Relating Pairs of Distance Domination Parameters

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ABSTRACT. Let $n \geq 1$ be an integer and let G be a graph of order p. A set $\mathcal{D}$ of vertices of G is a n-dominating set (total n-dominating) set of G if every vertex of $V(G) - \mathcal{D}$ ($V(G)$, respectively) is within distance n from some vertex of $\mathcal{D}$ other than itself. The minimum cardinality among all n-dominating sets (respectively, total n-dominating sets) of G is called the n-domination number (respectively, total n-domination number) and is denoted by $\gamma_n(G)$ (respectively, $\gamma_n^t(G)$). A set $\mathcal{I}$ of vertices of G is n-independent if the distance (in G) between every pair of distinct vertices of $\mathcal{I}$ is at least $n+1$. The minimum cardinality among all maximal n-independent sets of G is called the n-independence number of G and is denoted by $i_n(G)$. Suppose $\mathcal{I}_k$ is a n-independent sets of k vertices of G for which there exists a vertex v of G that is within distance n from every vertex of $\mathcal{I}_k$. Then a connected subgraph of minimum size that contains the vertices of $\mathcal{I}_k \cup \{v\}$ is called a n-generalized $K_{1,k}$ in G. It is shown that if G contains no n-generalized $K_{1,3}$, then $\gamma_n(G) = i_n(G)$. Further, it is shown if G contains no n-generalized $K_{1,k+1}$, $k \geq 2$, then $i_n(G) \leq (k-1)\gamma_n(G) - (k-2)$. It is shown that if G is a connected graph with at least $n+1$ vertices, then there exists a minimum n-dominating set $\mathcal{D}$ of G such that for each $d \in \mathcal{D}$, there exists a vertex $v \in V(G) - \mathcal{D}$ at distance n from d and distance at least $n+1$ from every vertex of $\mathcal{D} - \{d\}$. Using this result, it is shown if G is a connected graph on $p \geq 2n+1$ vertices, then $\gamma_n(G) \leq p/(n+1)$ and that $i_n(G) + n\gamma_n(G) \leq p$. Finally, it is shown that if T is a tree on $p \geq 2n+1$ vertices, then $i_n(G) + n\gamma_n^t(G) \leq p$.

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1. Introduction

For graph theory terminology not presented here we follow [12]. Specifically, $p(G)$ and $q(G)$ will denote, respectively, the number of vertices (also called the order) and number of edges (also called the size) of a graph G with vertex set $V(G)$ and edge set $E(G)$. For a connected graph G, the distance $d(u, v)$ between two vertices u and v is the length of a shortest u-v path. The eccentricity $e_G(v)$ of a vertex v of G is defined as $\max_{u \in V(G)} d(u, v)$. The radius rad G of G is $\min_{v \in V(G)} e(v)$, while the diameter *diam* G of G is $\max_{v \in V(G)} e(v)$. If S is a set of vertices of G and v is a vertex of G, then the *distance from v to S*, denoted by $d_G(v, S)$, is the shortest distance from v to a vertex of S. The nth power G^n of a connected graph G, where $n \geq 1$, is that graph with $V(G^n) = V(G)$ for which $uv \in E(G^n)$ if and only if $1 \leq d_G(u, v) \leq n$.

Let v be a vertex of a graph G. The degree of v in G, written as deg v, is the number of edges incident with v. Equivalently, the degree of v is the number of vertices different from v that are at distance at most 1 from v in G. This observation suggests a generalization of the degree of a vertex. In [14], for n a positive integer, the set of all vertices of G different from v and at distance at most n from v in G is defined as the *n-neighbourhood* of v in G and is denoted by $N_n(v)$. If $u \in N_n(v)$, then we say that u and v are *n-adjacent* vertices. The *n-degree*, $\deg_n v$, of v in G is given by $|N_n(v)|$. Hence $N_1(v) = N(v)$ and $\deg_1 v = \deg v$.

