# On Covering Designs with Block size 5 and Index 7

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ABSTRACT. Let $V$ be a finite set of order $v$. A $(v, k, \lambda)$ covering design of index $\lambda$ and block size $k$ is a collection of $k$-element subsets, called blocks, such that every 2-subset of $V$ occurs in at least $\lambda$ blocks. The covering problem is to determine the minimum number of blocks, $\alpha(v, k, \lambda)$ in a covering design. It is well known that $\alpha(v, k, \lambda) \geq \lceil \frac{v}{k} \lceil \frac{v-1}{k-1} \lambda \rceil \rceil = \phi(v, k, \lambda)$, where $\lceil x \rceil$ is the smallest integer satisfying $x \leq \lceil x \rceil$. It is shown here that $\alpha(v, 5, 7) = \phi(v, 5, 7)$ for all positive integers $v \geq 5$ with the possible exception of $v = 22$, 28, 142, 162.

## 1. Introduction

A $(v, k, \lambda)$ covering design (or respectively packing design) of order $v$, block size $k$ and index $\lambda$ is a collection $\beta$ of $k$-element subsets, called blocks, of a $v$-set $V$ such that every 2-subset of $V$ occurs in at least (at most) $\lambda$ blocks.

Let $\alpha(v, k, \lambda)$ denote the minimum number of blocks in a $(v, k, \lambda)$ covering design; and $\sigma(v, k, \lambda)$ denote the maximum number of blocks in a $(v, k, \lambda)$ packing design. A $(v, k, \lambda)$ covering design with $|\beta| = \alpha(v, k, \lambda)$ is called a minimum covering design. Similarly a $(v, k, \lambda)$ packing design with $|\beta| = \sigma(v, k, \lambda)$ will be called a maximum packing design. It is well known that [26]

$$\alpha(v, k, \lambda) \geq \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \lambda \right\rceil \right\rceil = \phi(v, k, \lambda) \text{ and}$$

$$\sigma(v, k, \lambda) \left\lfloor \left\lfloor \frac{v-1}{k-1} \lambda \right\rfloor \right\rfloor = \psi(v, k, \lambda)$$

where $\lceil x \rceil$ is the smallest and $\lfloor x \rfloor$ is the largest integer satisfying $\lfloor x \rfloor \leq x \leq \lceil x \rceil$.

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When $\alpha(v,k,\lambda) = \phi(v,k,\lambda)$ the $(v,k,\lambda)$ covering design is called a minimal covering design. SImilarly when $\sigma(v,k,\lambda) = \psi(v,k,\lambda)$ the $(v,k,\lambda)$ packing design is called an optimal packing design.

Many researchers have been involved in determining the covering numbers known to date (see bibliography) most notably W.H. Mills and R.C. Mullin. In one of their paper they proved the following [24].

**Theorem 1.1.** *Let $v$ be an odd integer greater than 5.*

*(i) If $v \equiv 1 \pmod 4$ and $\lambda > 1$, then $\alpha(v,5,\lambda) + e$ where $e = 1$ if $\lambda(v-1) \equiv 0 \pmod 4$ and $\lambda v \frac{v-1}{k-1} \equiv -1 \pmod 5$ and $e = 0$ otherwise with the exceptions that $\alpha(9,5,2) = \phi(9,5,2)+1$, $\alpha(13,5,2) = \phi(13,5,2)+1$ and the possible exceptions of the pairs $(v,\lambda) \in \{(53,2),(73,2)\}$, and*

*(ii) If $v \equiv 3 \pmod 4$ and $\lambda \geq 1$ then $\alpha(v,5,\lambda) = \phi(v,5,\lambda)+e$ where $e$ is as in (i) with the exceptions that $\alpha(15,5,\lambda) = \phi(15,5,\lambda)+1$ for $\lambda = 1,2$ and the possible exception of the pairs $(v,\lambda) \in \{(63,2),(83,2)\}$.*

*Our interest here is in the case $k = 5$ and $\lambda = 7$. Since the case $v$ odd has been treated by Mills and Mullin we treat $v$ even. Our goal is to prove the following.*

**Theorem 1.2.** *Let $v \geq 5$ be an even integer. Then $\alpha(v,5,7) = \phi(v,5,7)$ with the possible exception of $v = 22$, 28, 142, 162.*

## 2. Recursive Constructions

In order to describe our recursive constructions we require several other types of combinatorial designs. A balanced incomplete block design, $B[v,k,\lambda]$, is a $(v,k,\lambda)$ covering design where every 2-subset of poitns is contained in precisely $\lambda$ blocks. If a $B[v,k,\lambda]$ exists then it is clear that $\alpha(v,k,\lambda) = \lambda v(v-1)/k(k-1) = \phi(v,k,\lambda)$ and Hanani, [16], has proved the following existence theorem for $B[v,5,\lambda]$.

**Theorem 2.1.** *Necessary and sufficient conditions for the existence of a $B[v,5,\lambda]$ are that $\lambda(v-1) \equiv 0 \pmod 4$ and $\lambda v(v-1) \equiv 0 \pmod{20}$ and $(v,\lambda) \neq (15,2)$.*

The following obvious lemma is most useful to us.

**Lemma 2.1.** *If there exists a $B[v,5,\lambda]$ and $\alpha(v,5,\lambda') = \phi(v,5,\lambda')$ then $\alpha(v,5,\lambda+\lambda') = \phi(v,5,\lambda+\lambda')$.*

**Corollary.** *Let $v \equiv 0$ or $16 \pmod{20}$ be a positive integer. Then $\alpha(v,5,7) = \phi(v,5,7)$.*

**Proof:** If $v \equiv 0$ or $16 \pmod{20}$ then by Theorem 2.1 there exists a $B[v,5,4]$. On the other hand for such $v, \alpha(v,5,3) = \phi(v,5,3)$, [4]. Hence $\alpha(v,5,7) = \phi(v,5,7)$.

A $(v,k,\lambda)$ covering design (or respectively packing design) with a hole of size $h$ is a triple $(V,H,\beta)$ where $V$ is a $v$-set, $H$ is a subset of $V$ of cardinality $h$, and $\beta$ is a collection of $k$-element subsets, called blocks, of $V$ such that

1. no 2-subset of $H$ appears in any block;

2. every 2-subset $\{x,y\}$ of $V$ where at least one of $x,y$ does not lie in $H$, appears in at least (at most) $\lambda$ blocks;

3. $|\beta| = \phi(v,k,\lambda) - \phi(h,v,\lambda)$, $(|\beta| = \psi(v,k,\lambda) - \psi(h,k,\lambda))$.

**Lemma 2.2.** *If there exists a $(v,k,\lambda)$ covering design with a hole of size $h \geq 5$ and $\alpha(h,k,\lambda) = \phi(h,k,\lambda)$ then $\alpha(v,k,\lambda) = \phi(v,k,\lambda)$.*

**Proof:** Form the blocks of an $(h,k,\lambda)$ minimal covering design on the points of the hole. Adding the blocks of the covering design with the hole gives a $(v,k,\lambda)$ minimal covering design.

In many places through the paper instead of constructing a $(v,5,7)$ minimal covering design we construct a $(v,5,7)$ covering design with a hole of size $h \geq 5$ where $\alpha(h,5,7) = \phi(h,5,7)$ and then apply lemma 2.2.

Let $k,\lambda$ and $v$ be positive integers and $M$ be a set of positive integers. A group divisible design $\mathrm{GD}[k,\lambda,M,v]$ is a triple $(V,\beta,\gamma)$ where $V$ is a set of points with $|V| = v$, and $\gamma = \{G_1,\dots,G_n\}$ is a partition of $V$ into $n$ sets called groups. The collection $\beta$ consists of $k$-subsets of $V$, called blocks, with the following properties.

1. $|B \cap G_i| \leq 1$ for all $B \in \beta$ and $G_i \in \gamma$;

2. $|G_i| \in M$ for all $G_i \in \gamma$;

3. every 2-subset $\{x,y\}$ of $V$ such that $x$ and $y$ belong to distinct groups is contained in exactly $\lambda$ blocks.

If $M = \{m\}$ then the group divisible design is denoted by $\mathrm{GD}[k,\lambda,m,v]$. A $\mathrm{GD}[k,\lambda,m,km]$ is called a transversal design and denoted by $\mathrm{T}[k,\lambda,m]$. It is well known that a $\mathrm{T}[k,1,m]$ is equivalent to $k-2$ mutually orthogonal Latin squares of side $m$.

In the sequel we shall use the following existence theorem for traversal designs. The proof of this result may be found in [1], [2], [12], [13], [16], [25], [27].

**Theorem 2.2.** *There exists a* $T[6,1,m]$ *for all positive integers* $m$ *with the exception of* $m \in \{2, 3, 4, 6\}$ *and the possible exception of* $m \in \{10, 14, 18, 22, 26, 34, 42\}$.

**Theorem 2.3.** *If there exists a* $GD[6,7,5,5n]$ *and a* $(20+h,5,7)$ *covering design with a hole of size* $h$ *then there exists a* $(20(n-1)+4u+h,5,7)$ *covering design with a hole of size* $4u+h$ *where* $0 \leq u \leq 5$.

**Proof:** Take a GD$[6,7,5,5n]$ and delete $5-u$ points from the last group. Inflate this design by a factor of 4. On the blocks of size 5 and 6 construct a GD$[5,1,4,20]$ and a GD$[5,1,4,24]$ respectively. Add $h$ points to the groups and on the first $n-1$ groups construct a $(20+h,5,7)$ covering design with a hole of size $h$, and take the $h$ points with the last group to be the hole. In a similar way we can prove the following:

**Theorem 2.4.** *If there exists a* $GD[6,7,5,5n]$, *a* $(20+h,5,7)$ *covering design with a hole of size* $h$ *and a* $(20+h,5,7)$ *minimal covering design then there exists a* $(20n+h,5,7)$ *minimal covering design.*

**Proof:** Take a GD$[6,7,5,5n]$ and inflate the design by a factor of 4. Replace the blocks of this design by the blocks of a GD$[5,1,4,24]$. Add $h$ points to the groups and on the first $(n-1)$ groups construct a $(20+h,5,7)$ covering design with a hole of size $h$ and on the last group construct a $(20+h,5,7)$ minimal covering design. It is readily checked that this construction yields a $(20n+h,5,7)$ minimal covering design.

