Total Chromatic Number of Graphs of High Maximum Degree

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ABSTRACT. The total chromatic number $\chi_T(G)$ of a graph G is the least number of colours needed to colour the edges and vertices of G so that no incident or adjacent elements receive the same colour. This paper shows that if G has maximum degree $\Delta(G) > \frac{3}{4}|V(G)| - \frac{1}{2}$, then $\chi_T(G) \leq \Delta(G) + 2$. A slightly weaker version of the result has earlier been proved by Hilton and Hind [9]. The proof here is shorter and simpler than the one given in [9].

Introduction

The graphs we shall consider are finite and simple. We denote the vertex set, edge set, maximum degree, minimum degree and chromatic index of a graph G by $V(G)$, $E(G)$, $\Delta(G)$ (or simply Δ), $\delta(G)$ (or simply δ) and $\chi_1(G)$ respectively. We denote the degree of a vertex x in G by $d_G(x)$ or simply $d(x)$.

A famous result of Vizing says that for any graph G,

$$\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1.$$

If $\chi_1(G) = \Delta(G)$, G is said to be class 1, and if $\chi_1(G) = \Delta(G) + 1$, G is said to be class 2. The subgraph induced by those vertices of maximum degree $\Delta(G)$ is denoted by G_Δ.

A *total colouring* of a graph G is a function

$$\pi : E(G) \cup V(G) \rightarrow \mathcal{C}$$

where $\mathcal{C}$ is a set of colours such that

1. no adjacent edges or vertices of G have the same image; and
2. the image of each vertex is distinct from the image of its incident edges.

The *total chromatic number* of a graph G is the least value of $|C|$ for which G has a total colouring $\pi : E(G) \cup V(G) \to C$. From its definition, $\chi_T(G) \geq \Delta(G) + 1$. Behzad [1] and Vizing [10] independently made the conjecture that for any graph G,

$$\chi_T(G) \leq \Delta(G) + 2.$$

This is known as the total colouring conjecture and no counterexample has been found. See [8] for a recent survey.

Preliminary Results

The first lemma is due to Erdös and Posá [6].

Lemma 1. *A graph G contains a matching of size at least* $\min\{\delta(G), \lfloor \frac{1}{2}|V(G)| \rfloor\}$.

The next result, although well-known, has yet to be formally stated until now and so has always been known in the literature as an argument, the Vizing's fan argument.

Lemma 2. *If all edges of G except one edge xy have been coloured with at most $\Delta(G)$ colours, and if all vertices which are adjacent to x, with the possible exception of y, have degree $< \Delta(G)$, then there is a colouring of all edges of G using $\Delta(G)$ colours.*

Proof: In the proof of Vizing's theorem presented in [7], it is first assumed that there is a $(\Delta + 1)$-colouring of all edges of G with the exception of one edge and it is then shown that this colouring can be extended to a colouring of all edges of G without using additional colours. We can adapt this proof of Vizing's theorem, using only Δ colours instead of $\Delta+1$ colours, but we must made sure that in a Δ-colouring of all edges of G with the exception of edge xy, there is at least one colour missing from x and from each vertex (including y) adjacent to x. This explains why we require all vertices adjacent to x, with the possible exception of y, to have degree $< \Delta$. (Actually in the proof, we need only require certain vertices and not all vertices adjacent to x to have degree $< \Delta$ in order to extend to a colouring of all edges of G.)

A related result of Lemma 2 is Lemma 5 of [2] which is the first to suggest adapting this argument of Vizing's theorem, as far as we know. Those graphs whose edges can be coloured using Vizing's fan argument are class 1 and have been studied by various authors. In particular, we will describe one such graph H, first mentioned in [3]. Let H be a graph with vertices $u_1, \ldots, u_r, v_t, \ldots, v_0, v_1, \ldots, v_r$ where $r \geq 1$ and $t \leq 0$ and such that $\{u_1v_1, \ldots, u_rv_r\}$ is a maximum matching of H and $E(H)$ contains edges only of the form: u_iv_j with $i \leq j$ and v_iv_j with $i \geq 1$ or $j \geq 1$. We shall say H has *class 1 structure.*

Lemma 3. *Let H be a subgraph of G induced by vertices such that H has class 1 structure. If $V(G_\Delta) \subseteq V(H)$, then G is class 1.*

Proof: Replace G_Δ with H throughout the proof of Theorem 5 of [3].