This definition of the n-degree of a vertex suggests a generalization of the domination, total domination and independent domination numbers of a graph. Let $n \geq 1$ be an integer and let G be a graph. In [19], a set $\mathcal{D}$ of vertices of G is defined to be an *n-dominating set (total n-dominating set)* of G if every vertex in $V(G) - \mathcal{D}$ ($V(G)$, respectively) is within distance n from some vertex of $\mathcal{D}$ other than itself. The minimum cardinality among all n-dominating sets (total n-dominating sets) of G is called the *n-domination number* (*total n-domination number*) of G and is denoted by $\gamma_n(G)$ ($\gamma_n^t(G)$). We note that the parameter $\gamma_n^t(G)$ is defined only for graphs with no isolated vertex. Observe that $\gamma(G) = \gamma_1(G)$ and $\gamma_t(G) = \gamma_1^t(G)$. For the graph shown in Figure 1, $D = \{u, w, y\}$ is a 2-dominating set of G with $\gamma_2(G) = |D|$, while $T = \{u, v, x, y\}$ is a total 2-dominating set with $\gamma_2^t(G) = |T|$.

Another domination parameter that has received considerable attention in the literature is the independent domination number. A set $\mathcal{I}$ of vertices of a graph G is defined to be *n-independent* in G if every vertex of $\mathcal{I}$ is at distance at least $n+1$ from every other vertex of I in G. Furthermore, I is defined to be an *n-independent dominating set* of G if $\mathcal{I}$ is n-independent and n-dominating in G. The *n-independent domination number* $i_n(G)$ of G is the minimum cardinality among all n-independent dominating sets of

G. Hence 1-independent dominating sets of G are independent dominating sets of G and $i_1(G) = i(G)$. For the graph G of Figure 1, $\mathcal{I} = \{v, x\}$ is a 3-independent dominating set of G with $|\mathcal{I}| = i_3(G)$.

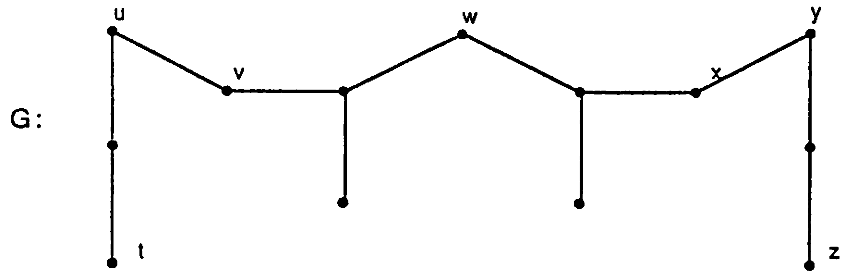

Figure 1. The graph G.

These concepts of distance domination in graphs find applications in many situations and structures which give rise to graphs. Consider, for instance, the following illustration. Let G be the graph associated with the road grid of a city where the vertices of G correspond to the street intersections and where two vertices are adjacent if and only if the corresponding street intersections are a block apart. A minimum n-dominating set in G may be used to locate a minimum number of facilities (such as utilities, police stations, waste disposal dumps, hospitals, blood banks, transmission towers) such that every intersection is within n city blocks of a facility. For practical reasons it may be desirable that each facility be sited within n blocks of some other facility (for instance to cope with emergencies and breakdowns), in which case the use of a total n-dominating set of minimum cardinality is indicated. To avoid interfernce and contamination, it may also be required that no two facilities be within n blocks of each other, and facilities should then be sited at points corresponding to vertices in a minimum n-independent dominating set. Corresponding applications to the design of computer nertworks and defence systems exist. For more applications see [13].

Results on the concept of n-domination in graphs have been presented by, among others, Bascó and Tuza [3, 4], Beineke and Henning [5], Bondy and Fan [7] , Chang [8], Chang and Nemhauser [9, 10, 11], Fraisse [14], Fricke, Hedetniemi, and Henning [15, 16], Hattingh and Henning [17, 18], Henning, Oellermann, and Swart [19, 20, 21, 22], Meir and Moon [23], Mo and Williams [24], Slater [25], Topp and Volkmann [26], and Xin He and Yesha [27].

2. The distance domination number γ_n.

We begin by stating a useful observation, the proof of which is immediate.

Proposition 1 *If G is a connected graph, then $\gamma_n(G) = \gamma(G^n)$, $\gamma_n^t(G) = \gamma_t(G^n)$, and $i_n(G) = i(G^n)$.*

Bollobás and Cockayne [6] established the following result.