It is clear that the application of the above theorems require the existence of a GD$[6,7,5,5n]$. Our authority for this is the following lemma of Hanani [16, p. 286].

**Lemma 2.3.** *There exists a* $GD[6,7,5,5n]$ *for* $n=7$.

If in the definition of GD$[k,\lambda,m,v]$ (similarly T$[k,\lambda,m]$) condition 3 is changed to be read as (3) every 2-subset $\{x,y\}$ of $V$ such that $x$ and $y$ are neither in the same group (column) nor in the same row is contained in exactly $\lambda$ blocks of $\beta$ and no block contains two elements of the same row. Then the resultant design is called a modified group divisible design (modified transversal design) and is denoted by MGD$[k,\lambda,m,v]$ (MT$[k,\lambda,m]$). (We may look at the point set of GD$[k,\lambda,m,v]$ as a matrix of size $m \times n$, then the groups of MGD$[k,\lambda,m,v]$ are precisely the columns of the matrix.

A resolvable modified group divisible design, RMGD$[k,\lambda,m,v]$, is a modified group divisible design the blocks of which can be partitioned into parallel classes. It is clear that a RMGD$[5,1,5,5m]$ is the same as RT$[5,1,m]$ with one parallel class of blocks singled out, and since RT$[5,1,m]$ is equivalent to T$[6,1,m]$ we have the following.

**Theorem 2.5.** *There exists a* $RMGD[5,1,5,5m]$ *for all positive integers* $m$ *with the exception of* $m \in \{2, 3, 4, 6\}$ *and the possible exception of*

$m \in \{10, 14, 18, 22, 26, 34, 42,\}$.

The next theorems are in the form most useful to us.

**Theorem 2.6.** [5] *If there exists a RMGD$[5,1,5,5m]$ and a GD$[5,7,\{4,s^*\}, 4m+s]$, where $*$ means there is exactly one group of size $s$, and there exists a $(20+h,5,7)$ covering design with a hole of size $h$ then there exists a $(20m+4u+h+s,5,7)$ covering design with a hole of size $4u+h+s$ where $0 \le u \le m-1$.*

The proof of the following theorem is very similar to the proof of Theorem 2.4 of [5].

**Theorem 2.7.** *If there exists (1) a RMGD$[5,1,5,5m]$ (2) a GD$[5,7,\{4,8^*\}, 4m+4]$ where $*$ means there is exactly one group of size 8 (3) a $(20,5,7)$ minimal covering design and a $(24,5,7)$ covering design with a hole of size 4. Then there exists a $(20m+4u+4,5,7)$ covering design with a hole of size $4u+4$ where $0 \le u \le m-1$.*

**Proof:** Take a RMGD$[5,1,5,5m]$ and inflate it by a factor of 4. To each of $u$, $0 \le u \le m-1$, parallel classes add 4 points and replace their blocks by the blocks of a GD$[5,7,4,24]$. On the remaining parallel classes construct a GD$[5,7,4,20]$. To the parallel class of block size $m$, after inflating by 4, add 4 new points $\{a,b,c,d\}$ to the last group and construct a GD$[5,7,\{4,8^*\},4m+4]$. Finally on the first $(m-1)$ groups construct a $(20,5,7)$ minimal covering design and on the last group construct a $(24,5,7)$ covering design with a hole of size 4, say, $\{a,b,c,d\}$. Then it is clear that this construction yields a $(20m+4u+4,5,7)$ covering design with a hole of size $4u+4$.

**Theorem 2.8.** [5] *If there exists (1) a RMGD$[5,1,5,5m]$ (2) a GD$[5,7, \{4,s^*\},4(m-1)+s]$ and (3) a $(20+h,5,7)$ covering design with a hole of size $h$. Then there exists a $(24(m-1)+s+h,5,7)$ covering design with a hole of size $4(m-1)+s+h$.*

It is clear that the application of the above theorems requires the existence of a GD$[5,7,\{4,s^*\},4m+s]$. We observe that we may choose $s=0$ if $m \equiv 1 \pmod 5$; $s=4$ if $m \equiv 0$ or $4 \pmod 5$, and $s = \frac{4(m-1)}{3}$ if $m \equiv 1 \pmod 3$ (see [5]). We may also apply the following [15].

**Theorem 2.9.** *There exists a GD$[5,1,\{4,8^*\},4m+8]$ where $m \equiv 0$ or $2 \pmod 5$) $m \ge 7$ with the possible exception of $m = 10$.*

We close this section with the following notation that will be used later.

A block $\langle k, k+m, k+n, k+y, f(k)\rangle \pmod v$ where $f(k)=a$ if $k$ is even and $f(k)=b$ if $k$ is odd is denoted by $\langle 0\ m\ n\ y\rangle \cup \{a,b\} \pmod v$.

Similarly a block $\langle k, k+m, k+n, k+y, f(k)\rangle \pmod v$ where $f(k)=h_i$ if $k \equiv i \pmod 4$ is denoted by $\langle 0\ m\ n\ y\rangle \cup \{h_i\}_{i=0}^{3}$.

A block of the form $\langle (0,k)\ (0,k+m)\ (1,k+n)\ (1,k+y)\ f(k)\rangle$, $k \in Z_v$ where $f(k) = h_1$ if $k$ is even and $f(k) = h_2$ if $k$ is odd is denoted by $\langle (0,0)\ (0,m)\ (1,n)\ (1,y)\rangle \cup \{h_1, h_2\}$.

## 3. The Structure of Packing and Covering Designs

Let $(V,\beta)$ be a $(v,k,\lambda)$ packing design, for each 2-subset $e = \{x,y\}$ of $V$ define $m(e)$ to be the number of blocks in $\beta$ which contain $e$. Note that by the definition of a packing design we have $m(e) \le \lambda$ for all $e$.

The complement of $(V,\beta)$, denoted by $C(V,\beta)$ is defined to be the graph with vertex set $V$ and edges $e$ occurring with multiplicity $\lambda$-$m(e)$ for all $e$. The number of edges (counting multiplicities) in $C(V,\beta)$ is given by $\lambda\binom{v}{2} - |\beta|\binom{k}{2}$. The degree of the vertex $x$ in $C(V,\beta)$ is $\lambda(v-1) - r_x\ (k-1)$ where $r_x$ is the number of blocks containing $x$.

In a similar way we define the excess graph of a $(V,\beta)$ covering design denoted by $E(V,\beta)$, to be the graph with vertex set $V$ and edges $e$ occurring with multiplicity $m(e)-\lambda$ for all $e$. The number of edges in $E(V,\beta)$ is given by $|\beta|\binom{k}{2} - \lambda\binom{v}{2}$; and the degree of each vertex is $r_x(k-1) - \lambda(v-1)$ where $r_x$ is as before.

**Lemma 3.1.** *Let $(V,\beta)$ be a $(v,5,4)$ packing design with $\psi(v,5,4) - e$ blocks where $e = 1$ if $v \equiv 3 \pmod 5$ and 0 otherwise. Then the degree of each vertex of $C(V,\beta)$ is divisible by 4 and the number of edges in the graph is 0, 4 or 12 when $v \mod 5 \in \{0,1\}, \{2,4\}$, or $\{3\}$.*

The only graph with 4 edges and every vertex of a degree divisible by 4 is the graph with four parallel edges connecting two vertices and $v-2$ isolated vertices. Therefore, when $v \equiv 2$ or $4 \pmod 5$ a $(v,5,4)$ optimal packing design is the same as, a $(v,5,4)$ packing design with a hole of size 2.

**Lemma 3.2.** *Let $(V,\beta)$ be a $(v,5,2)$ optimal packing design where $v \equiv 3 \pmod{10}$. Then the degree of each vertex of $C(V,\beta)$ is divisible by 4 and the number of edges in the graph is 6. Hence, $C(V,\beta)$ consists of $v-3$ isolated vertices and 3 other vertices the pairs of which are connected by 2 edges.*

**Lemma 3.3.** *Let $(V,\beta)$ be a $(v,5,4)$ minimal covering design. Then the degree of each vertex of $E(V,\beta)$ is divisible by 4 and the number of edge in the graph is 0, 6 or 8 when $v \mod 5 \in \{0,1\}, \{2,4\}$, or $\{3\}$ respectively.*

The only graph with 6 edges and every vertex of a degree divisible by 4 is the graph with $v-3$ isolated vertices and 3 other vertices the pairs of which are connected by two edges.

The following is very simple but most useful to us.

**Theorem 3.1.** *If there exists*

1. *A* $(v,5,\lambda)$ *covering design with* $\phi(v,5,\lambda)$ *blocks*

2. *A* $(v,5,\lambda')$ *packing design with* $\psi(v,5,\lambda')$ *blocks*

3. $\phi(v,5,\lambda)+\psi(v,5,\lambda')=\phi(v,5,\lambda+\lambda')$

4. *The complement graph* $C(V,\beta)$ *of the packing design is isomorphic to a subgraph* $G$ *of the excess graph,* $E(V,\beta)$*, of the covering design.*

Then there exists a $(v,5,\lambda+\lambda')$ covering design with $\phi(v,5,\lambda+\lambda')$ blocks, that is, a $(v,5,\lambda+\lambda')$ minimal covering design.

## 4. Constructions

In this section we distinguish the following cases.

### 4.1 $v\equiv 4 \pmod{20}$

**Lemma 4.1.** *(a) Let* $v\equiv 4 \pmod{20}$ *be a positive integer greater than 4. Then* $\alpha(v,5,7)=\phi(v,5,7)$*. (b) There exists a* $(24,5,7)$ *covering design with a hole of size 4.*

**Proof:** (a) For $v\equiv 4 \pmod{20}$, $v\neq 64, 84$, the construction is as follows:

1. take a $(v-1,5,2)$ minimal covering design, [24]. In this design each pair appears in precisely two blocks except one pair, say, $(v-2,v-1)$ that appears in six blocks.

2. take a $(v,5,4)$ optimal packing design, [8]. In this design each pair appear in precisely four blocks except one pair, say, $(v-2,v-1)$ that appear in zero block. Furthermore, assume in this design we have the following two blocks:

   $$\langle 1\ 2\ 3\ 7\ 9\rangle \qquad \langle a\ b\ c\ 8\ 10\rangle$$

   where $\{1,2,3,a,b,c,7,\ldots,10\}$ are arbitrary numbers, $\{a,b,c\}$ and $\{1,2,3,7\}$ are not necessarily disjoint. In the first block change 9 to $v$ and in the second change 10 to $v$.

