Complete $(k-1)$-subgraphs of k-critical graphs

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ABSTRACT. A graph G is called k-critical if $\chi(G) = k$ and $\chi(G-e) < \chi(G)$ for each edge e of G, where χ denotes the chromatic number. It was conjectured in [4] that any k-critical graph G of order greater than k has an edge which is contained in at most one complete $(k-1)$-subgraph $(k \geq 4)$. In this paper, we prove the conjecture for $k \leq 7$. Consequently we obtain that the number of complete $(k-1)$-subgraphs of any k-critical graph G of order $n > k$ is at most $n-k+3$ if $k \leq 7$.

1 Introduction

We use standard notation. All graphs considered are finite, undirected and have neither loops nor multiple edges. A graph G is called k-critical if $\chi(G) = k$ and $\chi(G-e) < \chi(G)$ for each edge e of G, where χ denotes the chromatic number. For any graph G, we use $T_{k-1}(G)$ to denote the set of all complete $(k-1)$-subgraphs of G and $t_{k-1}(G)$ to denote the number of complete $(k-1)$-subgraphs of G; namely, $t_{k-1}(G) = |T_{k-1}(G)|$. Gallai conjectured that $t_{k-1}(G) \leq n$ for any k-critical graph G of order n. The case $k = 3$ is trivial. The case $k = 4$ was proved by Stiebitz [3]. In 1992, Abbott and Zhou [1] proved Gallai's conjecture for all $k \geq 5$. In fact, they proved the following stronger result. The case $k = 4$ for planar graphs was also obtained by Koester [2].

Theorem A. (Abbott and Zhou [1]) *Let G be a k-critical graph of order n $(k \geq 4)$. Then $t_{k-1}(G) \leq n$ and the equality holds if and only if $k = n$ and $G = K_k$.*

At end of their paper, Abbott and Zhou posed the following conjecture.

Conjecture 1: For any k-critical graph G of order $n > k$, $t_{k-1}(G) \leq n-k+3$ $(k \geq 4)$.

In trying to solve this conjecture, the present author raised a stronger conjecture in [4].

Conjecture 2: Any k-critical graph G of order greater than k has an edge which is contained in at most one complete $(k-1)$-subgraph $(k \geq 4)$.

Conjecture 2 is stronger than Conjecture 1 because we have

Theorem B. (Su [4]) *Let G be any k-critical graph of order $n > k$ $(k \geq 4)$. If G has an edge that is contained in at most one complete $(k-1)$-subgraph of G, then the number of complete $(k-1)$-subgraphs of G is at most $n-k+3$, i.e., $t_{k-1}(G) \leq n-k+3$.*

The main result of this paper is the following theorem that provides a positive answer to Conjecture 2 for $k \leq 7$.

Theorem 1. *Let G be any k-critical graph of order $n > k$ and let $4 \leq k \leq 7$. Then there exists an edge e of G such that e is contained in at most one complete $(k-1)$-subgraph of G. Consequently, the number of complete $(k-1)$-subgraphs of G is at most $n-k+3$, i.e., $t_{k-1}(G) \leq n-k+3$.*

2 Proofs

Lemma 2. *Let G be a r-chromatic graph satisfying the following conditions:*

(i) *every vertex of G is contained in at least two complete r-subgraphs, and*

(ii) *G has at most r complete r-subgraphs.*

If $r \leq 5$, then there exists an edge $e = xy$ such that any proper r-coloring of $G-e$ assigns different colors to x and y. (We call such edge e a required edge).

Proof: If $r \leq 3$, there is nothing to prove since no such graph G exists. The case $r = 4$ is easy. So here we consider only the case $r = 5$.

Let $r = 5$ and let G be a 5-chromatic graph satisfying (i) and (ii). Let $|G| = n$. Then by (i) and (ii),

$$2n \leq \sum_{v \in G} \text{number of complete 5-subgraphs containing } v \leq 5^2.$$

Hence $n \leq 12$. Suppose that the lemma is false and that G is a counterexample.