Theorem *A If G is a connected nontrivial graph, then there exists a minimum dominating set $\mathcal{D}$ of G such that for each $d \in \mathcal{D}$, there exists a vertex $v \in V(G) - \mathcal{D}$ such that $N(v) \cap D = \{d\}$.*

An immediate consequence of Theorem A and Proposition 1 is that if G is a connected nontrivial graph, then there exists a minimum n-dominating set $\mathcal{D}$ of G such that for each $d \in \mathcal{D}$, there exists a vertex $v \in V(G) - \mathcal{D}$ such that $N_n(v) \cap D = \{d\}$. We prove the following stronger result.

Theorem 1 *For $n \geq 1$, if G is a connected graph of order at least $n+1$, then there exists a minimum n-dominating set $\mathcal{D}$ of G such that for each $d \in \mathcal{D}$, there exists a vertex $v \in V(G) - \mathcal{D}$ at distance exactly n from d such that $N_n(v) \cap D = \{d\}$.*

In order to prove this result, we first state a useful known result from [19].

Lemma *A For $n \geq 1$, let $\mathcal{D}$ be an n-dominating set of a graph G. Then $\mathcal{D}$ is a minimal n-dominating set of G if and only if each $d \in \mathcal{D}$ has at least one of the following two properties:*

- P_1: *There exists a vertex $v \in V(G) - \mathcal{D}$ such that $N_n(v) \cap \mathcal{D} = \{d\}$;*
- P_2: *The vertex d is at distance at least $n+1$ from every other vertex of D in G.*

Before proceeding further, we introduce some notation. Let S be a set of vertices of a connected graph G. We will call a nondecreasing sequence $\ell_1, \ell_2, \ldots, \ell_{|S|}$ of integers the *distance sequence* of S in G if the vertices of S can be labelled $v_1, v_2, \ldots, v_{|S|}$ so that $\ell_i = d_G(v_i, S - \{v_i\})$ for all i. For example, for the graph G given in Figure 1, the set $\{u, w, y\}$ has distance sequence $3, 3, 3$ in G, while the distance sequence of the set $\{t, w, z\}$ in G is $5, 5, 5$. (Observe that both $\{u, w, y\}$ and $\{t, w, z\}$ are 2-dominating sets of G.) As a further example, let G be obtained from a connected graph H by attaching a path of length n to each vertex of H. (The graph G is shown in Figure 2.) Then the distance sequence of $V(H)$ in G is the sequence $1, 1, \ldots, 1$ of length $p(H)$. (Observe that $V(H)$ is a minimum n-dominating set of G.)

Suppose $s_1 : a_1, a_2, \ldots, a_m$ and $s_2 : b_1, b_2, \ldots, b_n$ are two nondecreasing sequences of positive integers. Then we say that s_1 *precedes* s_2 in *dictionary* *order* if either $m \leq n$ and $a_i = b_i$ for $1 \leq i \leq m$ or if there exists an i ($1 \leq i \leq min\{m, n\}$) such that $a_i < b_i$ and $a_j = b_j$ for $j < i$.

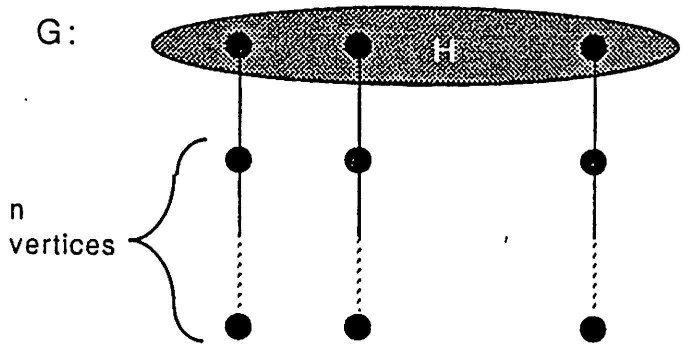

Figure 2. The graph H.

We are now in a position to present a proof of Theorem 1.

Proof of Theorem 1.

Among all minimum n-dominating sets of vertices of G, let $\mathcal{D}$ be one that has the smallest distance sequence in dictionary order. Let the distance sequence of $\mathcal{D}$ be given by $\ell_1, \ell_2, \dots, \ell_{\gamma_n(G)}$, where $\mathcal{D} = \{v_1, v_2, \dots, v_{\gamma_n(G)}\}$ and $\ell_i = d_G(v_i, \mathcal{D} - \{v_i\})$ for $1 \leq i \leq \gamma_n(G)$.