3. take a $(v+2,5,1)$ minimal covering design. This design is constructed by taking the blocks of B$[v+1,5,1]$ and then partioning the $v+1$ points into a set of quadruples, and to each quadruple adjoin the point $v+2$. Since $v+1\equiv 1 \pmod 4$ there is a block of size 2, say, $\langle v+1,v+2\rangle$ which we delete. We also assume that the pairs $(7,9)$, $(8,10)$, $(v-2,v-1)$ appear at least twice while the pairs

$(v, v+1)$, $(v, v+2)$ appear exactly once in the blocks of this design. Furthermore, assume we have the following two blocks

$$\langle 1\ 2\ 3\ v\ v+2 \rangle \qquad \langle a\ b\ c\ v\ v+1 \rangle$$

In the first block change $v+2$ to 9 and in the second block change $v+1$ to 10.

In all other blocks change $v+1$ and $v+2$ to $v$.

Now it is readily checked that the above three steps yield the blocks of a $(v, 5, 7)$ minimal covering design for all $v \equiv 4 \pmod{20}$, $v \neq 64$, 84.

For $v = 64, 84$ the constructions are given in the next table. In general, the construction in this table and other tables to follow is as follows. Let $X = Z_{v-n} \cup H_n$ or $X = Z_2 \times Z_{\frac{v-n}{2}} \cup H_n$ where $H_n = \{h_1, \ldots, h_n\}$ is the hole. The blocks are constructed by taking the orbits of the tabulated base blocks.

| $v$ | Point Set | Base Blocks |
|---|---|---|
| 64 | $Z_{56} \cup H_8$ | On $Z_{56}$ construct a B[56, 5, 4] and take the following blocks: |
| | | $\langle 0\ 1\ 2\ 28\ h_1 \rangle$ $\langle 0\ 2\ 14\ 38\ h_2 \rangle$ $\langle 0\ 7\ 22\ 32\ h_3 \rangle$ $\langle 0\ 9\ 20\ 26\ h_4 \rangle$ |
| | | $\langle 0\ 3\ 8\ 43\ h_5 \rangle$ $\langle 0\ 4\ 10\ 34\ h_6 \rangle$ $\langle 0\ 3\ 17\ 21\ h_7 \rangle$ $\langle 0\ 7\ 16\ 44\ h_8 \rangle$ |
| | | $\langle 0\ 5\ 21\ 34 \rangle \cup \{h_1, h_2\}$ $\langle 0\ 11\ 23\ 38 \rangle \cup \{h_3, h_4\}$ $\langle 0\ 5\ 13\ 36 \rangle \cup \{h_5, h_6\}$ |
| | | $\langle 0\ 4\ 11\ 19 \rangle \cup \{h_7, h_8\}$ $\langle 0\ 1\ 3\ 10 \rangle \cup \{h_i\}_{i=1}^{4}$ $\langle 0\ 6\ 23\ 37 \rangle \cup \{h_i\}_{i=5}^{8}$ |
| 84 | $Z_{76} \cup H_8$ | On $Z_{76}$ construct a B[76, 5, 4] and take the following blocks: |
| | | $\langle 0\ 1\ 4\ 9\ 24 \rangle$ $\langle 0\ 7\ 26\ 42\ 54 \rangle$ $\langle 0\ 10\ 21\ 46\ 59 \rangle$ $\langle 0\ 3\ 18\ 37 \rangle \cup \{h_i\}_{i=1}^{4}$ |
| | | $\langle 0\ 6\ 31\ 33 \rangle \cup \{h_i\}_{i=5}^{8}$ $\langle 0\ 1\ 5\ 12 \rangle \cup \{h_1, h_2\}$ $\langle 0\ 9\ 23\ 48 \rangle \cup \{h_3, h_4\}$ |
| | | $\langle 0\ 13\ 29\ 56 \rangle \cup \{h_5, h_6\}$ $\langle 0\ 14\ 31\ 55 \rangle \cup \{h_7, h_8\}$ $\langle 0\ 6\ 32\ 64\ h_i \rangle$ |
| | | $\langle 0\ 8\ 30\ 44\ h_2 \rangle$ $\langle 0\ 10\ 31\ 33\ h_3 \rangle$ $\langle 0\ 1\ 3\ 9\ h_4 \rangle$ |
| | | $\langle 0\ 4\ 16\ 38\ h_5 \rangle$ $\langle 0\ 5\ 15\ 41\ h_6 \rangle$ $\langle 0\ 7\ 20\ 59\ h_7 \rangle$ $\langle 0\ 11\ 29\ 57\ h_8 \rangle$ |

(b) For a $(24, 5, 7)$ covering design with a hole of size 4 proceed as follows:

1. take a $(23, 5, 2)$ packing design on $Z_{20} \cup H_3$ with a hole of size 3, say $H_3$ [9].
2. take two copies of a B[25, 5, 1] on $Z_{20} \cup H_5$. Assume in both copies we have the block $\langle h_1, h_2, h_3, h_4, h_5 \rangle$. Delete this block and in all other blocks change $h_5$ to $h_4$. The above two steps yield a $(24, 5, 4)$ covering design with a hole of size 4.
3. take a $(24, 5, 3)$ covering design with a hole of size 4 on $X = Z_{20} \cup H_4$

$$\langle 0\ 4\ 8\ 12\ 16 \rangle + i, i \in Z_4$$

$\langle 0\ 1\ 3\ 10 \rangle \cup \{h_i\}_{i=1}^{4} \pmod{20}$ $\quad$ $\langle 0\ 3\ 8\ 13 \rangle \cup \{h_1, h_2\} \pmod{20}$

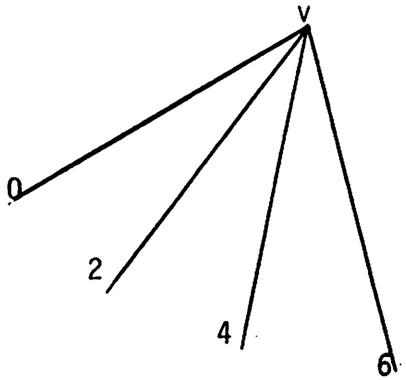
$\langle 0\ 3\ 9\ 14 \rangle \cup \{h_3, h_4\} \pmod{20}$ $\quad$ $\langle 0\ 1\ 2\ 4\ 8 \rangle \pmod{20}$

## 4.2 $v \equiv 8 \pmod{20}$

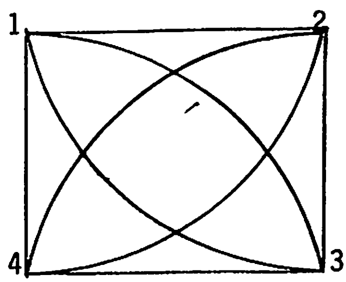
**Lemma 4.2.** *Let $v \equiv 8 \pmod{20}$ be a positive integer. If there exist (1) a $(v,5,3)$ covering design with a hole of size 8 (2) a $(v-1,5,2)$ covering design with a hole of size 7 (3) a $(v+1,5,2)$ optimal packing design with a hole of size 9 then there exists a $(v,5,7)$ minimal covering design.*

**Proof:** Assume the above three conditions hold. Then a $(v,5,7)$ minimal covering design can be constructed as follows:

1. take a $(v,5,3)$ covering design with a hole of size 8. On the hole of size 8 construct an $(8,5,3)$ minimal covering design as follow: $X = Z_8$ and blocks are $\langle 0\ 1\ 2\ 4\ 5 \rangle$ mod 8, $\langle 0\ 2\ 4\ 6 \rangle + i$, $i \in Z_2$. Careful inspection of this design shows that its excess graph contains the following subgraph on the vertices $\{0,2,4,6,v\}$ where $v \in Z_8$ is an arbitrary number different from $\{0,2,4,6\}$. Now by lemma 2.2 follows that the $(v,5,3)$ minimal covering design, $v \equiv 8 \pmod{20}$, has a subgraph as the above.

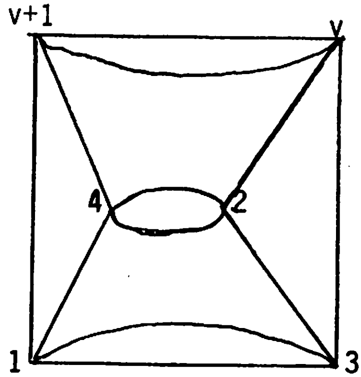



2. take a $(v-1,5,2)$ covering design with a hole of size 7. But the excess graph of the $(7,5,2)$ minimal covering design, [23], consists of 3 isolated vertices and the following graph on the remaining four vertices, say, $\{1,2,3,4\}$. Hence the excess graph of the $(v-1,5,2)$ minimal covering design, ($v \equiv 8 \pmod{20}$), consists of $v-5$ isolated vertices and the same graph as the above on the remaining 4 vertices.



3. take a $(v+1,5,2)$ packing design with a hole of size 9. The complement graph of the $(9,5,2)$ packing design, [9] consists of $v-5$ isolated vertices and the following graph on the remaining 6 vertices say, $\{1,2,3,4,v,v+1\}$. Hence, the complement graph of the $(v+1,5,2)$ packing design Consist of $v-5$ isolated vertices and the following graph as the above on the remaining 6 vertices.

In this design change $v+1$ to $v$. Since we changed $v+1$ to $v$ it is easy to see that the complement graph of the design in (3) is isomorphic to a subgraph of the excess graphs of the designs in (1) and (2). Now apply theorem 3.1 and the result follows.




**Lemma 4.3.**

1. *There exists a* $(v,5,3)$ *covering design with a hole of size 8 for* $v =$ 48, 68, 88.
2. *There exists a* $(v,5,2)$ *covering design with a hole of size 7 for* $v =$ 47, 67, 87.
3. *There exists a* $(v,5,2)$ *packing design with a hole of size 9 for* $v =$ 49, 69, 89.