Claim 1. *G contains no two complete 5-subgroups H_1 and H_2 such that $H_1 \cap H_2$ is a complete 4-subgraph.*

Suppose $|H_1 \cap H_2| = 4$. Let $V(H_1) - V(H_1 \cap H_2) = \{x\}$ and $V(H_2) - V(H_1 \cap H_2) = \{y\}$. Then $c(x) = c(y)$ for any proper 5-coloring c of $H_1 \cup H_2$. Hence, for any $u \in V(G) - V(H_1) \cup V(H_2)$, ux and uy cannot be both in $E(G)$ (otherwise both ux and uy are required edges of G). Since x is contained in two complete 5-subgraphs, there exists a complete 5-subgraph $H_3 \neq H_1$ such that $x \in H_3$. Similarly, there exists a complete 5-subgraph $H_4 \neq H_2$ such that $y \in H_4$. Let $W_1 = V(H_3) - V(H_1)$ and $W_2 = V(H_4) - V(H_2)$. Then $W_1 \cap W_2 = \emptyset$ ($w \in W_1 \cap W_2$ would imply that both wx and wy are in $E(G)$). Now, since every vertex of $W_1 \cup W_2$ is contained in two complete 5-subgraphs and since G has at most five complete 5-subgraphs, $W_1 \cup W_2$ is contained in a complete 5-subgraph H_5. Clearly $x, y \notin H_5$. Let c be a proper 5-coloring of G. Then there exists $z \in H_5$ such that $c(z) = c(x) = c(y)$. So $z \notin W_1 \cup W_2 \cup V(H_1) \cup V(H_2)$. It follows $z \notin H_i$ for each $i \in \{1, \dots, 4\}$. However, by (i) z is contained in at least two complete 5-subgraphs. Hence there exists $H_6 \notin \{H_1, \dots, H_5\}$ such that $v \in H_6$. This implies that G has six complete 5-subgraphs, contrary to (ii). Claim 1 is proved.

Claim 2. *G contains no two complete 5-subgraphs H_1 and H_2 such that $H_1 \cap H_2$ is a triangle.*

Suppose that $H_1 \cap H_2$ is a triangle. Let $V(H_1 \cap H_2) = \{z_1, z_2, z_3\}$. $V(H_1) - V(H_1 \cap H_2) = \{x_1, x_2\}$, and $V(H_2) - V(H_1 \cap H_2) = \{y_1, y_2\}$. We distinguish two cases.

Case 1. Both x_1 and x_2 (or both y_1 and y_2) are contained in a complete 5-subgraph $H_3 \neq H_1$ (or $H_4 \neq H_2$).

W.l.o.g. we suppose $x_1, x_2 \in H_3$. Clearly $y_1, y_2 \notin H_3$. (otherwise G would contain a K_6). Hence $|V(H_3) - V(H_1) \cup V(H_2)| = |V(H_3) - V(H_1)| \geq 2$ by Claim 1. Say $\{u_1, u_2\} \subseteq V(H_3) - V(H_1) \cup V(H_2)$. Since $\{c(y_1), c(y_2)\} = \{c(x_1), c(x_2)\}$ for any proper 5-coloring c of $H_1 \cup H_2$, we have $u_i y_j \notin E(G)$ for $1 \leq i, j \leq 2$ (otherwise $u_i y_j$ would be a required edge of G). By (i), $y_1 \in H_4$ and $u_1 \in H_5$ for some complete 5-subgraphs H_4 and H_5 of G with $H_4, H_5 \notin \{H_1, H_2, H_3\}$. Since $u_1 y_1 \notin E(G)$, we have $H_4 \neq H_5$. By (ii) $H_1, H_2, \dots, H_5$ are all the complete 5-subgraphs of G. Hence $y_2 \in H_4$ and $u_2 \in H_5$ (by (i)). So $|V(H_4) - V(H_1) \cup V(H_2)| = |V(H_4) - V(H_2)| \geq 2$ by Claim 1. Say $\{v_1, v_2\} \subseteq V(H_4) - V(H_1) \cup V(H_2)$. Since both v_1 and v_2 are contained in two complete 5-subgraphs, it follows that $v_1, v_2 \in H_5$. Let c be any proper 5-coloring of G. W.l.o.g. we assume $\{c(z_1), c(z_2), c(z_3)\} = \{1, 2, 3\}$ and $\{c(x_1), c(x_2)\} = \{c(y_1), c(y_2)\} = \{4, 5\}$. Then $\{c(u_1), c(u_2), c(v_1), c(v_2)\} \subseteq \{1, 2, 3\}$, contrary to the fact $\{u_1, u_2, v_1, v_2\} \subset V(H_5)$.