If $H = G_\Delta$, then Lemma 3 is a special case of Theorem 5 of [3] and it is this special case that is used in the proof of one of the main results (Theorem 1) of [3]. In fact, we can replace G_Δ in Theorems 4 and 5 of [3] by a subgraph H of G induced by vertices and need only $V(G_\Delta) \subseteq V(H)$. The edges of graphs in [3] are coloured using Vizing's fan argument with the final edges to be coloured have endvertices x and y each of degree Δ, a condition which we do not impose in this paper.

Following [11], a graph G is said to be *maximal* if whenever x and y are nonadjacent vertices, then either $d(x) = \Delta$ or $d(y) = \Delta$. To prove the total colouring conjecture, it suffices to prove it for maximal graphs. The concept of maximal graphs although not crucial in the proof of our main result, makes the argument more specific and thus shorter.

Lemma 4. *Let G be a maximal graph on n vertices with s vertices of degree Δ. Let $m = n - \Delta - 1$ where $n \geq 3m$. Then $\delta \geq \Delta + m - s$ where $s > m$ and there exist m disjoint pairs $S_i = \{x_i, y_i\}$ $(1 \leq i \leq m)$ of nonadjacent vertices in G such that either*

(1) there are at least m vertices of degree Δ not belonging to any of the S_i, or

(2) the x_i are all vertices of degree Δ and the y_i are all vertices of degree $< \Delta$.

Proof: There are $n - s$ vertices of degree $< \Delta$. Since G is maximal, these vertices induce a complete graph of order $n - s$ in G. Thus $\Delta \geq \delta \geq n - s - 1 = \Delta + m - s$ and so $s \geq m$. If $s = m$, then $\Delta = \delta$ and G is regular and so $s = n = m$, contradicting that $m = n - \Delta - 1$. Thus $s > m$. The complementary graph $\overline{G}$ has minimum degree $m = n - \Delta - 1 < \frac{n}{2}$ since $3m \leq n$. By Lemma 1, $\overline{G}$ has a matching $P = \{x_1y_1, \ldots, x_my_m\}$ of size m. Then $S_i = \{x_i, y_i\}$ $(1 \leq i \leq m)$ are m disjoint pairs of nonadjacent vertices.

If G is a regular graph, (1) follows since $n \geq 3m$. We suppose that G is not regular throughout the rest of the proof. Let $S^* = V(G) - (S_1 \cup \cdots \cup S_m)$. Among sets of m pairs of nonadjacent vertices, we choose $S_1, \ldots, S_m$ such that the set $Z = \{z \in S^* \mid d(z) = \Delta\}$ has maximum number of elements. If $|Z| \geq m$, then (1) follows.

Suppose now $|Z| < m$. Then $2m + |Z| < n$ and so there is a vertex $x \in S^*$ with $d(x) \leq \Delta - 1$. If x is nonadjacent to both x_i and y_i, then $d(x_i) = d(y_i) = \Delta$ since G is maximal. We can replace S_i by $S'_i = \{x_i, x\}$ and $Z' = \{z \in (S^* \cup S'_i) - S_i \mid d(z) = \Delta\}$ has one more element than Z contradicting that Z is maximal. Thus x can be nonadjacent to at most m vertices of $S_1 \cup \cdots \cup S_m$, and so x must be nonadjacent to a vertex $y \in S^*$, where $d(y) = \Delta$. If for some j, S_j consists of two vertices of degree Δ, then we can similarly replace S_j by $S'_j = \{x, y\}$, thereby contradicting that Z is maximal. Thus each S_i consists of one vertex, say x_i of degree Δ and one vertex, y_i of degree $< \Delta$. Thus (2) is established.

Main Results

The main ideas of the method are essentially the same as in [5, 9], some of which are set out in Lemma 5. Once Lemma 5 is proved, we can assign a single colour to edges in that matching and the two nonadjacent vertices that the matching misses, thereby using a total of m colours. We now just need $\Delta + 2 - m$ colours to assign to the remaining edges and $|V(G)| - 2m$ vertices, a problem which we will reformulate as an edge colouring problem. By applying Vizing's theorem or Lemma 3, we can obtain an edge colouring that uses $\Delta + 2 - m$ colours which can be modified to colour the remaining edges and vertices, with a distinct colour for each vertex. Unlike the proofs in [5, 9], we need not divide the proof according to whether G is of odd or even order and need not assume any known results in total colourings.