**Proof:** (1) For a $(v,5,3)$ covering with a hole of size 8, $v = 48,68,88$ see the next table. For a $(v,5,2)$ covering design with a hole of size 7, $v = 47,67,87$, see [9]. For a $(v,5,2)$ packing design with a hole of size 9, $v = 49,69,89$, see [9].

| $v$ | Point Set | Base Blocks |
|---|---|---|
| 48 | $Z_{40} \cup H_8$ | $\langle 0\ 8\ 16\ 24\ 32 \rangle + i, i \in Z_8$ twice. |
| | | $\langle 0\ 1\ 10\ 23 \rangle \cup \{h_i\}_{i=1}^{4} \langle 0\ 3\ 14\ 21 \rangle \cup \{h_i\}_{i=5}^{8} \langle 0\ 1\ 4\ 13 \rangle \cup \{h_1, h_2\}$ |
| | | $\langle 0\ 2\ 7\ 17 \rangle \cup \{h_3, h_4\} \langle 0\ 5\ 11\ 26 \rangle \cup \{h_5, h_6\} \langle 0\ 5\ 14\ 29 \rangle \cup \{h_7, h_8\}$ |
| | | $\langle 0\ 1\ 3\ 7\ 20 \rangle\ \langle 0\ 2\ 8\ 12\ 20 \rangle$ |
| 68 | $Z_{60} \cup H_8$ | $\langle 0\ 12\ 24\ 36\ 48 \rangle + i,\ i \in Z_{12}$, twice |
| | | $\langle 0\ 5\ 14\ 47 \rangle \cup \{h_i\}_{i=1}^{4} \langle 0\ 5\ 22\ 39 \rangle \cup \{h_i\}_{i=5}^{8} \langle 0\ 10\ 25\ 41 \rangle \cup \{h_1, h_2\}$ |
| | | $\langle 0\ 5\ 11\ 46 \rangle \cup \{h_3, h_4\} \langle 0\ 7\ 24\ 39 \rangle \cup \{h_5, h_6\} \langle 0\ 9\ 27\ 40 \rangle \cup \{h_7, h_8\}$ |
| | | $\langle 0\ 1\ 3\ 9\ 38 \rangle \langle 0\ 4\ 11\ 19\ 32 \rangle \langle 0\ 4\ 18\ 34\ 44 \rangle \langle 0\ 1\ 3\ 7\ 30 \rangle \langle 0\ 1\ 3\ 11\ 23 \rangle$ |
| 88 | $Z_{80} \cup H_8$ | On $Z_{80} \cup H_5$ construct a B[85, 5, 1] such that $H_5$ is a block which we delete, and take the following blocks |
| | | $\langle 0\ 16\ 32\ 48\ 64 \rangle + i,\ i \in Z_{16}$ |
| | | $\langle 0\ 20\ 40\ 60\ h_6 \rangle + i,\ i \in Z_{20}$ |
| | | $0\ 13\ 40\ 53 \rangle \cup \{h_7, h_8\}$ half orbit |
| | | $\langle 0\ 1\ 3\ 7\ 53 \rangle \langle 0\ 8\ 22\ 39\ 50 \rangle \langle 0\ 9\ 19\ 42\ 68 \rangle \langle 0\ 1\ 3\ 7\ 19 \rangle$ |
| | | $\langle 0\ 8\ 22\ 32\ 45 \rangle \langle 0\ 5\ 29\ 44 \rangle \cup \{h_1, h_2\} \langle 0\ 17\ 35\ 60 \rangle \cup \{h_3, h_4\}$ |
| | | $\langle 0\ 5\ 25\ 34 \rangle \cup \{h_5, h_6\} \langle 0\ 11\ 26\ 47 \rangle \cup \{h_7, h_8\}$ |

**Corollary.** $\alpha(v,5,7) = \phi(v,5,7)$ *for* $v = 8, 48, 68, 88$.

**Proof:** The proof follows from lemmas 4.2 and 4.3.

**Lemma 4.4.** *Let* $v \equiv 8 \pmod{20}$ *be a positive integer. Then* $\alpha(v,5,7) = \phi(v,5,7)$ *with the possible exception of* $v = 28$.

**Proof:** For $v = 8, 48, 68, 88$ the result follows from the corollary. For $v \geq 108$, $v \neq 128$, 168, 208, 268, simple calculations show that $v$ can be written in the form $v = 20m + 4u + h + s$ where $m, u, h$ and $s$ are chosen so that

1. there exists a RMGD[$5, 1, 5, 5m$];
2. $4u + h + s \equiv 8 \pmod{20}$, $8 \leq 4u + h + s \leq 88$, $4u + h + s \neq 28$;
3. $0 \leq u \leq m - 1$, $s \equiv 0 \pmod 4$, $h = 0$ or 4;
4. there exists a GD[$5, 7, \{4, s^*\}, 4m + s$].

Now apply theorem 2.6 to give the result.

For $v = 128$ apply theorem 2.3 with $n = 7$, $h = 0$ and $u = 2$. For $v = 168$ apply theorem 2.7 with $m = 8$ and $u = 1$. For $v = 208$, take a T[$5, 7, 40$], [16]. Add 8 points to the groups and on the first group construct a $(48, 5, 7)$ minimal covering design and on the other groups construct a $(48, 5, 7)$ covering design with a hole of size 8. Such design can be contracted on $X = Z_{40} \cup H_8$ as follows: On $X$ construct a $(48, 5, 4)$ covering design with a hole of size 8, say, $H_8$, [10], and take the following blocks: $\langle 0\ 8\ 16\ 24\ 32 \rangle + i$, $i \in Z_8$, twice $\langle 0\ 1\ 3\ 18 \rangle \cup \{h_i\}_{i=1}^{4} \pmod{40}$ $\langle 0\ 6\ 17\ 27 \rangle \cup \{h_i\}_{i=5}^{8} \pmod{40}$ $\langle 0\ 5\ 12\ 31 \rangle \cup \{h_1, h_2\} \pmod{40}$ $\langle 0\ 4\ 11\ 21 \rangle \cup \{h_3, h_4\} \pmod{40}$ $\langle 0\ 3\ 9\ 16 \rangle \cup \{h_5, h_6\} \pmod{40}$ $\langle 0\ 5\ 20\ 25 \rangle \cup \{h_7, h_8\} \pmod{40}$ $\langle 0\ 1\ 2\ 4\ 13 \rangle \pmod{40}$ $\langle 0\ 4\ 10\ 18\ 32 \rangle \pmod{40}$.

For $v = 268$ take a RGD[5, 1, 5, 65] [14] and inflate the design by a factor of 4. Add 4 points to each of two parallel classes and replace their blocks by the blocks of GD[5, 7, 4, 24] and on the remaining parallel classes construct a GD[5, 7, 4, 20]. Finally, on the groups construct a (20, 5, 7) minimal covering design. This construction gives a (268, 5, 7) covering design with a hole of size 8. But $\alpha(8,5,7) = \phi(8,5,7)$ hence, $\alpha(268,5,7) = \phi(268,5,7)$.

### 4.3 $v \equiv 12 \pmod{20}$

**Lemma 4.5.** *$\alpha(v,5,7) = \phi(v,5,7)$ for $v$ = 12, 32, 52, 72, 92.*

**Proof:** The required constructions are given in the following table.

| $v$ | Point Set | Base Blocks |
|---|---|---|
| 12 | $Z_{12}$ | $\langle 0\ 1\ 4\ 6\ 9\rangle$ twice $\langle 0\ 1\ 2\ 3\ 5\rangle\langle 0\ 1\ 2\ 6\ 8\rangle$ |
| 32 | $Z_{32}$ | $\langle 0\ 1\ 2\ 4\ 24\rangle$ 3 times $\langle 0\ 3\ 8\ 15\ 21\rangle$ twice $\langle 0\ 4\ 9\ 19\ 25\rangle$ twice |
| | | $\langle 0\ 5\ 11\ 15\ 20\rangle\langle 0\ 1\ 7\ 14\ 18\rangle\langle 0\ 2\ 5\ 14\ 18\rangle\langle 0\ 3\ 10\ 16\ 24\rangle$ |
| 52 | $Z_{40} \cup H_{12}$ | On $Z_{40} \cup H_{11}$ construct a B[51, 5, 2] with a hole of size 11. Such design can be constructed by taking a T[5, 2, 10], [16], adjoin a point to the groups and on the first four groups construct a B[11, 5, 2] and take the last group with the point to be the hole. Take also the following blocks: |
| | | $\langle 0\ 8\ 16\ 24\ 32\rangle + i$, $i \in Z_8$, 3 times $\langle 0\ 7\ 20\ 27\ h_{12}\rangle$ half orbit |
| | | $\langle 0\ 1\ 6\ 15\rangle \cup \{h_i\}_{i=1}^{4}\langle 0\ 3\ 13\ 22\rangle \cup \{h_i\}_{i=5}^{8}\langle 0\ 2\ 7\ 17\rangle \cup \{h_i\}_{i=9}^{12}$ |
| | | $\langle 0\ 1\ 3\ 21\ h_1\rangle\langle 0\ 4\ 10\ 31\ h_2\rangle\langle 0\ 4\ 11\ 26\ h_3\rangle\langle 0\ 5\ 17\ 28\ h_4\rangle$ |
| | | $\langle 0\ 1\ 3\ 6\ h_5\rangle\langle 0\ 4\ 12\ 26\ h_6\rangle\langle 0\ 4\ 13\ 29\ h_7\rangle\langle 0\ 7\ 17\ 28\ h_8\rangle$ |
| | | $\langle 0\ 1\ 2\ 11\ h_9\rangle\langle 0\ 2\ 6\ 18\ h_{10}\rangle\langle 0\ 3\ 8\ 23\ h_{11}\rangle\langle 0\ 6\ 13\ 27\ h_{12}\rangle$ |
| 72 | $Z_{60} \cup H_{12}$ | On $Z_{60} \cup H_{11}$ construct a B[71, 5, 4] with a hole of size 11. Such design can be constructed by taking a T[6, 1, 12]. Delete 3 points from last group and replace all its blocks by the blocks of B[6, 5, 4] and B[5, 5, 4]. Adjoin two points to the groups and on the first five groups construct a (14, 5, 4) covering design with a hole of size 2, [8], and take these 2 points with the last group to be the hole of size 11. Take also the following blocks: |
| | | $\langle 0\ 1\ 3\ 10\rangle \cup \{h_i\}_{i=1}^{4}\langle 0\ 5\ 19\ 34\rangle \cup \{h_i\}_{i=5}^{8}\langle 0\ 6\ 17\ 39\rangle \cup \{h_i\}_{i=9}^{12}$ |
| | | $\langle 0\ 4\ 12\ 28\ h_{12}\rangle\langle 0\ 18\ 25\ 41\rangle \cup \{h_1, h_2\}\langle 0\ 13\ 33\ 42\rangle \cup \{h_3, h_4\}$ |
| | | $\langle 0\ 1\ 3\ 24\rangle \cup \{h_5, h_6\}\langle 0\ 4\ 11\ 33\rangle \cup \{h_7, h_8\}\langle 0\ 5\ 13\ 30\rangle \cup \{h_9, h_{10}\}$ |
| | | $\langle 0\ 6\ 15\ 25\rangle \cup \{h_{11}, h_{12}\}\langle 0\ 1\ 3\ 13\ 43\rangle\langle 0\ 4\ 12\ 26\ 40\rangle\langle 0\ 5\ 16\ 31\ 37\rangle$ |
| 92 | $Z_{80} \cup H_{12}$ | On $Z_{80} \cup H_{11}$ construct a B[91, 5, 4] with a hole of size 11. Such designs can be constructed by taking a T[6, 1, 16]. Delete 10 point from the last group, and replace all its blocks by the blocks of a B[6, 5, 4] and B[5, 5, 4]. Adjoin five points to the groups and on the first five groups construct a B[21, 5, 4] such that these five points are a block, which we delete, and take these five points with the last group to be the hole of size 11. Take also the blocks of a (80, 5, 1) covering design, [18], and take the following blocks: |
| | | $\langle 0\ 2\ 14\ 40\ 50\rangle\langle 0\ 4\ 20\ 28\ 62\rangle\langle 0\ 17\ 39\ 50\ h_{12}\rangle\langle 0\ 1\ 7\ 26\rangle \cup \{h_i\}_{i=1}^{4}$ |
| | | $\langle 0\ 2\ 23\ 37\rangle \cup \{h_i\}_{i=5}^{8}\langle 0\ 13\ 31\ 46\rangle \cup \{h_i\}_{i=9}^{12}\langle 0\ 1\ 4\ 7\rangle \cup \{h_1, h_2\}$ |
| | | $\langle 0\ 5\ 10\ 29\rangle \cup \{h_3, h_4\}\langle 0\ 8\ 31\ 51\rangle \cup \{h_5, h_6\}\langle 0\ 9\ 21\ 48\rangle \cup \{h_7, h_8\}$ |
| | | $\langle 0\ 9\ 25\ 36\rangle \cup \{h_9, h_{10}\}\langle 0\ 13\ 28\ 63\rangle \cup \{h_{11}, h_{12}\}$ |