We show firstly that each vertex of $\mathcal{D}$ has property P_1. If this is not the case, then let i be the smallest integer such that the vertex v_i does not have property P_1. By Lemma A, v_i has property P_2, and so $\ell_i \geq n + 1$. Now let $v_i' \in N_n(v_i)$ and consider the set $\mathcal{D}' = (\mathcal{D} - \{v_i\}) \cup \{v_i'\}$. Necessarily $\mathcal{D}'$ is minimum n-dominating set of G. Furthermore, the vertex v_i' is within distance n from some vertex of $\mathcal{D} - \{v_i\}$; consequently, $\ell_i' = d_G(v_i', \mathcal{D}' - \{v_i'\}) < \ell_i$. Now let j be the largest integer for which $\ell_j < \ell_i$, and consider the value $\ell_k' = d_G(v_k, \mathcal{D}' - \{v_k\})$ for each k with $1 \leq k \leq j$. Since $\ell_k < \ell_i$, a shortest path from the vertex v_k to a vertex of $\mathcal{D} - \{v_i\}$ does not contain v_i. It follows, therefore, that $\ell_k' \leq \ell_k$ for all k ($1 \leq k \leq j$). This, together with the observation that $\ell_i' < \ell_r$ for all $r > j$, implies that the distance sequence of $\mathcal{D}'$ precedes that of $\mathcal{D}$ in dictionary order. This produces a contradiction. Hence every vertex of $\mathcal{D}$ has property P_1.

For each vertex v_i of $\mathcal{D}$, let w_i be a vertex of $V(G) - \mathcal{D}$ at maximum distance from v_i in G satisfying $N_n(w_i) \cap \mathcal{D} = \{v_i\}$ ($1 \leq i \leq \gamma_n(G)$). We show that $d(v_i, w_i) = n$ for all i. If this is not the case, then let i be the smallest integer for which $d(v_i, w_i) < n$. We observe, therefore, that every vertex of $V(G) - \mathcal{D}$ at distance greater than $n - 1$ from v_i is within distance n from some vertex of $\mathcal{D} - \{v_i\}$. We now consider a shortest path from the vertex v_i to a vertex of $\mathcal{D} - \{v_i\}$ in G. Let v_i^* denote the vertex adjacent to v_i on such a path. Further, let $\mathcal{D}^* = (\mathcal{D} - \{v_i\}) \cup \{v_i^*\}$. Necessarily $\mathcal{D}^*$ is minimum n-dominating set of G. Now let j be the largest integer for which $\ell_j < \ell_i$, and consider the value $\ell_k^* = d_G(v_k, \mathcal{D}^* - \{v_k\})$ for each k with $1 \le k \le j$. Necessarily, $\ell_k^* \le \ell_k$ for all k $(1 \le k \le j)$. Furthermore, $d_G(v_i^*, \mathcal{D}^* - \{v_i^*\}) = \ell_i - 1 < \ell_r$ for all $r > j$. It follows, therefore, that the distance sequence of $\mathcal{D}^*$ precedes that of $\mathcal{D}$ in dictionary order. This produces a contradiction. Hence $d(v_i, w_i) = n$ for all i, which completes the proof of the theorem. □

As an immediate corollary of Theorem 1, we have the following result which was established in [19].

Corollary 1 *For $n \ge 1$, if G is a connected graph of order $p \ge n+1$, then* $\gamma_n(G) \le \frac{p}{n+1}$.

3. Bounds relating i_n and γ_n.

We begin this section with the following theorem, which is in fact a corollary of Theorem 1. This result was established in [21] using entirely different techniques to those presented in this paper.

Theorem 2 *For $n \ge 1$, if G is a connected graph of order $p \ge n+1$, then* $i_n(G) + n\gamma_n(G) \le p$.