Lemma 5. *Let G be a maximal graph on n vertices with $\Delta > \frac{3n}{4} - \frac{1}{2}$ and s vertices of degree Δ. Let $m = n - \Delta - 1$ and $p = \min\{m, s - m\}$. Then the following hold:*

(1) There exist m disjoint pairs $S_i = \{x_i, y_i\}$ $(1 \leq i \leq m)$ of nonadjacent vertices in G and a set Z of p vertices of degree Δ not belonging to any of the S_i.

(2) There exist m edge-disjoint matchings $F_1, \ldots, F_m$ of G such that each F_i misses both vertices in S_i and the subgraph of $G - (F_1 \cup \cdots \cup F_m) = G^*$ induced by Z has class 1 structure.

(3) All vertices have degree $\leq \Delta + 1 - m$ in G^* and if a vertex has degree $\Delta + 1 - m$ in G^*, then that vertex must belong to $Z \cup S_1 \cup \cdots \cup S_m$.

Proof: We first note that $3m = 3(n - \Delta - 1) < n$. So (1) follows directly from Lemma 4, with $p = m$ in (1) of Lemma 4 and $p = s - m < m$ in (2) of Lemma 4.

We now prove (2). Consider the subgraph Z^* of G induced by $Z = \{z_1, \ldots, z_p\}$, where Z is as in (1). We first show that there exist p edge-disjoint matchings $M_1, \ldots, M_p$ of Z^* such that $Z^* - (M_1 \cup \cdots \cup M_p)$ has class 1 structure and $(i + |M_i|) \leq p$ and M_i misses z_i for each i. Let $M = \{u_1v_1, \ldots, u_rv_r\}$ be a maximum matching in Z^*. Let the remaining vertices of Z^* be $v_t, \ldots, v_0$ which induce a null graph in G. Note that $p = 2r + 1 - t$. We now define p edge-disjoint matchings $M_1, \ldots, M_p$ of Z^* as follows:

1. For $1 \leq i \leq r$, M_i is the matching consisting of edges u_ju_k of G such that $j + k \equiv i \pmod r$.
2. For $r + 1 \leq i \leq p$, M_i is the matching consisting of edges u_jv_k of G such that $j - k = i - r$.

Observe that $M_1 \cup \cdots \cup M_p$ contains all edges of Z^* except for edges which join u_i to v_j with $i \leq j$ and which join v_i to v_j for any i and j. We see that M is also a maximum matching of $Z^* - (M_1 \cup \cdots \cup M_p)$ and so $Z^* - (M_1 \cup \cdots \cup M_p)$ has class 1 structure. Next note that

$$|M_i| \leq \begin{cases} \frac{r}{2} & \text{if } 1 \leq i \leq r, \\ r & \text{if } r+1 \leq i \leq r+1-t, \\ p-i & \text{if } r+2-t \leq i \leq p. \end{cases}$$

Thus $(i + |M_i|) \leq p$ for all $i \in \{1, \ldots, p\}$. By labelling $z_1, \ldots, z_p$ as follows:

$$z_i = \begin{cases} v_{i+t-1} & \text{if } 1 \leq i \leq r, \\ v_{2r-i+1} & \text{if } r+1 \leq i \leq r+1-t, \\ u_{i+t-r-1} & \text{if } r+2-t \leq i \leq p, \end{cases}$$

it can be checked that M_i misses z_i for $1 \leq i \leq p$. At this stage, we divide the proof of (2) by considering the cases when $p = m$ and when $p < m$ separately. The proof for the case when $p = m$ is easier than for the other case and so is first considered. We may then adapt the proof of the first case to cope with the second case.

Case 1: $p = m$.

We claim that there are m edge-disjoint matchings $F_1, \ldots, F_m$ of G such that for $1 \leq k \leq m$,

(a) $M_k - (F_1 \cup \cdots \cup F_{k-1} \cup M) = M_k^* \subseteq F_k$;

(b) $G_k = (G - (F_1 \cup \cdots \cup F_{k-1} \cup M)) - (V(M_k^*) \cup S_k)$ has a matching C_k such that the set $B_k = \{v \in V(G_k) \mid d_{G_k}(v) \geq \Delta - k - |V(M_k^*)| - 2$ and v is C_k-unsaturated$\}$ has at most one element and if $B_k \neq \emptyset$, then $B_k = \{z_k\}$;

(c) $F_k = M_k^* \cup C_k$.