**Lemma 4.6.** *Let $v \equiv 12 \pmod{20}$ be a positive integer. Then $\alpha(v,5,7) = \phi(v,5,7)$.*

**Proof:** For $v$ = 12, 32, 52, 72, 92 the result follows from lemma 4.5. For $v \geq 112$, $v \neq 132$, write $v = 20m + 4u + h + s$ where $m, u, h$ and $e$ are chosen as in lemma 4.4 with the difference that $4u+h+s \equiv 12 \pmod{20}$, $12 \le 4u+h+s \le 92$. Now apply theorem 2.6 to get the result.

For $v = 132$ apply theorem 2.3 with $n = 7$, $u = 2$ and $h = 4$.

### 4.4 $v \equiv 2 \pmod{20}$

**Lemma 4.7.**

*(a)* $\alpha(v,5,7) = \phi(v,5,7)$ *for* $v = 42, 62, 82, 102, 122$.

*(b) There exists a* $(22,5,7)$ *covering design with a hole of size* 2.

*(c) There exists a* $(26,5,7)$ *covering design with a hole of size* 6.

**Proof:** (a) The required constructions are given in the following table. The first construction is a $(22,5,7)$ covering design with a hole of size 2, and the second one is a $(26,5,7)$ covering design with a hole of size 6.

| $v$ | Point Set | Base Blocks |
|---|---|---|
| 22 | $Z_{20} \cup H_2$ | $\langle 0\ 4\ 8\ 12\ 16\rangle + i,\ i \in Z_4$, 3 times |
| | | $\langle 0\ 3\ 10\ 13\rangle \cup \{h_1, h_2\}$ half orbit $\langle 0\ 1\ 5\ 11\ h_1\rangle$ |
| | | $\langle 0\ 3\ 10\ 13\ h_2\rangle \langle 0\ 4\ 9\ 15\rangle \cup \{h_1, h_2\} \langle 0\ 1\ 2\ 3\ 8\rangle$ |
| | | $\langle 0\ 2\ 5\ 11\ 13\rangle \langle 0\ 1\ 2\ 3\ 7\rangle \langle 0\ 2\ 6\ 9\ 14\rangle$. |
| 26 | $Z_{20} \cup H_6$ | $\langle 0\ 4\ 8\ 12\ 16\rangle + i;\ i \in Z_4$, twice $\langle 0\ 3\ 10\ 13\rangle \cup \{h_5, h_6\}$ half orbit |
| | | $\langle 0\ 1\ 3\ 14\rangle \cup \{h_i\}_{i=1}^4 \langle 0\ 4\ 9\ 15\rangle \cup \{h_1, h_2\} \langle 0\ 1\ 7\ 10\rangle \cup \{h_3, h_4\}$ |
| | | $\langle 0\ 1\ 3\ 8\rangle \cup \{h_5, h_6\} \langle 0\ 1\ 2\ 3\ h_1\rangle \langle 0\ 1\ 3\ 7\ h_2\rangle \langle 0\ 2\ 7\ 12\ h_3\rangle$ |
| | | $\langle 0\ 2\ 8\ 12\ h_4\rangle \langle 0\ 3\ 8\ 14\ h_5\rangle \langle 0\ 4\ 9\ 13\ h_6\rangle$ |
| 42 | $Z_{36} \cup H_6$ | On $Z_{36} \cup H_5$ construct a $B[41,5,4]$ with a hole of size 5, say, $H_5$ and take the following blocks: |
| | | $\langle 0\ 7\ 18\ 25\rangle \cup \{h_5, h_6\}$ half orbit $\langle 0\ 3\ 13\ 22\rangle \cup \{h_i\}_{i=1}^4$ $\langle 0\ 1\ 5\ 26\rangle \cup \{h_1, h_2\} \langle 0\ 1\ 13\ 20\rangle \cup \{h_3, h_4\} \langle 0\ 2\ 5\ 27\rangle \cup \{h_5, h_6\}$ $\langle 0\ 2\ 8\ 17\ h_6\rangle \langle 0\ 1\ 3\ 24\ 31\rangle \langle 0\ 4\ 8\ 18\ 24\rangle$ |
| 62 | $Z_{56} \cup H_6$ | On $Z_{56} \cup H_5$ construct a $B[61,5,5]$ such that $H_5$ is a block which we delete and take the following blocks: |
| | | $\langle 0\ 14\ 28\ 42\ h_6\rangle + i,\ i \in Z_{14}$, $\langle 0\ 1\ 3\ 10\rangle \cup \{h_1, h_2\}$ $\langle 0\ 5\ 13\ 32\rangle \cup \{h_3, h_4\} \langle 0\ 5\ 17\ 28\rangle \cup \{h_5, h_6\} \langle 0\ 6\ 21\ 37\ h_6\rangle$ |
| | | $\langle 0\ 1\ 3\ 7\ 25\rangle \langle 0\ 4\ 17\ 26\ 40\rangle \langle 0\ 8\ 18\ 29\ 44\rangle$ |
| 82 | $Z_{76} \cup H_6$ | On $Z_{76} \cup H_5$ construct a $B[81,5,5]$ with a hole of size 5, say, $H_5$ and take the following blocks: |
| | | $\langle 0\ 19\ 38\ 57\ h_6\rangle + i,\ i \in Z_{19}$ |
| | | $\langle 0\ 1\ 3\ 7\ 31\rangle \langle 0\ 5\ 16\ 39\ 49\rangle \langle 0\ 8\ 17\ 38\ 58\rangle \langle 0\ 12\ 25\ 40\ 54\rangle$ |
| | | $\langle 0\ 2\ 14\ 20\ 25\rangle \langle 0\ 7\ 15\ 31\ h_6\rangle \langle 0\ 1\ 10\ 37\rangle \cup \{h_1, h_2\}$ |
| | | $\langle 0\ 3\ 22\ 35\rangle \cup \{h_3, h_4\} \langle 0\ 4\ 21\ 47\rangle \cup \{h_5, h_6\}$ |
| 102 | $Z_{96} \cup H_5$ | On $Z_{96} \cup H_5$ construct a $B[101,5,5]$ with a hole of size 5, say, $H_5$ and take the following blocks: |
| | | $\langle 0\ 24\ 48\ 72\ h_6\rangle + i,\ i \in Z_{24} \langle 0\ 2\ 12\ 28\ 42\rangle \langle 0\ 1\ 4\ 9\ 35\rangle$ |
| | | $\langle 0\ 6\ 21\ 48\ 64\rangle \langle 0\ 7\ 20\ 44\ 56\rangle \langle 0\ 11\ 29\ 57\ 74\rangle \langle 0\ 1\ 3\ 7\ 18\rangle$ |
| | | $\langle 0\ 5\ 14\ 37\ 67\rangle \langle 0\ 8\ 33\ 55\ h_6\rangle \langle 0\ 10\ 35\ 55\rangle \cup \{h_1, h_2\}$ |
| | | $\langle 0\ 13\ 36\ 57\rangle \cup \{h_3, h_4\} \langle 0\ 19\ 38\ 65\rangle \cup \{h_5, h_6\}$ |
| 122 | $Z_{116} \cup H_6$ | On $Z_{116} \cup H_5$ construct a $B[112,5,5]$ with a hole of size 5, say, $H_5$ and take the following blocks: |
| | | $\langle 0\ 29\ 58\ 87\ h_6\rangle + i,\ i \in Z_{29} \langle 0\ 4\ 16\ 34\ 58\rangle \langle 0\ 1\ 3\ 8\ 21\rangle$ |
| | | $\langle 0\ 6\ 30\ 53\ 78\rangle \langle 0\ 10\ 36\ 50\ 85\rangle \langle 0\ 11\ 43\ 65\ 82\rangle \langle 0\ 9\ 37\ 64\ 83\rangle$ |
| | | $\langle 0\ 1\ 3\ 7\ 12\rangle \langle 0\ 8\ 33\ 56\ 84\rangle \langle 0\ 14\ 41\ 63\ 80\rangle \langle 0\ 15\ 35\ 61\ h_6\rangle$ |
| | | $\langle 0\ 10\ 31\ 69\rangle \cup \{h_1, h_2\} \langle 0\ 13\ 29\ 72\rangle \cup \{h_3, h_4\} \langle 0\ 15\ 52\ 71\rangle \cup \{h_5, h_6\}$ |

**Lemma 4.8.** *Let $v \equiv 2 \pmod{20}$ be a positive integer greater than 2. Then $\alpha(v,5,7) = \phi(v,5,7)$ with the possible exception of $v = 22$, 142, 162.*

**Proof:** For $v = 42, 62, 82, 102, 122$ see lemma 4.7. For $v \geq 222$, $v \neq 242$, write $v = 20m + 4u + h + s$ where $m, u, h$ and $s$ are chosen as follows:

1. there exists a RMGD$[5, 1, 5, 5m]$;
2. $4u + h + s = 42, 62, 82$;
3. $0 \leq u \leq m - 1$, $s \equiv 0 \pmod{4}$, $h = 2$ or 6;
4. there except a GD$[5, 7, \{4, s^*\}, 4m + s]$.

