_4+7m_3A_3+11m_2B_2+4m_1C_1)\, ln\,(m_4\mu_4+7n_3A_3+11m_2B_2+4m_1C_1)+\\
&\quad(m_4\mu_4+7m_3A_3+11m_2B_2+4m_1C_1)\bullet\\
&\quad\begin{bmatrix} ln\; m_1C_1+ln(m_2B_2+m_1C_1)+\\ ln(m_3A_3+3m_2B_2+m_1C_1)+ln(m_4\mu_4+6m_3A_3+7m_2B_2+m_1C_1)\end{bmatrix}\\
&\quad -2[m_1C_1\, ln\, m_1C_1+(m_2B_2+m_1C_1)\, ln\,(m_2B_2+m_1C_1)+\\
&\quad(m_3A_3+3m_2B_2+m_1C_1)\, ln\,(m_3A_3+3m_2B_2+m_1C_1)+(m_4\mu_4+6m_3A_3+\\
&\quad 7m_2B_2+m_1C_1)\, ln\,(m_4\mu_4+6m_3A_3+7m_2B_2+m_1C_1)] \qquad (2.9)
\end{aligned}$$

Proof: In order to derive (2.9), we consider the following inequality (See Mitrinović, [11])

$$(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4)^{\sum_{i=1}^4 a_i} \ge \left(\sum a_i\right)^{\sum a_i}\bullet$$
$$(a_1^{a_2+a_3+a_4-a_1})(a_2^{a_3+a_4+a_1-a_2})(a_3^{a_1+a_2+a_4-a_3})(a_4^{a_1+a_2+a_3-a_4})$$

where each $a_i > 0$. In this inequality set $a_k = \sum_{j=0}^m j^k X_j$, $k = 1, 2, 3$, and 4. Using (2.2)-(2.5), taking logs on both sides we obtain (2.9) after some simplification.

Some observations

1. It must be noted that the inequalities (2.6) through (2.9) are necessary conditions for the existence of B-arrays for given values of m and $\underline{\mu}'$. If one or more of these conditions are contradicted for a certain value of m (say, $m = k + 1$), then k is an upper bound for the number of constraints for the array T. Since (2.6)-(2.9) are merely necessary existence conditions, the array T with the given $\underline{\mu}'$ and $m = k$ may or may not exist. In this sense, the results presented here could be interpreted as results on the non-existence of arrays. Nonetheless, the results given here provide us with useful information on an upper bound for the number of constraints of T.

2. The inequalities (2.6)-(2.9) are merely polynomial functions and/or logarithms of the polynomial functions of m and $(\mu_0, \mu_1, \mu_2, \mu_3,$ and $\mu_4)$. For a given $\underline{\mu}'$, the above inequalities (2.6)-(2.9) merely involve the number of constraints m. It is not difficult to prepare a computer program to check if (2.6) through (2.9) are satisfied for a given $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ and for every $m \geq 5$. The program is terminated as soon as we get a contradiction for a value of m. If a B-array T with a given value of the index set $\underline{\mu}'$ does not exist for a certain value of $m = k$ (say), it is quite obvious T will not exist for any $m \geq k + 1$.

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