Case 2. x_1 and x_2 are contained in distinct complete subgraphs H_3 and H_4 with $H_3 \neq H_1$, $H_4 \neq H_1$, and similarly for y_1 and y_2.

Since G has at most five complete 5-subgraphs, we may assume w.l.o.g.

that $x_1, y_1 \in H_3$. Then $x_2y_1, y_2x_1 \notin E(G)$ since G is 5-colorable. Also, $x_2y_2 \notin E(G)$; otherwise it would be a required edge of G. Let H_5 ($\neq H_2$) be a complete 5-subgraph containing y_2. Then $H_1, \ldots, H_5$ are all the complete 5-subgraphs of G by (ii). Suppose $V(H_3) - V(H_1) \cup V(H_2) \neq \emptyset$. Let $u \in V(H_3) - V(H_1) \cup V(H_2)$. Then $u \in H_4$ or $u \in H_5$ by (i). So $ux_2, uy_1 \in E(G)$ or $uy_2, ux_1 \in E(G)$. Hence ux_2 or uy_2 is a required edge of G. Therefore we have $V(H_3) \subseteq V(H_1) \cup V(H_2)$. Hence $V(H_3) = \{x_1, y_1, z_1, z_2, z_3\}$ so that $H_3 \cap H_1$ is a complete 4-subgraph of G. This is impossible by Claim 1 and so Claim 2 is proved.