Now apply theorem 2.6 to get the result. For $v = 182$ apply theorem 2.8 with $m = s = 8$ and $h = 6$. For $v = 202, 242$ take a T$[6, 7, 10]$ and delete $s$ points from the last group where $s = 10$ or 0 respectively. Inflate the resultant design by a factor of 4. Replace the blocks of size 5 by the blocks of GD$[5, 1, 4, 20]$; and the blocks of size 6 by the blocks of GD$[5, 1, 4, 24]$. Finally, add two points to the groups and on the first group construct a $(42, 5, 7)$ minimal covering design, and on each other group construct a $(42, 5, 7)$ covering design with a hole of size 2. Such design can be constructed as follows on $X = Z_{40} \cup \{h_1, h_2\}$.

1. take a $(42, 5, 4)$ covering design with a hole of size 2, say, $\{h_1, h_2\}$, [8].
2. take the following blocks:

$\langle 0\ 8\ 20\ 28\ h_1 \rangle$ half orbit
$\langle 0\ 4\ 10\ 15\ 22 \rangle \pmod{40}$
$\langle 0\ 1\ 2\ 4\ 18 \rangle \pmod{40}$
$\langle 0\ 5\ 10\ 19\ 25 \rangle \pmod{40}$
$\langle 0\ 1\ 3\ 14 \rangle \pmod{40} \cup \{h_1, h_2, h_2, h_2\}$
$\langle 0\ 4\ 12\ 23\ 31 \rangle \pmod{40}$
$\langle 0\ 3\ 9\ 16\ 33 \rangle \pmod{40}$

### 4.5 $v \equiv 6, 10$ or $14 \pmod{20}$

In this section we first show that $\alpha(v,5,3) = \phi(v,5,3)$ for few values of $v \equiv 2 \pmod{4}$.

**Lemma 4.9.** *Let $v \equiv 6, 10, 14$ or $18 \pmod{20}$ be a positive integer less than 100, $v \neq 18, 26$. Then $\alpha(v,5,3) = \phi(v,5,3)$.*

**Proof:** For $v = 10$ let $X = \{1, \ldots, 10\}$ then the required blocks are $\langle 1\ 2\ 3\ 8\ 9 \rangle$ $\langle 2\ 3\ 4\ 8\ 10 \rangle$ $\langle 1\ 2\ 4\ 5\ 6 \rangle$ $\langle 1\ 2\ 5\ 9\ 10 \rangle$ $\langle 1\ 2\ 6\ 7\ 8 \rangle$ $\langle 1\ 3\ 4\ 7\ 9 \rangle$ $\langle 1\ 3\ 5\ 6\ 10 \rangle$ $\langle 1\ 4\ 7\ 8\ 10 \rangle$ $\langle 2\ 3\ 4\ 5\ 7 \rangle$ $\langle 2\ 6\ 7\ 9\ 10 \rangle$ $\langle 3\ 4\ 6\ 9\ 10 \rangle$ $\langle 3\ 5\ 6\ 7\ 8 \rangle$ $\langle 4\ 5\ 6\ 8\ 9 \rangle$ $\langle 5\ 7\ 8\ 9\ 10 \rangle$.

For $v = 50, 70$ take a T[5, 3, 10] and a T[5, 3, 14] [16] and replace their groups by the blocks of a $(v, 5, 3)$ minimal covering design where $v = 10, 14$ respectively.

For $v = 90$ take a GD[5, 1, 5, 45] [11] and inflate it by a factor of two, that is, replace each of its blocks by the blocks of a GD[5, 3, 2, 10], [16] and on the groups construct a (10, 5, 3) minimal covering design.

For all other values see the following table. In this table, in case the point set is $Z_2 \times Z_n$ then a block of the form $\langle(0,0)\ (0,k)\ (1,r)\ (1,s)\ (1,t)\rangle$ means the block $\langle(0,0)\ (0,k)\ (1,r)\ (1,s)\ (1,t)\rangle$ (mod $-, n$).

| $v$ | Point Set | Base Blocks |
|---|---|---|
| 6 | $Z_5 \cup H_1$ | $\langle 0\ 1\ 2\ 3\ h_1\rangle$ |
| 14 | $Z_2 \times Z_7$ | $\langle(0,0)\ (0,1)\ (0,2)\ (0,5)\ (1,3)\rangle\langle(0,0)\ (0,1)\ (1,0)\ (1,1)\ (1,3)\rangle$ |
| | | $\langle(0,0)\ (0,2)\ (1,1)\ (1,5)\ (1,6)\rangle\langle(0,0)\ (0,3)\ (1,0)\ (1,4)\ (1,5)\rangle$ |
| 30 | $Z_2 \times Z_{15}$ | $\langle(\alpha,0)\ (\alpha,3)\ (\alpha,6)\ (\alpha,9)\ (\alpha,12)\rangle + (-,i) i \in Z_3$, twice, $\alpha = 0, 1$ |
| | | $\langle(0,0)\ (0,1)\ (0,5)\ (0,7)\ (1,9)\rangle\langle(0,0)\ (0,1)\ (0,5)\ (1,6)\ (1,11)\rangle$ |
| | | $\langle(0,0)\ (0,2)\ (0,10)\ (1,3)\ (1,14)\rangle\langle(0,0)\ (0,1)\ (1,0)\ (1,1)\ (1,3)\rangle$ |
| | | $\langle(0,0)\ (0,3)\ (1,0)\ (1,2)\ (1,10)\rangle\langle(0,0)\ (0,2)\ (1,6)\ (1,10)\ (1,11)\rangle$ |
| | | $\langle(0,0)\ (0,4)\ (1,7)\ (1,11)\ (1,13)\rangle\langle(0,0)\ (0,7)\ (1,5)\ (1,12)\ (1,13)\rangle$ |
| 34 | $Z_{34}$ | $\langle 0\ 3\ 4\ 11\ 17\rangle\langle 0\ 10\ 12\ 15\ 26\rangle\langle 0\ 1\ 2\ 6\ 12\rangle\langle 0\ 2\ 9\ 18\ 21\rangle\langle 0\ 4\ 9\ 17\ 24\rangle$ |
| 38 | $Z_{32} \cup H_6$ | $\langle 0\ 1\ 2\ 4\ 9\rangle\langle 0\ 2\ 6\ 14\ 22\rangle\langle 0\ 3\ 7\ 18\rangle \cup \{h_5, h_6\}$ |
| | | $\langle 0\ 3\ 13\ 22\rangle \cup \{h_3, h_4\}\langle 0\ 5\ 11\ 20\rangle \cup \{h_1, h_2\}\langle 0\ 5\ 11\ 18\rangle \cup \{h_i\}_{i=1}^4$ |
| | | $\langle 0\ 1\ 16\ 17\rangle \cup \{h_5, h_6\}$ half orbit |
| 46 | $Z_{40} \cup H_6$ | $\langle 0\ 8\ 16\ 24\ 32\rangle + i,\ i \in Z_8$ |
| | | $\langle 0\ 1\ 2\ 4\ 8\rangle\langle 0\ 4\ 11\ 20\ 26\rangle\langle 0\ 10\ 12\ 19\ 24\rangle\langle 0\ 3\ 14\ 25\rangle \cup \{h_5, h_6\}$ |
| | | $\langle 0\ 5\ 15\ 28\rangle \cup \{h_3, h_4\}\langle 0\ 8\ 17\ 27\rangle \cup \{h_1, h_2\}$ |
| | | $\langle 0\ 3\ 20\ 23\rangle \cup \{h_5, h_6\}$ half orbit $\langle 0\ 1\ 6\ 19\rangle\{h_i\}_{i=1}^4$ |
| 54 | $Z_2 \times Z_{27}$ | $\langle(0,0)\ (0,3)\ (0,4)\ (0,9)\ (0,16)\rangle\langle(0,0)\ (0,2)\ (0,10)\ (0,14)\ (1,0)\rangle$ |
| | | $\langle(0,0)\ (0,1)\ (1,2)\ (1,3)\ (1,5)\rangle\langle(0,0)\ (0,11)\ (1,5)\ (1,9)\ (1,17)\rangle$ |
| | | $\langle(0,0)\ (1,7)\ (1,13)\ (1,16)\ (1,24)\rangle\langle(0,0)\ (1,12)\ (1,19)\ (1,21)\ (1,26)\rangle$ |
| | | $\langle(0,0)\ (0,2)\ (0,11)\ (1,10)\ (1,22)\rangle\langle(0,0)\ (0,4)\ (1,12)\ (1,18)\ (1,23)\rangle$ |
| | | $\langle(0,0)\ (0,1)\ (0,7)\ (1,0)\ (1,1)\rangle\langle(0,0)\ (0,3)\ (0,12)\ (1,1)\ (1,14)\rangle$ |
| | | $\langle(0,0)\ (0,5)\ (0,13)\ (1,10)\ (1,23)\rangle\langle(0,0)\ (0,2)\ (0,10)\ (1,14)\ (1,18)\rangle$ |
| | | $\langle(0,0)\ (0,3)\ (0,10)\ (1,6)\ (1,7)\rangle\langle(0,0)\ (0,5)\ (1,11)\ (1,20)\ (1,22)\rangle$ |
| | | $\langle(0,0)\ (0,6)\ (1,9)\ (1,14)\ (1,22)\rangle\langle(0,0)\ (1,7)\ (1,12)\ (1,15)\ (1,19)\rangle$ |
| 58 | $Z_2 \times Z_{26} \cup H_6$ | $\langle(1,0)\ (1,2)\ (1,9)\ (1,14)\ (1,22)\rangle\langle(1,0)\ (1,2)\ (1,12)\ (1,13)\ (1,22)\rangle$ |
| | | $\langle(0,0)\ (0,4)\ (0,16)\ (1,18)\ (1,25)\rangle\langle(0,0)\ (0,2)\ (0,8)\ (1,8)\ (1,18)\rangle$ |
| | | $\langle(0,0)\ (0,5)\ (0,13)\ (1,7)\ (1,8)\rangle\langle(0,0)\ (0,2)\ (1,12)\ (1,17)\ (1,20)\rangle$ |
| | | $\langle(0,0)\ (0,21)\ (1,11)\ (1,19)\ (1,22)\rangle\langle(0,0)\ (0,7)\ (0,16)\ (0,20)\ (1,5)\rangle$ |
| | | $\langle(0,0)\ (0,3)\ (0,11)\ (0,12)\ (1,25)\rangle\langle(0,0)\ (0,1)\ (0,7)\ (0,11)\ (1,4)\rangle$ |
| | | $\langle(0,0)\ (0,1)\ (1,23)\ (1,25)\ h_1\rangle\langle(0,0)\ (0,2)\ (1,19)\ (1,23)\ h_2\rangle$ |
| | | $\langle(0,0)\ (0,3)\ (1,8)\ (1,17)\ h_3\rangle\langle(0,0)\ (0,3)\ (1,3)\ (1,9)\ h_4\rangle$ |
| | | $\langle(0,0)\ (0,5)\ (1,11)\ (1,12)\ h_5\rangle\langle(0,0)\ (0,12)\ (1,1)\ (1,16)\ h_6\rangle$ |
| | | $\langle(0,0)\ (0,7)\ (1,7)\ (1,12)\rangle \cup \{h_1, h_2\}$ |
| | | $\langle(0,0)\ (0,9)\ (1,10)\ (1,13)\rangle \cup \{h_3, h_4\}$ |
| | | $\langle(0,0\ (0,11)\ (1,13)\ (1,20)\rangle \cup \{h_5, h_6\}$ |

