Some New Results on (g, f)-Factorizations of Graphs

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ABSTRACT. Let g and f be integer-valued functions defined on $V(G)$ with $f(v) \geq g(v) \geq 1$ for all $x \in V(G)$. A graph G is called a (g, f)-graph if $g(v) \leq d_G(v) \leq f(v)$ for each vertex $v \in V(G)$, and a (g, f)-factor of a graph G is a spanning (g, f)-subgraph of G. A graph is (g, f)-factorable if its edges can be decomposed into (g, f)-factors. The purpose of this paper is to prove the following three theorems: (i) If $m \geq 2$, every $((2mg+2m-2)t+(g+1)s, (2mf-2m+2)t+(f-1)s)$-graph G is (g, f)-factorable. (ii) Let $g(x)$ be even and $m > 2$. (1) If m is even, and G is a $((2mg+2)t+(g+1)s, (2mf-2m+4)t+(f-1)s)$-graph. Then G is (g, f)-factorable; (2) if m is odd, and G is a $((2mg+4)t+(g+1)s, (2mf-2m+2)t+(f-1)s)$-graph. Then G is (g, f)-factorable. (iii) Let $f(x)$ be even and $m > 2$. (1) If m is even, and G is a $((2mg+2m-4)t+(g+1)s, (2mf-2)t+(f-1)s)$-graph, then G is (g, f)-factorable; (2) if m is odd, and G is a $((2mg+2m-2)t+(g+1)s, (2mf-4)t+(f-1)s)$-graph, then G is (g, f)-factorable, where t and m are integers and s is a nonnegative integer.

1. Introduction

All graphs under consideration are undirected and finite. Multiple edges are allowed but loops are not.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An edge joining vertices u and v is denoted by uv. For a vertex $v \in V(G)$, we write $d_G(v)$ for the degree of v in G. Let $g(x)$ and $f(x)$ be integer-valued functions defined on $V(G)$ with $f(x) \geq g(x) \geq 1$ for all $x \in V(G)$. A (g, f)-factor of a graph is a spanning (g, f)-subgraph of G. The graph G is said to be (g, f)-factorable if $E(G)$ can be partitioned into (g, f)-factors

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$F_1, F_2, \ldots, F_n$ of G, and $\{F_1, F_2, \ldots, F_n\}$ is called a (g, f)-factorization of G.

Given a subset $X \subset V(G)$, we denote by $G - X$ the subgraph obtained from G by deleting the vertices in X together with the edges incident to vertices in X. For $E' \subset E(G)$, $G - E'$ denotes the graph obtained from G by deleting the edges in E'. If X and Y are disjoint subsets of $V(G)$, we write $E(X, Y) = \{xy \in E(G) : x \in X, y \in Y\}$ and $e(X, Y) = |E(X, Y)|$. The necessary and sufficient condition that a graph has (g, f)-factor was given by Lovasz [1] in 1970. Then Kano studied some sufficient *condition* [2] for an n-edge connected graph to have (g, f)-factor; Akiyama and Kano survived some *results* [3] regarding to the factors and factorizations of graphs in 1985. Recently, Cai discussed $[a, b]$-*factorizations* [4] of some graphs, Liu obtained some *results* [6] on (g, f)-factorizations. The purpose of this paper is to prove the following three theorems.

Let t and m denote positive integers and s denote a nonnegative integer. Let $g(x)$ and $f(x)$ be integer-valued functions defined on $V(G)$ with $f(x) \geq g(x) \geq 1$ for all $x \in V(G)$.