Proof. Among all the n-dominating sets of vertices of G with cardinality $\gamma_n(G)$, let $\mathcal{D}$ be one which comes first in dictionary order. Using the notation introduced in the proof of Theorem 1, let Q_i denote a v_i-w_i path of length n in G for each i with $1 \le i \le \gamma_n(G)$. We show that this collection $\{Q_1, Q_2, \ldots, Q_{\gamma_n(G)}\}$ of paths is disjoint. If this is not the case, then for some i and j with $1 \le i < j \le \gamma_n(G)$, we have $V(Q_i) \cap V(Q_j) \ne \emptyset$. This implies, however, that at least one of w_i and w_j is within distance n from both v_i and v_j, which produces a contradiction. Hence the collection $\{Q_1, Q_2, \ldots, Q_{\gamma_n(G)}\}$ of paths is disjoint.

let $\mathcal{I}$ be a minimum n-independent dominating set of vertices of G. Then $\mathcal{I}$ contains at most one vertex from each path Q_i $(1 \le i \le \gamma_n(G))$. Let W_i be a set of n vertices of Q_i that are not in $\mathcal{I}$ for all i. Then $(\cup_{i=1}^{\gamma_n(G)} W_i) \cap \mathcal{I} = \emptyset$ and $|\cup_{i=1}^{\gamma_n(G)} W_i| = n\gamma_n(G)$. Hence we have

$$\begin{aligned} i_n(G) + n\gamma_n(G) &= |\mathcal{I}| + |\cup_{i=1}^{\gamma_n(G)} W_i| \\ &= |\mathcal{I} \cup (\cup_{i=1}^{\gamma_n(G)} W_i)| \\ &\le |V(G)| \\ &= p. \qquad \square \end{aligned}$$

Since every n-independent dominating set of a graph G is an n-dominating set of G, we have the following proposition.

Proposition 2 *For $n \geq 1$ and for every graph G, $\gamma_n(G) \leq i_n(G)$.*

We note that strict inequality may occur in Proposition 2. Consider for instance the graph G constructed as follows. For $n, m \geq 1$, we recall that the double star $S(m, n)$ is obtained from the (disjoint) union of two stars $K_{1,n}$ and $K_{1,m}$ by joining a vertex of maximum degree in $K_{1,n}$ to a vertex of maximum degree in $K_{1,m}$. The graph G is obtained from the double star $S(2, 2)$ by subdividing each edge $n-1$ times. Then G is a graph for which $\gamma_n(G) = 2$ and $i_n(G) = 3$.

Allan and Laskar [1] established the following sufficient condition for the independent domination number of a graph to equal its domination number.

Theorem *B If a graph G has no induced subgraph isomorphic to $K_{1,3}$, then $\gamma(G) = i(G)$.*

In order to present the next two results, we need to define a generalization of $K_{1,k}$ for $k \geq 3$. Let G be a graph that contains a n-independent set $\mathcal{I}_k$ of k vertices and a vertex v of G that is within distance n from every vertex of $\mathcal{I}_k$. Then we shall refer to a connected subgraph of G of minimum size that contains all the vertices in $\mathcal{I}_k \cup \{v\}$ as a n-generalized $K_{1,k}$ in G. The next result follows immediately from Theorem B and Proposition 1 and the fact that if a graph G contains no n-generalized $K_{1,3}$, then G^n contains no induced $K_{1,3}$.

Theorem 3 *For $n \geq 1$, if G is a graph containing no n-generalized $K_{1,3}$, then $\gamma_n(G) = i_n(G)$.*

Bollobás and Cockayne [6] established the next result.

Theorem *C If G is a graph containing no induced subgraph isomorphic to $K_{1,k+1}$ ($k \geq 2$), then $i(G) \leq (k-1)\gamma(G) - (k-2)$.*

Theorem C may be generalized as in Theorem 4. The proof is immediate from Theorem C and Proposition 1 and the fact that if a graph G contains no n-generalized $K_{1,k+1}$ ($k \geq 2$), then G^n contains no induced $K_{1,k+1}$. However in order to characterize the extremal graphs, we offer a direct proof.

Theorem 4 *For $n \geq 1$ and $k \geq 2$, if G is a graph containing no n-generalized $K_{1,k+1}$, then $i_n(G) \leq (k-1)\gamma_n(G) - (k-2)$.*

Proof. Let $\mathcal{D}$ be a minimum n-dominating set of vertices of G and let $\mathcal{I}$ be a maximal n-independent set of vertices of $\mathcal{D}$ in G. Further, let Y denote the set of all vertices in $V(G) - \mathcal{D}$ that are at distance at least $n+1$ from every vertex of $\mathcal{I}$ in G. Let X be a maximal n-independent set of vertices of Y in G. Then $\mathcal{I} \cup X$ is a maximal n-independent set in G; or, equivalently, $\mathcal{I} \cup X$ is an n-independent dominating set of G.