| $v$ | Point Set | Base Blocks |
|---|---|---|
| 66 | $Z_2 \times Z_{30} \cup H_6$ | $\langle (0,0)\ (0,6)\ (0,12)\ (0,18)\ (0,24) \rangle + (-,i),\ i \in Z_6$ |
| | | $\langle (1,0)\ (1,6)\ (1,12)\ (1,18)\ (1,24) \rangle + (-,i),\ i \in Z_6$ |
| | | $\langle (0,0)\ (0,2)\ (0,6)\ (0,14)\ (0,21) \rangle \langle (0,0)\ (0,8)\ (1,20)\ (1,22)\ (1,27) \rangle$ |
| | | $\langle (0,0)\ (0,1)\ (1,1)\ (1,10)\ (1,18) \rangle \langle (0,0)\ (0,3)\ (1,13)\ (1,26)\ (1,28) \rangle$ |
| | | $\langle (0,0)\ (0,11)\ (1,2)\ (1,5)\ (1,6) \rangle \langle (0,0)\ (0,8)\ (1,7)\ (1,17)\ (1,29) \rangle$ |
| | | $\langle (0,0)\ (0,10)\ (0,13)\ (0,14)\ (1,18) \rangle \langle (0,0)\ (0,10)\ (0,15)\ (0,24)\ (1,8) \rangle$ |
| | | $\langle (0,0)\ (0,10)\ (0,12)\ (0,17)\ (1,28) \rangle \langle (0,0)\ (1,13)\ (1,15)\ (1,23)\ (1,29) \rangle$ |
| | | $\langle (0,0)\ (1,0)\ (1,3)\ (1,4)\ (1,15) \rangle \langle (0,0)\ (1,7)\ (1,12)\ (1,16)\ (1,26) \rangle$ |
| | | $\langle (0,0)\ (0,1)\ (1,22)\ (1,27) \rangle \cup \{h_1, h_2\} \langle (0,0)\ (0,2)\ (1,24)\ (1,27)\ h_1 \rangle$ |
| | | $\langle (0,0)\ (0,3)\ (1,4)\ (1,11) \rangle \cup \{h_3, h_4\} \langle (0,0)\ (0,4)\ (1,11)\ (1,20)\ h_2 \rangle$ |
| | | $\langle (0,0)\ (0,5)\ (1,5)\ (1,6) \rangle \cup \{h_5, h_6\} \langle (0,0)\ (0,7)\ (1,13)\ (1,24)\ h_3 \rangle$ |
| | | $\langle (0,0)\ (0,9)\ (1,12)\ (1,19)\ h_4 \rangle \langle (0,0)\ (0,11)\ (1,14)\ (1,20)\ h_5 \rangle$ |
| | | $\langle (0,0)\ (0,13)\ (1,2)\ (1,15)\ h_6 \rangle$ |
| 74 | $Z_{64} \times H_{10}$ | $\langle 0\ 3\ 32\ 35 \rangle \cup \{h_9, h_{10}\}$ half orbit |
| | | $\langle 0\ 9\ 11\ 50 \rangle \cup \{h_i\}_{i=1}^{4} \langle 0\ 5\ 10\ 31 \rangle \cup \{h_i\}_{i=5}^{8}$ |
| | | $\langle 0\ 13\ 28\ 47 \rangle \cup \{h_1, h_2\} \langle 0\ 11\ 27\ 46 \rangle \cup \{h_3, h_4\}$ |
| | | $\langle 0\ 9\ 22\ 33 \rangle \cup \{h_5, h_6\} \langle 0\ 7\ 30\ 39 \rangle \cup \{h_7, h_8\}$ |
| | | $\langle 0\ 17\ 18\ 21 \rangle \cup \{h_9, h_{10}\} \langle 0\ 2\ 24\ 31\ 47 \rangle$ |
| | | $\langle 0\ 1\ 22\ 28\ 38 \rangle \langle 0\ 5\ 13\ 25\ 49 \rangle \langle 0\ 1\ 3\ 7\ 15 \rangle \langle 0\ 4\ 10\ 18\ 30 \rangle$ |
| 78 | $Z_{72} \cup H_6$ | $\langle k\ k+6\ k+36\ k+42\ f(k) \rangle$ where $k = 0, \ldots, 35$ and $f(k) = h_5$<br>if $k \equiv 0$ or $1 \pmod 4$ otherwise $f(k) = h_6$ |
| | | $\langle 0\ 3\ 11\ 23\ 37 \rangle \langle 0\ 4\ 13\ 29\ 31 \rangle \langle 0\ 10\ 15\ 22\ 39 \rangle \langle 0\ 1\ 3\ 9\ 24 \rangle$ |
| | | $\langle 0\ 5\ 16\ 36\ 50 \rangle \langle 0\ 7\ 19\ 37\ 47 \rangle \langle 0\ 1\ 5\ 15\ 45 \rangle \langle 0\ 2\ 8\ 21\ 46 \rangle$ |
| | | $\langle 0\ 7\ 16\ 39 \rangle \cup \{h_1, h_2\} \langle 0\ 4\ 17\ 43 \rangle \cup \{h_3, h_4\}$ |
| | | $\langle 0\ 11\ 31\ 48 \rangle \cup \{h_5, h_6\} \langle 0\ 1\ 19\ 22 \rangle \cup \{h_i\}_{i=1}^{4}$ |
| 86 | $Z_{80} \cup H_6$ | Take a block $(80, 5, 1)$ covering design on $Z_{80}$, [18], and take the<br>following blocks: |
| | | $\langle 0\ 16\ 32\ 48\ 64 \rangle + i,\ i \in Z_{16} \langle k\ k+12\ k+40\ k+52\ f(k) \rangle$<br>where $k = 0, \ldots, 39$<br>and $f(k) = h_5$ if $k \equiv 0, 1, 2$ or $3 \pmod 8$ otherwise $f(k) = h_6$ |
| | | $\langle 0\ 1\ 3\ 7\ 26 \rangle \langle 0\ 1\ 3\ 7\ 21 \rangle \langle 0\ 8\ 17\ 29\ 47 \rangle \langle 0\ 8\ 24\ 34\ 63 \rangle$ |
| | | $\langle 0\ 11\ 24\ 46\ 60 \rangle \langle 0\ 5\ 15\ 42 \rangle \cup \{h_i\}_{i=1}^{4} \langle 0\ 5\ 33\ 48 \rangle \cup \{h_1, h_2\}$ |
| | | $\langle 0\ 9\ 31\ 44 \rangle \cup \{h_3, h_4\} \langle 0\ 11\ 30\ 53 \rangle \cup \{h_5, h_6\}$ |
| 94 | $Z_{80} \cup H_{14}$ | $\langle 0\ 16\ 32\ 48\ 64 \rangle + i,\ i \in Z_{16}$ 3 times $\langle 0\ 1\ 20\ 24\ 38 \rangle \langle 0\ 2\ 8\ 29\ 36 \rangle$ |
| | | $\langle 0\ 1\ 3\ 7\ 27 \rangle \langle 0\ 8\ 23\ 42\ 52 \rangle \langle 0\ 2\ 10\ 36\ 49 \rangle \langle 0\ 11\ 33\ 50 \rangle \cup \{h_i\}_{i=1}^{4}$ |
| | | $\langle 0\ 5\ 14\ 35 \rangle \cup \{h_i\}_{i=5}^{8} \langle 0\ 11\ 33\ 50 \rangle \cup \{h_i\}_{i=9}^{12} \langle 0\ 3\ 13\ 68 \rangle \cup \{h_1, h_2\}$ |
| | | $\langle 0\ 5\ 31\ 40 \rangle \cup \{h_3, h_4\} \langle 0\ 12\ 25\ 43 \rangle \cup \{h_5, h_6\} \langle 0\ 3\ 7\ 66 \rangle \cup \{h_7, h_8\}$ |
| | | $\langle 0\ 15\ 35\ 58 \rangle \cup \{h_9, h_{10}\} \langle 0\ 11\ 29\ 38 \rangle \cup \{h_{11}, h_{12}\} \langle 0\ 1\ 6\ 25 \rangle \cup \{h_{13}, h_{14}\}$ |
| | | $\langle k\ k+12\ k+40\ k+52\ f(k) \rangle$ where $k = 0, \ldots, 39$<br>and $f(k) = h_{13}$ if $k \equiv 0, 1, 2$ or $3 \pmod 8$ otherwise $f(k) = h_{14}$ |
| 98 | $Z_2 \times Z_{46} \cup H_6$ | $\langle (0,0)\ (0,1)\ (0,3)\ (0,14)\ (0,23) \rangle \langle (0,0)\ (0,5)\ (0,12)\ (0,20)\ (0,30) \rangle$ |
| | | $\langle (1,0)\ (1,1)\ (1,6)\ (1,10)\ (1,30) \rangle \langle (1,0)\ (1,2)\ (1,14)\ (1,25)\ (1,33) \rangle$ |
| | | $\langle (0,0)\ (0,4)\ (0,21)\ (0,27)\ (1,2) \rangle \langle (0,0)\ (1,0)\ (1,1)\ (1,3)\ (1,7) \rangle$ |
| | | $\langle (0,0)\ (1,0)\ (1,10)\ (1,28)\ (1,30) \rangle \langle (0,0)\ (0,1)\ (0,3)\ (1,9)\ (1,17) \rangle$ |
| | | $\langle (0,0)\ (0,4)\ (0,8)\ (1,28)\ (1,45) \rangle \langle (0,0)\ (0,5)\ (0,11)\ (1,4)\ (1,37) \rangle$ |
| | | $\langle (0,0)\ (0,6)\ (0,19)\ (1,29)\ (1,44) \rangle \langle (0,0)\ (0,2)\ (1,15)\ (1,38)\ (1,43) \rangle$ |
| | | $\langle (0,0)\ (0,7)\ (0,24)\ (1,12)\ (1,42) \rangle \langle (0,0)\ (0,8)\ (1,19)\ (1,30)\ (1,39) \rangle$ |
| | | $\langle (0,0)\ (0,9)\ (0,24)\ (1,0)\ (1,3) \rangle \langle (0,0)\ (0,10)\ (1,2)\ (1,6)\ (1,14) \rangle$ |
| | | $\langle (0,0)\ (0,10)\ (0,26)\ (1,33)\ (1,44) \rangle \langle (0,0)\ (0,12)\ (1,20)\ (1,33)\ (1,39) \rangle$ |
| | | $\langle (0,0)\ (0,12)\ (0,28)\ (1,17)\ (1,41) \rangle \langle (0,0)\ (0,19)\ (1,9)\ (1,16)\ (1,30) \rangle$ |
| | | $\langle (0,0)\ (0,14)\ (0,28)\ (1,15)\ (1,40) \rangle \langle (0,0)\ (0,21)\ (1,31)\ (1,40)\ (1,45) \rangle$ |
| | | $\langle (0,0)\ (0,5)\ (1,13)\ (1,25)\ h_1 \rangle \langle (0,0)\ (0,7)\ (1,21)\ (1,43)\ h_2 \rangle$ |
| | | $\langle (0,0)\ (0,11)\ (1,7)\ (1,26)\ h_3 \rangle \langle (0,0)\ (0,13)\ (1,19)\ (1,29)\ h_4 \rangle$ |
| | | $\langle (0,0)\ (0,15)\ (1,3)\ (1,32)\ h_5 \rangle \langle (0,0)\ (0,17)\ (1,28)\ (1,35)\ h_6 \rangle$ |
| | | $\langle (0,0)\ (0,1)\ (1,2)\ (1,5) \cup \{h_1, h_2\}$<br>$\langle (0,0)\ (0,3)\ (1,12)\ (1,27) \rangle \cup \{h_3, h_4\}$ |
| | | $\langle (0,0)\ (0,9)\ (1,31)\ (1,32) \rangle \cup \{h_5, h_6\}$ |