Theorem 1. *If $m \geq 2$, every $((2mg + 2m - 2)t + (g + 1)s, (2mf - 2m + 2)t + (f - 1)s)$-graph G is (g, f)-factorable.*

Theorem 2. *Let $g(x)$ be even and $m > 2$. (1) If m is even, every $((2mg + 2)t + (g + 1)s, (2mf - 2m + 4)t + (f - 1)s)$-graph G is (g, f)-factorable; (2) if m is odd, every $((2mg + 4)t + (g + 1)s, (2mf - 2m + 2)t + (f - 1)s)$-graph G is (g, f)-factorable.*

Theorem 3. *Let $f(x)$ be even and $m > 2$. (1) If m is even, every $((2mg + 2m - 4)t + (g + 1)s, (2mf - 2)t + (f - 1)s)$-graph G is (g, f)-factorable; (2) if m is odd, every $((2mg + 2m - 2)t + (g + 1)s, (2mf - 4)t + (f - 1)s)$-graph G is (g, f)-factorable.*

In order to prove these theorems, we need the following Lemmas:

Lemma A. [2] *Let G be an n-edge-connected graph ($n > 0$), θ be a real number such that $0 \leq \theta \leq 1$, and g and f be two integer-valued functions defined on $V(G)$ such that $g(v) \leq f(v)$ for all $v \in V(G)$. If one of (Ia), (Ib), (II) and one of (IIIa), (IIIb), (IIIc), (IIId), (IIIe), (IIIf) hold, then G has a (g, f)-factor.*

(Ia) $g(v) \leq \theta d_G(v) \leq f(x)$ *for all* $v \in V(G)$.

(Ib) $\sum_{x \in V(G)} [\max 0, g(v) - \theta d_G(v) + \max 0, \theta d_G(v) - f(v)] < 1$.

(II) *G has at least one vertex v such that $g(v) < f(v)$; or $g(v) = f(v)$ for all $v \in V(G)$ and $\sum_{x \in V(G)} f(v) \equiv 0 \pmod 2$.*

(IIIa) $n\theta \geq 1$ *and* $n(1 - \theta) \geq 1$.

(IIIb) $\{d_G(v) : g(v) = f(v),\ v \in V(G)\}$ and $\{f(v) : g(v) = f(v),\ v \in V(G)\}$ both consist of even numbers.

(IIIc) $\{d_G(v) : g(v) = f(v),\ v \in V(G)\}$ consist of even numbers, n is odd, $(n+1)\theta \geq 1$, and $(n+1)(1-\theta) \geq 1$.

(IIId) $\{f(v) : g(v) = f(v),\ v \in V(G)\}$ consist of even numbers and $l(1-\theta) \geq 1$, where $l \in \{n, n+1\}$ and $l \equiv 1 \pmod 2$.

(IIIe) $\{d_G(v) : g(v) = f(v),\ v \in V(G)\}$, and $\{f(v) : g(v) = f(v),\ v \in V(G)\}$ both consist of odd numbers and $ln \geq 1$, where $l \in \{n, n+1\}$ and $l \equiv 1 \pmod 2$.

(IIIf) $g(v) < f(v)$ for every $v \in V(G)$.

Lemma B. [2] *Let $a(x)$ and $b(x)$ be integer-valued functions defined on $V(G)$, and n is a positive integer. Then graph G is $(2a(x), 2b(x))$-factorable if and only if G is a $(2a(x)n, 2b(x)n)$-graph.*

In the following proofs we always assume that the graphs concerned are connected, for otherwise we consider each of its complements.

2. Proofs of the Theorems

Lemma 1.1. *If $m \geq 2$, every $(mg + m - 2, mf - m + 2)$-graph G with at most one vertex u of degree $mg(u) + m - 2$ and with at most one vertex w of degree $mf(w) - m + 2$ is (g, f)-factorable.*

Proof: We may assume that $g(x) < f(x)$ for some $v \in V(G)$. Since if $g(x) = f(x)$ for all $x \in V(G)$, then $m = 2$, G is a $(2f, 2f)$-graph with two vertices, which implies that the lemma holds. We apply induction on m.