We show that each vertx of $\mathcal{D} - \mathcal{I}$ is n-adjacent to at most $k-1$ vertices of X in G. If this is not the case, then there is a vertex v of $\mathcal{D} - \mathcal{I}$ that is n-adjacent to (at least) k vertices of X in G. Furthermore, v is n-adjacent with some vertex of $\mathcal{I}$. Hence there exists an n-independent set of $k+1$ vertices and a vertex v within distance n from every vertex of that set. This implies that G contains an n-generalized $K_{1,k+1}$, which produces a contradiction. We deduce, therefore, that each vertex of $\mathcal{D} - \mathcal{I}$ is n-adjacent to at most $k-1$ vertices of X in G. This, together with the observation that every vertex of Y (and hence of X) is n-adjacent with some vertex of $\mathcal{D} - \mathcal{I}$ in G, implies that $|X| \leq (k-1)(\gamma_n(G) - |\mathcal{I}|)$. It follows that

$$\begin{aligned} i_n(G) &\leq |\mathcal{I}| + |X| \\ &\leq |\mathcal{I}| + (k-1)(\gamma_n(G) - |\mathcal{I}|) \\ &\leq (k-1)\gamma_n(G) - (k-2)|\mathcal{I}| \\ &\leq (k-1)\gamma_n(G) - (k-2). \qquad \ldots(1) \end{aligned}$$

□

We now attempt to characterize graphs G for which $i_n(G) = (k-1)\gamma_n(G) - (k-2)$. If $\gamma_n(G) = 1$, then G is a graph with maximum n-degree equal to $p(G) - 1$ and equality holds in (1). Hence in what follows let G be a graph with $\gamma_n(G) \geq 2$ and for which $i_n(G) = (k-1)\gamma_n(G) - (k-2)$.

Using the notation introduced in the proof of Theorem 4, $|\mathcal{D}| = \gamma_n(G)$ and equality holds at each point in the sequence of inequalities of (1). Hence $|\mathcal{I}| = 1$ for every choice $\mathcal{I}$ of maximal n-independent sets of vertices of $\mathcal{D}$ in G. It follows that the vertices of $\mathcal{D}$ are pairwise n-adjacent in G. Furthermore, equality in the above sequence (1) implies that every maximal n-independent set X of vertices of Y in G is of cardinality $(k-1)(\gamma_n(G)-1)$, with exactly $k-1$ vertices of X that are n-adjacent to each vertex of $\mathcal{D} - \mathcal{I}$ and with each vertex of X being n-adjacent to exactly one vertex of $\mathcal{D} - \mathcal{I}$ in G. It follows, therefore, that each vertex of Y is n-adjacent with exactly one vertex of $\mathcal{D} - \mathcal{I}$; for otherwise, if there is a vertex of Y that is n-adjacent with at least two vertices of $\mathcal{D} - \mathcal{I}$, then X can be chosen to contain such a vertex, which would produce a contradiction.

We show next that each vertex of $V(G) - \mathcal{D}$ is either n-adjacent to every of $\mathcal{D}$ or n-adjacent to exactly one vertex of $\mathcal{D}$. Suppose v is a vertex of $V(G) - \mathcal{D}$ that is at distance at least $n+1$ from some vertex u of $\mathcal{D}$ in G. We now choose $\mathcal{I} = \{u\}$, and so $v \in Y$. Moreover, we may further choose X to contain the vertex v. It follows, then, by our earlier observations, that v is n-adjacent to exactly one vertex of $\mathcal{D}$.

Now let Z be the set of all vertices of $V(G) - \mathcal{D}$ that are n-adjacent to every vertex of $\mathcal{D}$ in G. Then, by our earlier observations, every vertex of

$V(G) - (\mathcal{D} \cup Z)$ is n-adjacent to exactly one vertex of $\mathcal{D}$ in G. Moreover for each $d \in \mathcal{D}$, if we let $N_d = N_n(d) - (\mathcal{D} \cup Z)$, then every maximal n-independent set of vertices of $\langle N_d \rangle_G$ in G contains exactly $k-1$ vertices.