Suppose we have already constructed matchings $F_1, \ldots, F_m$ according to conditions (a) to (c). Then each F_k misses both vertices in S_k, and since $F_k \cap M = \emptyset$ and $(M_1 \cup \cdots \cup M_m) \subseteq (F_1 \cup \cdots \cup F_m)$, $Z^* - (F_1 \cup \cdots \cup F_m)$ has maximum matching M and so has class 1 structure, thereby proving (2) in this case. To prove such matchings do exist, we suppose edge-disjoint matchings $F_1, \ldots, F_{k-1}$ have been constructed according to conditions (a) to (c) where $1 \le k \le m$. We first prove that G_k has a matching C_k such that $|B_k| \le 1$. Let

$$\lambda_k = \Delta - k - |V(M_k^*)| - 2$$

and $v \in B_k$ with $d_{G_k}(v) = d$. There are d vertices $v_1, \ldots, v_d$ adjacent to v in G_k and another d vertices $v_1', \ldots, v_d'$ such that $v_j v_j' \in C_k$ $(1 \le j \le d)$ as we want B_k to have as few elements as possible. (It is possible that $v_j' \in \{v_1, \ldots, v_d\}$ for some j.) We claim that for any vertex w of G_k with $d_{G_k}(w) \ge \lambda_k$, there is a $v - w$ alternating path (with respect to C_k) of odd length ≤ 3 in G_k, that is, either $vw \in E(G_k)$ or $wv_j' \in E(G_k)$ for some $j \in \{1, \ldots, d\}$ where $v \in B_k$. Suppose not. Then $v, v_1', \ldots, v_d'$ are each nonadjacent to w in G_k and so $d_{G_k}(w) \le |V(G_k)| - d - 2$. Since $|V(G_k)| = n - |V(M_k^*)| - 2$ and $d \ge \lambda_k$ we have

$$\lambda_k \le d_{G_k}(w) \le n - |V(M_k^*)| - \lambda_k - 4$$

which simplifies to

$$2\Delta \le n + 2k + |V(M_k^*)|.$$

Since

$$k + \frac{1}{2}|V(M_k^*)| \le k + |M_k| \le p = m,$$

it follows that

$$2\Delta \le n + 2m = n + 2(n - \Delta - 1)$$

or $\Delta \le \frac{3n}{4} - \frac{1}{2}$ contradicting that $\Delta > \frac{3n}{4} - \frac{1}{2}$. Hence there is a $v - w$ alternating path of odd length ≤ 3 in G_k if $d_{G_k}(w) \ge \lambda_k$. If w is C_k-unsaturated, then we can replace C_k by

$$C_k' = \begin{cases} C_k \cup \{vw\} & \text{if } vw \in E(G_k), \\ (C_k \cup \{vv_j, wv_j'\}) - \{v_j v_j'\} & \text{if } wv_j' \in E(G_k), \end{cases}$$

where now v and w are C_k'-saturated. This shows that $|B_k| \le 1$.

We next show that if $B_k \ne \emptyset$, then $B_k = \{z_k\}$. Since $d_G(z_k) = \Delta$ and M_k misses z_k, z_k belongs to $V(G_k)$ and so $d_{G_k}(z_k) \ge \lambda_k$. Thus we may assume that z_k is C_k-saturated. Let $z_k z_k' \in C_k$. If $d_{G_k}(z_k') \ge \lambda_k$, then there is a $v - z_k'$ alternating path of odd length ≤ 3 in G_k where $v \in B_k$. Then as above, we can similarly modify C_k by C_k' so that $B_k = \{z_k\}$ (with respect to C'_k). Suppose now $d_{G_k}(z_k) < \lambda_k$. Since there is a $v - z_k$ alternating path of odd length (≤ 3), we can again modify C_k by C'_k so that B_k is now empty (with respect to C'_k). Therefore, we have shown that F_k can be constructed according to conditions (a) to (c), thereby proving (2) in this case.

Case 2: $p = s - m < m$.