When $m = 2$, G is a $(2g, 2f)$-graph with at most one vertex u of degree $2g$ and with at most one vertex w of degree $2f$. Obviously, the vertex set $\{x : 2g < d_G(x) < 2f\}$ is not empty because G is loopless. For each $x \in V(G)$, define

$$p(x) = \max\{g(x), d_G(x) - f(x)\}$$
$$q(x) = \min\{f(x), d_G(x) - g(x)\}$$

Then

$$2p(x) \leq d_G(x) \leq 2q(x) \tag{1.1}$$
$$p(x) = q(x) \text{ if and only if } d_G(x) = 2g(x) \text{ or } 2f(x) \tag{1.2}$$

By taking $n = 1$ and $\theta = \frac{1}{2}$, then (Ia), (II), and (IIIc) of Lemma A hold. Hence G contains a (p, q)-factor F. F is a (g, f)-factor. Set $F' = G - E(F)$.

Then F' also is a (g, f)-factor. Therefore, Lemma 1.1 is proved when $m = 2$. Suppose that $m \geq 3$ and that the assertion is true for all smaller values of m. For each $x \in V(G)$, put

$$p'(x) = \max\{g(x), d_G(x) - [(m-1)f(x) - m + 2]\}$$
$$q'(x) = \min\{f(x), d_G(x) - [(m-1)g(x) + m - 2]\}$$

If G has vertex u with degree $mg(u) + m - 2$, we modify $q'(u) = q'(u) + 1$; if G has vertex w with degree $mf(w) - m + 2$, we modify $p'(w) = p'(w) - 1$. Then

$$mp'(x) \leq d_G(x) \leq mq'(x) \tag{1.3}$$
$$p'(x) \neq q'(x) \text{ for each } x \in V(G) \tag{1.4}$$

Set $n = 1$, and $\theta = \frac{1}{m}$, then (Ia), (II) and (IIIf) of Lemma A holds. So G contains a (p', q')-factor H. Clearly, H is a (g, f)-factor. It is easy to prove that remaining subgraph $G - E(H)$ is a $((m-1)g + m - 3, (m-1)f - m + 3)$-graph with at most one vertex u of degree $(m-1)g + m - 3$ and with at most one vertex w of degree $(m-1)f - m + 3$. Therefore, by induction hypothesis, $G - E(H)$ is (g, f)-factorable. The proof is completed.

The following result is proved in [5]:

Lemma C. [5]: *Let G be a graph, and m be a positive integer, then every $(mg + m - 1, mf - m + 1)$-graph is (g, f)-factorable.*

Lemma 1.2. *Every $(2mg + 2m - 2, 2mf - 2m + 2)$-graph G is (g, f)-factorable.*

Proof: For each $x \in V(G)$, define

$$p(x) = \max\{mg + m - 1, d_G(x) - (mf - m + 1)\}$$
$$q(x) = \min\{mf - m + 1, d_G(x) - (mg + m - 1)\}$$

If G has vertices with degre $2mg + 2m - 2$, then we choose one such vertex x_1 and modify $p(x_1) = p(x_1) - 1$; if G has vertices with degree $2mf - 2m + 2$, then we choose one such vertex x_2 and modify $q(x_2) = q(x_2) + 1$. Then

$$2p(x) \leq d_G(x) \leq 2q(x) \text{ for each } x \in V(G)$$

$p(x) = q(x)$ if and only if $d_G(x) = 2mg + 2m - 2$ or $2mf - 2m + 2$ but $x \neq x_1, x_2$. By a similar argument used in the proof of Lemma 1.1, we can show that G has a (p, q)-factor F. Clearly, F is a $(mg + m - 2, mf - m + 2)$-factor with at most one vertex x_1 of degree $mg(x_1) + m - 2$ and with at most one vertex x_2 of degree $mf(x_2) - m + 2$. Set $G' = G - E(F)$, G is a $(mg + m - 1, mf - m + 1)$-graph, then both F and G' are (g, f)-factorable by Lemma 1.1 and Lemma C. Therefore G is (g, f)-factorable. The Lemma is proved.

Now we prove Theorem 1 by induction on s.

