Another Construction for the (25,4,1) Design

R.G. Stanton and D.W. Kroeker
Department of Computer Science
University of Manitoba
Winnipeg, Canada
R3T 2N2

1 Geometric Background

In [3], a construction for the (25,4,1) BIBD, which may also be designated as the Steiner System S(2,4,25), was given; that construction was based on the geometry PG(2,4). We present another construction that is obtained from PG(2,8).

Suppose that a, b, c, d, are 4 points in the 73-point geometry and that no three of these are collinear. Then they generate partial lines abe, cde; acf, bdf: adg, bcg; then the three diagonal points e, f, g, are collinear, and so generate a seventh partial line efg.

Each of these seven partial lines contains 6 further points, and so the total number of distinct points on these 7 lines is $42 + 7 = 49$. Thus we have determined a set L of 24 further points such that the points in L do not lie on any of the seven base lines determined by abe, cde, acf, bdf, adg, bcg, efg.

Now the lines of the geometry can be considered as comprising the seven base lines, together with 42 further lines (6 other lines through each of a, b, c, d, e, f, g), and 24 additional lines that do not contain any of a, b, c, d, e, f, g. The points of L that occur with a must give rise to at least 36 pairs from L (if all the points of L are evenly distributed in 4-sets with a); a similar remark holds for each of b, c, d, e, f, g. Also, the points of L occur 48 times in the 24 "additional lines", and so must give rise to at least 24 pairs from L. But this creates a minimum of $36(7) + 24 = 276$ pairs from L, and the number of pairs from L is exactly 276. So we conclude that the points from L do appear in 42 sets of 4 (6 sets occurring with each of a, b, c, d, e, f, g) and in 24 pairs. The 24 pairs all have the property that the occurrence of pairs xy and xz implies the occurrence of yz. So these 24 pairs can be formed into 8 triples of the form xyz. In turn, these triples can be used to form 8 quadruples $Sxyz$ by adjoining an ideal point S to each of the 8 triples. This gives a total of 50 quadruples on the 24 points of L, together with S. So we have constructed a design (25,50,8,4,1).

2 Implementation of the Construction

Let us now perform the actual construction. We generate PG(2,8) by taking the line $\{1,2,4,8,16,32,64,55,37\}$ and the 72 lines obtained from it by translation by $1,2,3,4,\ldots,72$. For the set of 4 points, no-3-collinear, we select the points 1, 2, 3, 6. Then our 7 base lines are determined by the partial lines:

$$(1,2,4),(3,6,4);(1,3,0),(2,6,0);(1,6,33),(2,3,33);(0,4,33).$$

The set L is easily determined as comprising the points 11, 13, 19, 20, 21, 22, 23, 25, 26, 27, 29, 40, 41, 42, 45, 47, 48, 49, 52, 59, 61, 67, 68, 69.

The pairs from L that occur in the 24 "additional lines" can be arranged in 8 triples that, with the adjunction of point S generate the following 8 quadruples:

$$(S,22,61,69),(S,13,40,45),(S,11,23,68),(S,20,47,48),$$
$$(S,25,27,42),(S,19,29,59),(S,21,26,67),(S,41,49,52).$$

The quadruples from L that occur with 1 are:

$$(11,26,42,47),(20,21,23,27),(19,41,45,69),$$
$$(22,40,49,59),(25,48,67,68),(13,29,52,61).$$

The quadruples from L that occur with 2 are:

$$(13,19,27,48),(11,21,22,52),(20,29,40,42),$$
$$(25,45,47,59),(23,41,61,67),(26,49,68,69).$$

The quadruples from L that occur with 3 are:

$$(13,20,49,67),(22,23,25,29),(21,40,41,47),$$
$$(26,45,48,52),(19,42,61,68),(11,27,59,69).$$

The quadruples from L that occur with 6 are:

$$(20,41,59,68),(13,21,42,69),(19,23,47,52),$$
$$(25,26,40,61),(11,29,48,49),(22,27,45,67).$$

The quadruples from L that occur with 0 are:

$$(11,13,25,41),(19,20,22,26),(27,40,52,68),$$
$$(23,42,45,49),(21,48,59,61),(29,47,67,69).$$

The quadruples from L that occur with 4 are:

$$(11, 19, 40, 67), (21, 29, 45, 68), (13, 23, 26, 59),$$
$$(22, 41, 42, 48), (27, 47, 49, 61), (20, 25, 52, 69).$$

The quadruples from L that occur with 33 are:

$$(19, 21, 25, 49), (11, 20, 45, 61), (26, 27, 29, 41),$$
$$(13, 22, 47, 68), (23, 40, 48, 69), (42, 52, 59, 67).$$

The 50 quadruples that we have displayed give the complete design S(2,4,25).

3 Properties of the Design

We employ the Groups & Graphs package (cf. [1]) as we did in [3]. The automorphism group has order 504. As might be expected, S is an invariant point. One set of generators of the group is given as follows. Write

$A = (11, 20, 25)(13, 61, 52)(21, 67, 26)(22, 49, 40)$,
$B = (23, 48, 42)(27, 68, 47)(41, 45, 69)$,
$C = (11, 52, 48, 25, 19, 22, 45, 68, 49, 20, 27, 29, 69, 13, 23, 41, 47, 42, 59, 61, 40)$,
$D = (21, 26, 67)$,
$E = (11, 61)(13, 47)(19, 21)(20, 45)(22, 68)(23, 69)$,
$F = (25, 49)(26, 29)(27, 41)(40, 48)(42, 52)(59, 67)$.

Then the group is generated by $a = AB$, $b = CD$, $c = EF$.

All the designs S(2,4,25) with non-trivial automorphism groups are known (cf [2]). The design that we have obtained is isomorphic to that listed as Design 1 in [2].

References

[1] W.L. Kocay, Groups & Graphs, A Macintosh Application for Graph Theory, *Journal of Combinatorial Maths and Combinatorial Computing* **3** (1988), 195–206.

[2] Earl S. Kramer, S.S. Magliveras, and V.D. Tonchev, On the Steiner Systems S(2,4,25) invariant under a group of order 9, *Annals of Discrete Maths* **34** (1987), 307–314.

[3] R.G. Stanton, Note on a Construction for a (25,4,1) Design, *Journal of Combinatorial Maths and Combinatorial Computing* **5** (1989), 61–62.