Let w_i ($1 \leq i \leq m$) be distinct vertices of $V(G) - (S_1 \cup \cdots \cup S_m)$ such that $w_i = z_{i+p-m}$ if $m - p + 1 \leq i \leq m$. If $1 \leq i \leq m - p$, then by Lemma 4, $\Delta - 1 \geq d_G(w_i) \geq \delta \geq \Delta - p$ and if $m - p + 1 \leq i \leq m$, $d_G(w_i) = \Delta$ where $w_i \in Z$. We define

$$M_i^+ = \begin{cases} \emptyset & \text{if } 1 \leq i \leq m - p, \\ M_{i+p-m} & \text{if } m - p + 1 \leq i \leq m. \end{cases}$$

Then M_i^+ misses w_i for $1 \leq i \leq m$. Let

$$\alpha_i = \begin{cases} 1 & \text{if } 1 \leq i \leq m - p, \\ 0 & \text{if } m - p + 1 \leq i \leq m. \end{cases}$$

We claim that there are m edge-disjoint matchings $F_1, \ldots, F_m$ of G such that for $1 \leq k \leq m$,

(a) $M_k^+ - (F_1 \cup \cdots \cup F_{k-1} \cup M) = M_k^* \subseteq F_k$;

(b) $G_k = (G - (F_1 \cup \cdots \cup F_{k-1} \cup M)) - (V(M_k^*) \cup S_k)$ has a matching C_k such that the set $B_k = \{v \in V(G_k) \mid d_{G_k}(v) \geq \Delta - p\alpha_k - k - |V(M_k^*)| - 2$ and v is C_k-unsaturated$\}$ has at most one element and if $B_k \neq \emptyset$, then $B_k = \{w_k\}$;

(c) $F_k = M_k^* \cup C_k$.

Once we have constructed such matchings $F_1, \ldots, F_m$, then as in Case 1, result (2) follows. We suppose edge-disjoint matchings $F_1, \ldots, F_{k-1}$ have been constructed according to conditions (a) to (c) where $1 \leq k \leq m$. Let

$$\lambda'_k = \Delta - p\alpha_k - k - |V(M_k^*)| - 2$$

and $v \in B_k$. Since $d_G(w_k) \geq \Delta - p\alpha_k$ and M_k^+ misses w_k, w_k belongs to $V(G_k)$ and so $d_{G_k}(w_k) \geq \lambda'_k$. Thus as in Case 1 to prove F_k exists, we need only to prove that for any vertex w of G_k with $d_{G_k}(w) \geq \lambda'_k$ there is a $v - w$ alternating path (with respect to C_k) of odd length ≤ 3 in G_k. Suppose not. Then as in Case 1, we have

$$\lambda'_k \leq d_{G_k}(w) \leq n - |V(M_k^*)| - \lambda'_k - 4$$

which simplifies to

$$2\Delta \le n + 2k + |V(M_k^*)| + 2p\alpha_k.$$

If $1 \le k \le m - p$, $M_k^* = \emptyset$, $\alpha_k = 1$ and so

$$k + \frac{1}{2}|V(M_k^*)| + p\alpha_k \le (m-p) + p = m.$$

If $m - p + 1 \le k \le m$, $M_k^* \subseteq M_{k+p-m}$, $\alpha_k = 0$ and so

$$k + \frac{1}{2}|V(M_k^*)| + p\alpha_k \le k + |M_{k+p-m}| \le k + p - (k+p-m) = m.$$

It follows that

$$2\Delta \le n + 2m = n + 2(n - \Delta - 1)$$

or $\Delta \le \frac{3n}{4} - \frac{1}{2}$, which is a contradiction. Therefore, we have shown that F_k can be constructed according to conditions (a) to (c). This completes the proof of (2).

We next prove (3). If $x \in V(G)$ is F_j-saturated for all $j \in \{1, \ldots, m\}$, we have $d_{G^*}(x) \le \Delta - m$. Let $x \in V(G)$ be F_k-unsaturated but F_j-saturated for all $j \in \{k+1, k+2, \ldots, m\}$ where $1 \le k \le m$. Since $F_k = M_k^* \cup C_k$ and x is F_k-unsaturated, x must belong to $V(G_k) \cup S_k$. Let $G_k^* = G - (F_1 \cup \cdots \cup F_{k-1})$. We divide the proof according to the following cases.

Case 1: $d_{G_k^*}(x) \le \Delta - k$. Since x is F_j-saturated for all $j \in \{k+1, k+2, \ldots, m\}$, we have

$$\begin{aligned} d_{G^*}(x) &\le d_{G_k^*}(x) - (m-k) \\ &\le \Delta - m. \end{aligned}$$

Case 2: $d_{G_k^*}(x) \ge \Delta - k + 1$. We know that $x \in V(G_k) \cup S_k$. If $x \in V(G_k)$, then x is C_k-unsaturated and

$$\begin{aligned} d_{G_k}(x) &\ge d_{G_k^*}(x) - 1 - |V(M_k^*)| - 2 \\ &\ge \Delta - k - |V(M_k^*)| - 2. \end{aligned}$$

Thus if $p = m$, then $d_{G_k}(x) \ge \lambda_k$ and so by condition (b), $x = z_k$. If $p < m$, then $d_{G_k}(x) \ge \lambda'_k$ and so by condition (b), $x = w_k$. We consider two subcases.