If $s = 0$, then G is $(2mg+2m-2, 2mf-2m+2)$-factorably by Lemma B. According to Lemma 1.2, every $(2mg+2m-2, 2mf-2m+2)$-graph is also (g, f)-factorable. Therefore, Theorem 1 is proved when $s = 0$. Suppose the assertion is true for all smaller values of s. For each $v \in V(G)$, define

$$g'(x) = \max\{g(x), d_G(x) - [(2mf(x) - 2m + 2)t + (f(x) - 1)(s - 1)]\}$$
$$f'(x) = \min\{f(x), d_G(x) - [(2mg(x) + 2m - 2)t + (f(x) + 1)(s - 1)]\}$$

Let

$$b = \max_{x \in V(G)} mf(x), \lambda = \frac{b}{(2mb - 2m + 2)t + bs}.$$

Then it follows easily from the inequality $(2mg + 2m - 2)t + (g + 1)s \leq (2mf - 2m + 2)t + (f - 1)s$ that $g + 1 \leq f$, we prove $g'(x) \leq \lambda d_G(x) \leq f'(x)$ for every $x \in V(G)$.

We first show that $g'(x) \leq \lambda d_G(x)$.

1. Assume that $g'(x) = g(x)$, we need only to prove $\lambda \geq \frac{g}{(2mg+2m-2)t+(g+1)s}$ which is equivalent to

$$\frac{b}{(2mb - 2m + 2)t + bs} \geq \frac{g}{(2mg + 2m - 2)t + (g + 1)s}.$$

Simplify the form mentioned above, we have

$$2bt(m - 1) + bs \geq 2gt(1 - m).$$

Clearly, the above mentioned form is right as $m \geq 2$. Then

$$\begin{aligned} \lambda d_G(x) &\geq \frac{g}{(2mg + 2m - 2)t + (g + 1)s} \times [(2mg + 2m - 2)t + (g + 1)s] \\ &= g(x) = g'(x). \end{aligned}$$

2. Assume that $g'(x) = d_G(x) - [(2mf(x) - 2m + 2)t + (f(x) - 1)(s - 1)]$, we can easily prove that

$$\lambda \geq \frac{f - 1}{(2mf - 2m + 2)t + (f - 1)s}.$$

by $b \geq mf$. Therefore

$$\begin{aligned}
\lambda &\geq \frac{f-1}{(2mf-2m+2)t+(f-1)s} \\
&= 1 - \frac{(2mf-2m+2)t+(f-1)(s-1)}{(2mf-2m+2)t+(f-1)s} \\
&\geq 1 - \frac{(2mf-2m+2)t+(f-1)(s-1)}{d_G(x)} \\
&= \frac{d_G(x) - [(2mf-2m+2)t+(f-1)(s-1)]}{d_G(x)} \\
&\geq \frac{g'(x)}{d_G(x)}.
\end{aligned}$$

Then $g'(x) \leq \lambda d_G(x)$. So $g'(x) \leq \lambda d_G(x)$ for every $x \in V(G)$ by 1 and 2.

It follows that $\lambda d_G(x) \leq f'(x)$ with similar method.

Thus

$$\begin{aligned}
&g'(x) < f'(x) \text{ for every } x \in V(G) \text{ and} \\
&g'(x) \leq \lambda d_G(x) \leq f'(x)
\end{aligned}$$

Taking $n = 1$; and $\theta = \lambda$, then (Ia), (II), and (IIIf) of Lemma B hold. So G contains (g', f')-factor F, F is a (g, f)-factor. The remaining subgraph $G-E(F)$ is a $((2mg+2m-2)t+(g+1)(s-1), (2mf-2m+2)t+(f-1)(s-1))$-graph. By induction hypothesis $G - E(F)$ if (g, f)-factorably. Thus G is (g, f)-factorable.

Proof of Theorem 2: Since the proof is very similar to that of Theorem 1, we only give an outline of the proof and leave the details to the reader.