Thus G has the following structure: G is a graph with vertex set $V(G) = \mathcal{D} \cup Z \cup (\cup_{d \in \mathcal{D}} N_d)$ where $|\mathcal{D}| = \gamma_n(G)$, the vertices of $\mathcal{D}$ are pairwise n-adjacent in G to every vertex of Z. For every $d \in \mathcal{D}$, $N_d = N_n(d) - (\mathcal{D} \cup Z)$. Further, every maximal n-independent set of vertices of N_d in G has exactly $k-1$ elements and all maximal n-independent sets of vertices of the union of any $\gamma_n(G) - 1$ of the sets N_d in G have cardinality $(k-1)(\gamma_n(G) - 1)$.

4. Bounds relating i_n and γ_n^t.

Allan, Laskar and Hedetniemi [2] established the following relationship between the independent domination number and total domination number of a graph.

Theorem D *If G is a connected graph of order $p \geq 3$, then $i(G) + \gamma_t(G) \leq p$.*

The next result extends this result for all trees of sufficiently large order.

Theorem 5 *For an integer $n \geq 2$, if T is a tree of order $p \geq 2n+1$, then $i_n(T) + n\gamma_n^t(T) \leq p$.*

Proof. The following paragraphs outline the proof. (The details of the proof are left to the reader.) If $rad\, T \leq n$, then $i_n(T) + n\gamma_n^t(T) = 1 + 2n \leq p$. Assume thus that $rad\, T \geq n+1$. Suppose the theorem is false and let T be a counterexample of smallest possible order $p \geq 2n+1$. Let $diam\, T = d$ and let $P : u_0, u_1, \ldots, u_d$ be a longest path in T. We prove that $\gamma_n^t(T - w) = \gamma_n^t(T) - 1$ for every end-vertex of T.

Let i be the smallest integer such that deg $u_{n+i} \geq 3$. Choose the path P such that i is as small as possible. We show that deg $u_j = 2$ for $1 \leq j \leq n+1$ and $d - n - 1 \leq j \leq d$, and hence that $2 \leq i \leq d - 2n - 2$. We show further that $2 \leq i \leq n-1$. We then consider the component of $T - \{u_{n+i-1}u_{n+i}, u_{n+i}u_{n+i+1}\}$ that contains u_{n+i}. Call this component T_i. Then T_i is a nontrivial component, and by our choice of P, the eccentricity $e_{T_i}(u_{n+i})$ of u_{n+i} in T_i satisfies $e_{T_i}(u_{n+i}) \leq n+i$. We then prove that $e_{T_i}(u_{n+i}) \neq n+i$, and that each end-vertex of T_i (different from u_{n+i}) is at distance at least $i+2$ from u_{n+i}. In this way we establish that $e_{T_i}(u_{n+i}) \in \{i+2, i+3, \ldots, n+i-1\}$.

We show next that T_i is not a path with u_{n+i} as end-vertex. Otherwise, if $T_i : u_{n+i}, v_1, v_2, \ldots, v_j$ $(i+2 \leq j \leq n+i-1)$ is a path with u_{n+i} as end-vertex, then let T_1' and T_2' be the components of $T - \{u_{n+i}u_{n+i+1}\}$

containing u_{n+i} and u_{n+i+1}, respectively. We show that $p(T_1') \geq 2n$. We then consider two cases, depending on whether $p(T_2') \geq 2n$ or $p(T_2') < 2n$.

If $p(T_2') \geq 2n$, then we show that $i_n(T_1') + n\gamma_n^t(T_1') \leq p(T_1')$ and $i_n(T_2') + n\gamma_n^t(T_2') \leq p(T_2')$. Thus $i_n(T) + n\gamma_n^t(T) \leq i_n(T_1') + i_n(T_2') + n[\gamma_n^t(T_1') + \gamma_n^t(T_2')] \leq p(T_1') + p(T_2') = p$, which contradicts our assumption about T.