**Lemma 4.10.** *Let $v \equiv 6, 10$ or $14 \pmod{20}$, $v < 100$ be a positive integer. Then $\alpha(v, 5, 7) = \phi(v, 5, 7)$.*

**Proof:** For $v \equiv 6, 10$ or $14 \pmod{20}$ $v < 100$, $v \neq 26$ the blocks of a $(v, 5, 7)$ minimal covering design are the blocks of a $(v, 5, 4)$ [10] and $(v, 5, 3)$ minimal covering design. For a $(26, 5, 7)$ minimal covering design see lemma 4.7.

**Lemma 4.11.** *Let $v \equiv 6, 10$ or $14 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 7) = \phi(v, 5, 7)$.*

**Proof:** For $v \equiv 6, 10$ or $14 \pmod{20}$, $v < 100$ the result follows from the previous lemma. For $v > 100$, $v \neq 126$, 130, 134, 146 simple calculation shows that $v$ can be written in the form $v = 20m + 4u + h + s$ where $m, u, h$ and $s$ are chosen so that

1. there exist a RMGD$[5, 1, 5, 5m]$;
2. $0 \leq u \leq m - 1$, $s \equiv 0 \pmod{4}$ and $h = 2$ or 6;
3. $4u + h + s \equiv 6, 10$ or $14 \pmod{20}$ and $6 \leq 4u + h + s \leq 90$;
4. there exists a GD$[5, 7, \{4, s^*\}, 4m + s]$.

Now apply theorem 2.6 to get the result.

For $v = 126$, 130, 134 apply theorem 2.3 with $n = 7$, $h = 6$ and $u = 0, 1, 2$ respectively.

For $v = 146$ apply theorem 2.4 with $n = 7$ and $h = 6$.

### 4.6 $v \equiv 18 \pmod{20}$

**Lemma 4.12.** *Let $v \equiv 18 \pmod{20}$ be a positive integer and assume*

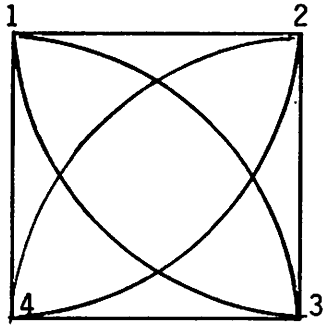
1. *there exists a $(v, 5, 3)$ covering design with a hole of size 6.*
2. *there exists a $(v, 5, 4)$ minimal covering design such that its excess graph consists of $v - 4$ isolated vertices and the following graph on the remaining 4 vertices, say, $\{1, 2, 3, 4\}$.*

*Then there exists a $(v, 5, 7)$ minimal covering design.*

**Proof:** Let $v \equiv 18 \pmod{20}$ be a positive integer such that conditions (1) and (2) hold. Then a $(v, 5, 7)$ minimal covering design can be constructed as follows:

1. take a $(v, 5, 3)$ covering design with a hole of size 6. On the hole construct a $(6, 5, 3)$ minimal covering design as follow: $X = Z_5 \cup \{h\}$ and blocks $\langle 0\ 1\ 2\ 3\ h\rangle \pmod{5}$. It is clear that the excess graph of

the $(v, 5, 3)$ minimal covering design contains the following subgraph. From this design delete the block $\langle 1\ 2\ 3\ 4\ h \rangle$.




2. take a $(v, 5, 4)$ minimal covering design on $Z_{v-1} \cup \{h\}$ such that its excess graph is the same as described in (2). It is easy to check that the above two steps yield the blocks of a $(v, 5, 7)$ minimal covering design.

**Lemma 4.13.**

1. *There exists a* $(v, 5, 3)$ *covering design with a hole of size* 6 *for* $v = 38$, 58, 78, 98.

2. *There exists a* $(v, 5, 4)$ *minimal covering design for* $v = 38$, 58, 78, 98 *such that its excees graph is the same as in condition (2) of the previous lemma.*

**Proof:**

1. For a $(v, 5, 3)$, $v = 38$, 58, 78, 98, covering design with a hole of size 6 see lemma 4.9.

2. To show that there exists a $(v, 5, 4)$, $v = 38$, 58, 78, 98, minimal covering design such that its excess graph satisfies the constraint of the lemma we show that for $v = 38$, 58, 78, 98, there exists a $(v, 5, 4)$ covering design with a hole of size 8 or 13. But the excess graph of both $(8, 5, 4)$ and $(13, 5, 4)$ minimal covering designs satisfy this constraint, [10]. Hence, the excess graph of a $(v, 5, 4)$ minimal covering design, for $v = 38$, 58, 78, 98, satisfies this constraint.

For $v = 38$, 58 see [10], and for $v = 78$, 98 see [7].

**Corollary.** $\alpha(v, 5, 7) = \phi(v, 5, 7)$ *for* $v = 38, 58, 78, 98$.

**Proof:** The result follows from lemma 4.12 and 4.13.

**Lemma 4.14.** *Let $v \equiv 18 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 7) = \phi(v, 5, 7)$.*

**Proof:** For $v = 18$ let $X = Z_{18}$ then the blocks are

$$\langle 0\ 1\ 3\ 11\ 15\rangle \pmod{18} \qquad \langle 0\ 1\ 2\ 3\ 7\rangle \pmod{18}$$
$$\langle 0\ 1\ 5\ 9\ 13\rangle \pmod{18} \qquad \langle 0\ 2\ 5\ 10\ 17\rangle \pmod{18}$$
$$\langle 0\ 1\ 3\ 9\ 12\rangle \pmod{18} \qquad \langle 0\ 2\ 6\ 11\ 13\rangle \pmod{18}.$$

For $v = 38, 58, 78, 98$ the result follows from the corollary.

For $v \geq 118$, $v \neq 138$ write $v = 20m + 4u + h + s$ where $m, u, h$ and $s$ are chosen so that

1. there except a RMGD$[5, 1, 5, 5m]$;
2. $4u + h + s \equiv 18 \pmod{20}$, $18 \leq 4u + h + s \leq 98$;
3. $0 \leq u \leq m - 1$, $s \equiv 0 \pmod 4$ and $h = 6$;
4. there exists a GD$[5, 7, \{4, s^*\}, 4m + s]$.

Now apply theorem 2.6 to get the result.

For $v = 138$ apply theorem 2.3 with $n = 7$, $u = 3$ and $h = 6$.

## 5. Conclusion

We have shown that $\alpha(v, 5, 7) = \phi(v, 5, 7)$ for all positive integers $v \geq 5$ with the possible exception of $v = 22, 28, 142, 162$.

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