Lemma D. [5] *Let m be a positive integer. (1) If $g(x)$ is even, and G is a $(mg, mf-m+1)$-graph, then G is (g, f)-factorable; (2) If $f(x)$ is even, and G is a $(mg+m-1, mf)$-graph, then G is (g, f)-factorable.*

Lemma 2.1. *Let $g(x)$ be even and $m > 2$. (1) If m is even, and G is $(mg, mf-m+2)$-graph with at most one vertex of degree mg, then G is (g, f)-factorable. (2) If m is odd, and G is a $(mg+1, mf-m+2)$-graph with at most one vertex u of degree $mg(u)+1$ and with at most one vertex w of degree $mf(w)-2m+2$, then G is (g, f)-factorable.*

Proof: (1) Let $m = 2r$, where r is a positive integer. Put

$$\begin{aligned}
p(x) &= \max\{rg, d_G(x) - (rf - r + 1)\} \\
q(x) &= \min\{rf - r + 1, d_G(x) - rg\}
\end{aligned}$$

Case 1: If the vertex set $\{x \in V(G) : mg < d_G(x) < mf - m + 2\} \neq \emptyset$, then we have

$$2p(x) \leq d_G(x) \leq 2q(x) \tag{2.1}$$
$$p(x) = q(x) \text{ if and only if } d_G(x) = mg \text{ or } mf - m + 2 \tag{2.2}$$

By a similar argument used in the proof of Lemma 1.1, we can show that G has a (p, q)-factor F. Clearly, both F and $G - E(F)$ are $(rg, rf - r + 1)$-graph. So G is (g, f)-factorable by Lemma D.

Case 2: If the vertex set $\{x \in V(G) : mg < d_G(x) < mf - m + 2\} = \emptyset$, we can choose one such vertex x_0 with degree $mf(x_0) - m + 2$, and modify $p(x_0) = p(x_0) - 1$. Then for each $x \in V(G)$, we have

$$2p(x) \leq d_G(x) \leq 2q(x)$$

$p(x) = q(x)$ if and only if $d_G(x) = mg$ or $mf - m + 2$ but $x \neq x_0$.

So G has a $(rg, rf - r + 1)$-factor F using a similar method of Lemma 1.1, and F is (g, f)-factorably by Lemma D. Define $G' = G - E(F)$, then G' is a $(rg, rf - r + 2)$-graph with at most one vertex u of degree $rg(u)$ and with at most one vertex x_0 of degree $rf(x_0) - r + 2$. Moreover, the vertex set $\{x \in V(G') : rg < d_{G'}(x) < rf - r + 2\} = \{x \in V(G') : d_{G'}(x) = rf - r + 1\}$. Now we prove that G' is (g, f)-factorable by induction on r as follows.

When $r = 2$, G' is a $(2g, 2f)$-graph with at most one vertex u of degree $2g(u)$ and with at most one vertex x_0 of degree $2f$. Clearly, the assertion is right by the Lemma 1.1. Suppose that $r \geq 3$ and that the assertion is true for all smaller values of r, for each $x \in V(G')$, set

$$p'(x) = \max\{g(x), d_{G'}(x) - [(r-1)f(x) - r + 2]\}$$
$$q'(x) = \min\{f(x), d_{G'}(x) - (r-1)g(x)\}.$$

If G' has vertex x_0 with degree $rf(x_0) - r + 2$, then we modify $p'(x_0) = p'(x_0) - 1$. Thus we have

$$rp'(x) \leq d_{G'}(x) \leq rq'(x) \tag{2.3}$$
$$\text{and } p'(x) = q'(x) \text{ if and only if } d_{G'}(x) = rg(x) \tag{2.4}$$

Taking $n = 1$, and $\theta = \frac{1}{r}$, then (Ia), (II) and (IIIb) of Lemma A hold. Therefore G' has a (p', q')-factor F', F' if a (g, f)-factor. Set $F'' = G' - E(F')$. Then F'' is a $[(r-1)g, (r-1)f - r + 3]$-graph with at most one vertex u of degree $(r-1)g(u)$ and with at most one vertex x_0 of degree $(r-1)f(x_0) - r + 3$. Thus F'' is (g, f)-factorable by induction hypothesis. So G is (g, f)-factorable.