Subcase 2a: x is F_j-saturated for all $j \in \{1, \ldots, k-1\}$. Then $d_{G^*}(x) \le \Delta - (m-1) = \Delta + 1 - m$. If $d_{G^*}(x) = \Delta + 1 - m$, then we must have $d_G(x) = \Delta$ and $x \in Z \cup S_k$.

Subcase 2b: x is F_q-unsaturated but is F_j-saturated for all $j \in \{q+1, q+2, \ldots, k-1\}$ where $q < k$. Since either $x \in S_k \cup \{z_k\}$ or $x \in S_k \cup \{w_k\}$, x belongs to $V(G_q)$ and is C_q-unsaturated. Then x cannot be z_k, since otherwise $d_G(x) = d_G(z_k) = \Delta$ and so $d_{G_q}(x) \geq \lambda_q$ which implies that $x = z_k = z_q$, a contradiction. Vertex x also cannot be w_k, since otherwise $d_G(x) = d_G(w_k) \geq \Delta - p\alpha_k$ and so $d_{G_q}(x) \geq \lambda'_q$ which implies that $x = w_k = w_q$, a contradiction. Thus x belongs to S_k. If $d_{G_q^*}(x) \geq \Delta - q + 1$ and since $x \in V(G_q)$, we may use Case 2 with k replaced by q to conclude that either $x = z_q$ or $x = w_q$, which is again a contradiction. Thus $d_{G_q^*}(x) \leq \Delta - q$. Since x is F_j-saturated for all $j \in \{q, q+1, \ldots, m\} - \{q, k\}$, we have

$$\begin{aligned} d_{G^*}(x) &\leq d_{G_q^*}(x) - (m - q - 1) \\ &\leq \Delta + 1 - m. \end{aligned}$$

Therefore we have shown that vertex x has degree $\leq \Delta + 1 - m$ in G^* and if x has degree $\Delta + 1 - m$ in G^*, then x must belong to $Z \cup S_k$. This proves (3) and completes the proof of Lemma 5.

We are now ready to prove the main theorem. Hilton and Hind [9] have proved that if $\Delta(G) \geq \frac{3}{4}|V(G)|$, then $\chi_T(G) \leq \Delta(G) + 2$.

Theorem 1. *Let G be a graph on n vertices with $\Delta(G) > \frac{3n}{4} - \frac{1}{2}$. Then $\chi_T(G) \leq \Delta(G) + 2$.*

Proof: We may assume that G is maximal. Then results (1) to (3) of Lemma 5 hold. We now construct a graph G^+ by introducing a new vertex v^* to $G^* = G - (F_1 \cup \cdots \cup F_m)$ and adding an edge joining v^* to each vertex in $V(G) - (S_1 \cup \cdots \cup S_m)$ where F_i and S_i are as in Lemma 5. We note that $d_{G^+}(v^*) = n - 2m = \Delta + 1 - m$. From Lemma 5, $\Delta(G^+) \leq \Delta(G^*) + 1 \leq \Delta + 2 - m$, and any vertex of degree $\Delta + 2 - m$ in G^+ must be adjacent to v^* and so must belong only to Z where Z is as in Lemma 5. By the construction of F_i in Lemma 5, the subgraph of G^+ (or G^*) induced by Z has class 1 structure. If $\Delta(G^+) = \Delta + 2 - m$, then G^+ is class 1 by Lemma 3 and so has an edge colouring π that uses $\Delta + 2 - m$ colours $1, 2, \ldots, \Delta + 2 - m$. If $\Delta(G^+) = \Delta + 1 - m$, then by Vizing's theorem, G^+ has an edge colouring π that uses $\Delta + 2 - m$ colours. We modify π to a total colouring θ of G that uses $\Delta + 2$ colours as follows:

$\theta(e) = \pi(e)$ if $e \in E(G) \cap E(G^+)$;
$\theta(v) = \pi(vv^*)$ if $v \in V(G) - (S_1 \cup \cdots \cup S_m)$;
$\theta(v) = \theta(f) = \Delta + 2 - m + j$ if $v \in S_j$ and $f \in F_j (1 \leq j \leq m)$.

It is easy to check that θ is indeed a total colouring of G.

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