Claim 3. *G contains no two vertex-disjoint complete 5-subgraphs.*

Suppose that H_1 and H_2 are two complete 5-subgraphs of G such that $V(H_1) \cap V(H_2) = \emptyset$. Let $V(H_1) = \{x_1, \ldots, x_5\}$ and $V(H_2) = \{y_1, \ldots, y_5\}$. Note that G has at most five complete 5-subgraphs. Hence, by (i) and Claims 1 and 2, there exists a complete 5-subgraph H_3 of G such that $|V(H_3) \cap V(H_1)| = 2 = |V(H_3) \cap V(H_2)|$, say $V(H_3) = \{x_1, x_2, y_1, y_2, u\}$. Since u is contained in two complete 5-subgraphs, there exists a complete 5-subgraph H_4 such that $u \in H_4$ and $H_4 \notin \{H_1, H_2, H_3\}$. By (i) and by Claims 1 and 2, $V(H_4) \cap \{x_3, x_4, x_5\} \neq \emptyset$ and $V(H_4) \cap \{y_3, y_4, y_5\} \neq \emptyset$ (otherwise, by (i) $\{x_3, x_4, x_5\}$ or $\{y_3, y_4, y_5\}$ is contained in a complete 5-subgraph H' so that $|V(H') \cap V(H_1)| \geq 3$ or $|V(H') \cap V(H_2)| \geq 3$, contrary to Claim 1 or Claim 2). W.l.o.g. we assume that $x_3, y_3 \in H_4$. If $|V(H_4) \cap \{x_3, x_4, x_5\}| \geq 2$ or $|V(H_4) \cap \{y_3, y_4, y_5\}| \geq 2$, say $x_3, x_4 \in H_4$, then $c(u) = c(x_5)$ for any proper 5-coloring c of G. Hence $\{c(y_1), c(y_2)\} = \{c(x_3), c(x_4)\}$ and so x_3y_3 is a required edge of G, a contradiction. Therefore we have $V(H_4) \cap \{x_3, x_4, x_5\} = \{x_3\}$ and $V(H_4) \cap \{y_3, y_4, y_5\} = \{y_3\}$. Then, there exists a complete 5-subgraph $H_5 \notin \{H_1, \ldots, H_4\}$ such that $V(H_5) \cap (V(H_1) \cup V(H_2)) = \{x_4, x_5, y_4, y_5\}$. Let $V(H_5) - V(H_1) \cup V(H_2) = \{v\}$. Then $v \neq u$ since G is 5-colorable. Since $|G| \leq 12$ and since both u and v are contained in two complete 5-subgraphs, we must have $u, v \in H_4$. Assume w.l.o.g. that $V(H_4) = \{x_3, y_3, u, v, x_2\}$. Then $vx_j \in E(G)$, $j = 2, \ldots, 5$. Hence $c(v) = c(x_1)$ for any proper 5-coloring c of G. So $\{c(y_4), c(y_5)\} = \{c(x_2), c(x_3)\}$. Then $\{c(y_1), c(y_2), c(y_3)\} = \{c(x_1), c(x_4), c(x_5)\}$ and hence $c(u) = c(x_2)$ or $c(u) = c(x_3)$, contrary to $ux_2, ux_3 \in E(G)$. This completes the proof of Claim 3.

Claim 4. *G has two complete 5-subgraphs H_1 and H_2 such that $|V(H_1 \cap H_2)| = 2$.*

Suppose it is not so. Then by Claims 1–3. $|V(H' \cap H'')| = 1$ for any distinct complete 5-subgraphs H' and H'' of G. Let H_0 be a complete 5-subgraph of G and let $V(H_0) = \{x_1, \ldots, x_5\}$. By (i), for every vertex x_i, there exists a complete 5-subgraph $H_i \neq H_0$ such that $x_i \in H_i$. Then $H_0, H_1, \ldots, H_5$ are six distinct complete 5-subgraphs of G, contrary to (ii). Claim 4 is proved.

Now let H_1 and H_2 be any two complete 5-subgraphs of G with $|H_1 \cap$

$H_2| = 2$. Let $V(H_1 \cap H_2) = \{z_1, z_2\}$, $V(H_1) - V(H_1 \cap H_2) = \{x_1, x_2, x_3\}$ and $V(H_2) - V(H_1 \cap H_2) = \{y_1, y_2, y_3\}$.

Claim 5. *For any complete 5-subgraph H' with $H' \notin \{H_1, H_2\}$, $|V(H') \cap \{x_1, x_2, x_3\}| \le 1$ and $|V(H') \cap \{y_1, y_2, y_3\}| \le 1$.*

Suppose that there is a complete 5-subgraph $H_3 \notin \{H_1, H_2\}$ such that either $|V(H_3) \cap \{x_1, x_2, x_3\}| \ge 2$ or $|V(H_3) \cap \{y_1, y_2, y_3\}| \ge 2$, say $|V(H_3) \cap \{x_1, x_2, x_3\}| \ge 2$ (it is impossible that both $|V(H_3) \cap \{x_1, x_2, x_3\}| \ge 2$ and $|V(H_3) \cap \{y_1, y_2, y_3\}| \ge 2$). By Claims 1 and 2 $|V(H_3) \cap \{x_1, x_2, x_3\}| = 2$, and by Claim 3 $|V(H_3) \cap \{y_1, y_2, y_3\}| = 1$. W.l.o.g. we may assume that $V(H_3) = \{x_1, x_2, y_3, u, v\}$. Then, for any proper 5-coloring c of G, $c(y_3) = c(x_3)$. So x_3 cannot be adjacent to y_j for each $j \in \{1, 2, 3\}$ (otherwise we would have a required edge) and x_i cannot be adjacent to both y_1 and y_2, $i = 1, 2$. Note that G has at most five complete 5-subgraphs and that x_3 and y_j cannot be in the same complete 5-subgraph, $j = 1, 2$. By (i) we have $x_3 \in H_4$ and $y_1, y_2 \in H_5$, where $H_1, H_2, \ldots, H_5$ are all the complete 5-subgraphs of G. Hence $V(H_5) \cap V(H_1) = \emptyset$, contrary to Claim 3. This completes the proof of Claim 5.