Proof: (2). Define

$$p(x) = \max\{g(x), d_G(x) - [(m-1)f(x) - m + 3]\},$$
$$q(x) = \min\{f(x), d_G(x) - [(m-1)g(x) - 1)]\}.$$

If G has vertex x_1 with degree of $mg + 1$, we modify $q(x_1) = q(x_1) + 1$. Then for each $x \in V(G)$ we have

$$p(x) \neq q(x)$$
$$mp(x) \leq d_G(x) \leq mq(x)$$

The rest of the proof is very similar to the proof of Lemma 1.1. Here we omit it. SO G has a (g, f)-factor F. Set $G' - G - E(F)$, then G' is a $[(m-1)g, (m-1)f - m + 3]$-graph with at most one vertex of degree $(m-1)g$, and $m - 1$ is even, therefore G' is (g, f)-factorable by Case 1. Thus G is (g, f)-factorable.

Lemma 2.2. *Let $g(x)$ be even and $m > 2$. (1) If m is even, every $(2mg + 2, 2mf - 2m + 4)$-graph G is (g, f)-factorable; (2) if m is odd, every $(2mg + 4, 2mf - 2m + 2)$-graph G is (g, f)-factorable.*

Proof: Here we omit the proof of being similar to that of Lemma 1.2.

We still use the method of induction to prove Theorem 2. If $s = 0$, then Theorem 2 holds by Lemma B and Lemma 2.2. Suppose that $s \geq 1$ and that the assertion is true for all smaller values of s. For each $x \in V(G)$, put

$$g'(x) = \max\{g(x), d_G(x) - [(2mf(x) - 2m + 4)t + (f(x) - 1)(s - 1)]\},$$
$$f'(x) = \min\{f(x), d_G(x) - [(2mg(x) + 2)t + (g(x) - 1)(s - 1)]\}.$$

Let

$$b = \max_{x \in V(G)} mf(x), \lambda = \frac{b}{(2mb - 2m + 4)t + bs}.$$

Then it follows easily from the inequality $(2mg + 2)t + (g + 1)s \leq (2mf - 2m + 4)t + (f - 1)s$ that $g + 1 \leq f$, we can derive $g'(x) < \lambda d_G(x) < f'(x)$ using the similar method of Theorem 1. Here we omit the detailed proof.

Therefore G contains a (g', f')-factor F, F is a (g, f)-factor, and also, $G - E(F)$ is a $[(2mg + 2)t + (g + 1)(s - 1), (2mf - 2m + 4)t + (f - 1)(s - 1)]$-graph. So Theorem 2 (1) is proved by induction hypothesis. Theorem 2 (2) can be proved by the same method. We omit it.

Finally, we give a brief proof of Theorem 3.

At first, we give two lemmas whose proof is omitted being very similar to that of Lemma 2.1 and Lemma 2.2, respectively.

Lemma 3.1. *Let $f(x)$ be even and $m > 2$. (1) If m is even, every $(mg + m - 2, mf)$-graph G with at most one vertex u of degree $mf(u)$ G is (g, f)-factorable; (2) If m is odd, every $(mg + m - 2, mf - 1)$-graph G with at most one vertex w of degree $mg(w) + m - 2$ and with at most one vertex z of degree $mf(z) - 1$ is (g, f)-factorable.*

Lemma 3.2. *Let $f(x)$ be even and $m > 2$. (1) If m is even, every $(2mg + 2m - 4, 2mf - 2)$-graph G is (g, f)-factorable; (2) If m is odd, every $(2mg + 2m - 2, 2mf - 4)$-graph G is (g, f)-factorable.*

The proof of Theorem 3 is similar to that of Theorem 2. here we omit it.

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