If $p(T_2') < 2n$, then we show firstly that T_2' is the path $u_{n+i+1}, u_{n+i+2}, \ldots, u_d$. (Hence T is obtained from the star $K_{1,3}$ by subdividing one edge $n+i-1$ times, one edge $j-1$ times, and the remaining edge $d-n-i+1$ times.) By our choice of i, the length of T_2' is at least $n+i-1$. We show that T_2' has at most $2n-2$ vertices and thus length at most $2n-3$. We then show that v_j is n-dominated by u_{d-n}, and that $d_T(u_n, u_{d-n}) > n$. Thus $\{u_n, u_{n+i}, u_{d-n}\}$ is a total n-dominating set of T and $\{u_n, u_{d-n}\}$ an n-independent dominating set of T. Hence $p = p(T) > 3(n+1)$, $i_n(T) = 2$, and $\gamma_n^t(T) = 3$. Consequently, $i_n(T) + n\gamma_n^t(T) \leq p$, once again producing a contradiction. Therefore $\deg_{T_i} u_{n+i} \geq 2$ or $\Delta(T_i) \geq 3$.

If there exists $w \in V(T_i) - \{u_{n+i}\}$ such that $\deg_{T_i} w \geq 3$, then choose such a w with $d_{T_i}(u_{n+i}, w)$ as large as possible. Let x be the vertex that precedes w on the u_{n+i}-w path. Denote the end-vertices of the component of $T - wx$ that contains w by $w_1, w_2, \ldots, w_m$, where $m \geq 2$. Assume that the vertices have been labelled in such a manner that $d(w, w_1) \geq d(w, w_2) \geq \cdots \geq d(w, w_m)$. Then a total n-dominating set of minimum cardinality in $T - w_2$ contains a vertex y that is within distance n from w_1 but not w_2. We may assume, without loss of generality, that $d_T(w_1, y) = n$. So y is an internal vertex on the w_1-w path and $d_T(w_2, y) > n$. Hence $d_T(w, w_1) > n$ and $d_T(w_1, w_2) \geq 2n+1$. We then consider four possibilities, depending on whether $d_T(w_1, w_2) = 2n+1$, $2n+2 \leq d_T(w_1, w_2) \leq 3n$, $d_T(w_1, w_2) = 3n+1$ and $d_T(w_1, w_2) \geq 3n+2$. All four possibilities, however, produce a contradiction.

Hence $\deg_{T_i} x \leq 2$ for all $x \in V(T_i) - \{u_{n+i}\}$ and $\deg_{T_i} u_{n+i} \geq 2$. Let $v_1, v_2, \ldots, v_r$ where $r \geq 2$ be the end-vertices of T_i, labelled in such a way that $d_T(v_j, u_{n+i}) \geq d_T(v_{j+1}, u_{n+i})$ for $1 \leq j < r$. Since P is a longest path, $d(v_1, u_{n+i}) \leq n+i \leq 2n-1$. Let Q_j be the v_j-u_{n+i} path in T $(1 \leq j \leq r)$. We show that $d(v_1, u_{n+i}) > n$ and that $d(v_2, u_{n+i}) > n$. We then consider the paths $Q_1 : v_1 = x_0, x_1, \ldots, x_n, \ldots, x_{n+i}$ and $Q_2 : v_2 = z_0, z_1, \ldots, z_n, \ldots, u_{n+i}$. We show that $d_T(z_n, x_n) > n$ and therefore that $d(v_1, v_2) \geq 3n+1$. We then prove that $d(v_1, v_2) = 3n+1$. Let S be the set of $2n+2$ vertices that consists of the vertices in $V(Q_1) - \{u_{n+i}\}$ together with the first $2n+2-|V(Q_1) - \{u_{n+i}\}|$ vertices on the v_2-u_{n+i} path Q_2. Then $2n+1 < p(T-S) = p-2n-2$. Suppose z' is the end-vertex of $T-S$ that belongs to Q_2. Then $d(u_{n+i}, z') = n-1$. Let $\mathcal{D}$ be a total n-dominating set of $T-S$ of minimum cardinality. Then there exists a vertex z in $\mathcal{D}$ such that $d_{T-S}(z, z') \leq n$. We may assume that z is not an internal vertex of Q_2 (otherwise replace z with u_{n+i}). However, then

$d_T(z, x_n) \leq n$. Thus $\mathcal{D} \cup \{x_n, z'\}$ is a total n-dominating set of T. Further if $\mathcal{I}$ is an n-independent dominating set of $T - S$, then we need to add at most two vertices to $\mathcal{I}$ to produce an n-independent dominating set of T. Since $i_n(T - S) + n\gamma_n^t(T - S) \leq p(T - S) = p - 2n - 2$, it follows that $i_n(T) + n\gamma_n^t(T) \leq p$, which contradicts our assumption about T. □

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