Now by (i), there exist three distinct complete 5-subgraphs $H_3, H_4, H_5 \notin \{H_1, H_2\}$ such that $\{x_1, x_2, x_3\} \cap V(H_{i+2}) = \{x_i\}$, $i = 1, 2, 3$. By (ii), $H_1, \ldots, H_5$ are all the complete 5-subgraphs of G. So, w.l.o.g. we may assume that $\{y_1, y_2, y_3\} \cap V(H_{i+2}) = \{y_i\}$, $i = 1, 2, 3$. Suppose $V(H_j) \cap \{z_1, z_2\} = \emptyset$ for each $j \in \{3, 4, 5\}$. Then, since $|G| \le 12$, we have $|V(G) - V(H_1) \cup V(H_2)| = 4$ by Claim 2, say $V(G) - V(H_1) \cup V(H_2) = \{u_1, \ldots, u_4\}$. Assume $V(H_3) = \{x_1, y_1, u_1, u_2, u_3\}$ and $V(H_4) = \{x_2, y_2, u_2, u_3, u_4\}$. Then $\{x_3, y_3, u_1, u_4\} \subseteq V(H_5)$ and also $u_2 \in V(H_5)$ or $u_3 \in V(H_5)$. Replacing H_1, H_2 by H_3, H_4 and $\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}$ by $\{u_1, x_1, y_1\}, \{u_4, x_2, y_2\}$, we then have that $V(H_5) \cap (V(H_3) - V(H_3 \cap H_4)) = \{u_1\}$, $V(H_5) \cap (V(H_4) - V(H_4)) = \{u_4\}$, and $V(H_5) \cap V(H_3 \cap H_4) \neq \emptyset$. Therefore, we may assume that $V(H_j) \cap \{z_1, z_2\} \neq \emptyset$ for some $j \in \{3, 4, 5\}$, say $V(H_3) \cap \{z_1, z_2\} = \{z_1\}$ (note that, by Claim 2, $\{z_1, z_2\} \not\subseteq V(H_j)$). Let $H = H_1 \cap H_2 \cup H_3$. Then H is a subgraph of G with

$$V(H) = \{u_i \mid i = 0, 1, 2, 3\} \cup \{v_i, w_i \mid i = 1, 2, 3\}$$

and

$$E(H) = \{u_i u_j \mid 0 \le i < j \le 3\} \cup \{v_i w_i \mid i = 1, 2, 3\}$$
$$\cup\, (\cup_{j=1}^{3} \{u_i v_j, u_i w_j \mid 0 \le i \le 3, i \neq j\}).$$

By (i) and (ii), we may assume w.l.o.g. that $\{v_1, v_2, v_3\} \subseteq V(H_4)$ and $\{w_1, w_2, w_3\} \subseteq V(H_5)$ (note that, by Claim 5, v_i and w_i cannot be contained in the same H_j for $i = 1, 2, 3$ and $j = 4, 5$). Let c be any proper 5-coloring of G, say $c(u_i) = i$, $i = 1, 2, 3$, and $c(u_0) = 4$. Then $c(v_i) = 5$ or $c(w_i) = 5$ for each $i \in \{1, 2, 3\}$. Hence $c(v_i) = c(v_j) = 5$ or $c(w_i) = c(w_j) =$

5 for some pair i, j ($i \neq j$). This contradicts the fact that $\{v_i, v_j\} \subset V(H_4)$ and $\{w_i, w_j\} \subset V(H_5)$. Lemma 2 is proved. □

Proof of Theorem 1: We use induction on k. The case $k = 4$ was settled by Abbott and Zhou (see [1, p.227]). Actually that $t_3(G) \leq n-1$ for $n > 4$ follows from Theorem A. (The cases $k = 5$ and $k = 6$ have been proved in [4], but we will give a unified treatment for all k with $4 \leq k \leq 7$.) Let $4 < k \leq 7$. Assume the theorem is true for all $(k-1)$-critical graphs. Let G be a k-critical graph of order $n > k$. By Theorem A, $t_{k-1}(G) < n$. Let $t(v; G)$ denote the number of complete $(k-1)$-subgraphs of G containing the vertex v. Then

$$\sum_{v \in V(G)} t(v; G) = (k-1)t_{k-1}(G) < (k-1)n.$$

Hence there exists a vertex u of G such that $t(u; G) \leq k-2$.

Suppose that Theorem 1 is false and suppose that G is a counterexample. Then every edge of G is contained in at least two complete $(k-1)$-subgraphs. Let H be the subgraph of G induced by the neighbor set $N(u; G)$ of u in G. Then every vertex of H is contained in at least two complete $(k-2)$-subgraphs of H and H has at most $k-2$ complete $(k-2)$-subgraphs. We claim that H is $(k-2)$-chromatic. To prove the claim, it is enough to show that $d(u; G) < n-1$ since G is k-critical. In fact, if $d(u; G) = n-1$, then $G - u$ is $(k-1)$-critical and so by induction hypothesis, there exists an edge e of $G - u$ such that e is contained in at most one complete $(k-2)$-subgraph of $G - u$. Hence e is contained in at most one complete $(k-1)$-subgraph of G, contrary to the choice of G. Therefore, $d(u; G) < n-1$ and so the subgraph of G induced by $N(u; G) \cup \{u\}$ is $(k-1)$-colorable since G is k-critical. It follows that H is $(k-2)$-colorable. Hence H is $(k-2)$-chromatic since it contains complete $(k-2)$-subgraphs. Now by Lemma 2, there exists an edge $e = xy$ of H such that any proper $(k-2)$-coloring of $H - e$ assigns different colors to x and y. However, since G is k-critical, there is a proper $(k-1)$-coloring c of $G - e$ such that $c(x) = c(y)$. Then c induces a proper $(k-2)$-coloring c' of $H - e$ such that $c'(x) = c'(y)$, a contradiction. Theorem 1 is proved. □

Remark. Lemma 2 might still hold for $r = 6$ or even $r = 7$ or 8. But it does not hold for $r \geq 9$. For example, let G_1, G_2 and G_3 be three copies of the graph $K_1 + P_4$. Let $G' = G_1 \cup G_2 \cup G_3$ such that $G_1 \cap G_2 \cap G_3 = \emptyset$ and such that, for each pair i, j, $1 \leq i < j \leq 3$, the intersection $G_i \cap G_j$ consists of a single vertex which is of degree 2 in both G_i and G_j. Clearly G' is 3-chromatic and contains precisely 9 triangles. Moreover, every vertex of G' is contained in at least two triangles. Let $G = G' + K_{r-3}$. Then, for any $r \geq 9$, G is a r-chromatic graph satisfying the conditions (i) and (ii) of Lemma 2. However, it is easy to verify that, for every edge $e = xy$ of G, there exists a proper r-coloring c of $G - e$ such that $c(x) = c(y)$. Therefore, to solve Conjecture 2, we need to develop more effective techniques.

References

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