Two cyclic supplementary difference sets and optimal designs in linear models

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ABSTRACT. D-optimal exact designs in a completely randomized statistical set-up are constructed, for comparing $n > 2$ qualitative factors (treatments), making r observations per treatment level in the presence of n (or less) quantitative or continuous factors (regression factors or covariates) of influence. Their relation with cyclic supplementary difference sets $2\text{-}\{u; k_1, k_2; \lambda\}$ is showed, when $n = 2u \equiv 2 \pmod 4$, $r \equiv 1 \pmod 2$, $r \neq 1$, $r < u$ and k_1, k_2, λ are defined by $1 \leq k_1 \leq k_2 \leq (u-1)/2$, $(u-2k_1)^2 + (u-2k_2)^2 = 2(ur+u-r)$, $\lambda = k_1 + k_2 - (u-r)/2$. Making use of known cyclic difference sets, the existence of a multiplier and the non-periodic autocorrelation function of two sequences, such supplementary difference sets are constructed for the first time. A list of all 201 supplementary difference sets $2\text{-}\{u; k_1, k_2; \lambda\}$ for $n = 2u < 100$ is given.

1. Introduction and Preliminary results

Let u, k_1, k_2 and λ be positive integers. Suppose that $C = \{c_1, \ldots, c_{k_1}\}$, $D = \{d_1, \ldots, d_{k_2}\}$ are two collections of k_1, k_2 residues mod u respectively,

JCMCC 18 (1995), pp. 33-56

such that the congruence

$$c_i - c_j \equiv a \pmod{u}, d_i - d_j \equiv a \pmod{u} \qquad (1)$$

has exactly λ solutions for any $a \not\equiv 0 \pmod{u}$. Such two residue sets C and D are called *supplementary difference sets*, denoted by 2-$\{u; k_1, k_2; \lambda\}$ (see in (Seberry) Wallis [17,18]).

From the definition it follows that

$$\lambda(u-1) = k_1(k_1 - 1) + k_2(k_2 - 1). \qquad (2)$$

If C, D are two supplementary difference sets, construct the incidence matrices R_1, R_2 of C, D respectively, which are $u \times u$ circulant $(+1, -1)$-matrices having in their first row -1 in the positions indicated by C, D respectively and $+1$ in the remaining positions of their first row. The following result is proved by Chadjipantelis and Kounias [4].

Theorem 1. *(i) If C, D are supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$ and R_1, R_2 the corresponding incidence matrices, then*

$$R_1 R_1^T + R_2 R_2^T = 4(k_1 + k_2 - \lambda)I_u + 2(u - 2(k_1 + k_2 - \lambda))J_u.$$

(ii) Given two $u \times u$ circulant matrices $R_1 R_2$ satisfying (2), then the corresponding sets C, D are supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$, where k_1, k_2 is the number of -1's in each row of R_1, R_2 respectively and λ satisfies (2).

Here, A^T denotes the transpose of a given matrix A, I_u is the identity matrix of order u and J_u is the $u \times u$ matrix of ones.

Let $n = 2u$ and W_1, W_2 are $u \times u$ commuting matrices, with elements ± 1 such that

$$W_1 W_1^T + W_2 W_2^T = 2(u - r)I_u + 2rJ_u, \quad |r| < u. \qquad (3)$$

Then for the matrix

$$W = \begin{bmatrix} W_1 & -W_2 \\ W_2^T & W_1^T \end{bmatrix} \qquad (4)$$

it holds that

$$W^T W = W W^T = [2(u - r)I_u + 2rJ_u] \otimes I_2. \qquad (4a)$$

It is known (Ehlich [7]), that if $u \equiv 1 \pmod{2}$ (i.e. $n \equiv 2 \pmod{4}$) and $r = 1$, the $n \times n$ matrix W has the maximum determinant among all $n \times n$ $(+1, -1)$-matrices. Such matrices W are used in optimum design problem and correspond to the D-optimal weighing designs for estimating n weights making n weighings in a chemical balance, as well as to the D-optimal 2^n fractional factorial resolution III designs for estimating the main effects of n two-level factors in n observations. These two statistical settings are closely related. For the construction of such D-optimal designs see in Ehlich [7], Yang [23–26], Chadjipantelis and Kounias [4], Chadjipantelis, Kounias and Moyssiadis [5], Cohn [6], Kharaghani [9], Whiteman [20], Trung [16], Koukouvinos, Kounias and Seberry [10].

In this paper we consider the more general case $n \equiv 2 \pmod 4$, $r \equiv 1 \pmod 2$, and facilitate the following two problems:

(a) Construct all possible two supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$ for $u \equiv 1 \pmod 2$ (i.e. for $n \equiv 2 \pmod 4$) and for all almost practical values of n, i.e. for $n < 100$.

(b) Find their connection with optimal experimental exact designs.

As stated above, for $r = 1$ these two supplementary difference sets can be used to construct D-optimal first order designs. In the sequel we consider the case $r \neq 1$.

It is proved in Section 3, that such matrices W can be used for constructing D-optimal complex linear designs in a covariates model without blocking, for estimating the effects of n treatments (or one treatment at n levels) in the presence of n (or less) continuous covariates with values on an n-cube, making r observations per treatment level.

If W_1, W_2 are circulant matrices, then pre- and post-multiplying both sides of (3) by e^T and e respectively, we get

$$(u - 2k_1)^2 + (u - 2k_2)^2 = 2(u(r+1) - r) \tag{4b}$$

where e is the $u \times 1$ vector of ones and k_1, k_2 is the number of -1's in every row of W_1, W_2 respectively.

If W_1, W_2 satisfy (3), so do $\pm W_1, \pm W_2$, i.e. we can always take $0 \leq k_1 \leq k_2 \leq (u-1)/2$. Now form the two sets

$$C = \{c_1, c_2, \ldots, c_{k_1}\}, D = \{d_1, d_2, \ldots, d_{k_2}\}$$

where c_i, d_j are the positions of -1's in the first row of W_1, W_2 respectively.

From Theorem 1 it follows that (3) holds if and only if the congruence (1) has exactly

$$\lambda = k_1 + k_2 - (u - r)/2 \tag{4c}$$

solutions for any $a \not\equiv 0 \pmod u$.

Hence the construction of the two circulant $u \times u$ $(+1, -1)$-matrices W_1, W_2 satisfying (3) is equivalent to the construction of two supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$ where k_1, k_2, λ satisfy (4b) and (4c).

In this paper we construct such supplementary difference sets for $3 \le r < u < 50$ and for $r = -1$.

The construction methods are given in Section 2. In Section 3, using the results of Section 2 we construct D-optimal complex linear designs for a covariates model, when $rn \equiv 2 \pmod 4$, $r \equiv 1 \pmod 2$, $r \ge 3$. In Section 4 a list of all 201 two supplementary difference sets for $u < 50$ is given.

2. Construction methods of 2-$\{u; k_1, k_2; \lambda\}$ SDS

In this section we give some construction methods of supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$ for $u \equiv 1 \pmod 2$, where k_1, k_2 satisfying (4b) with $r \equiv 1 \pmod 2$, $3 \le r < u$, $r = -1$ and $\lambda = k_1 + k_2 - (u-r)/2$. Before giving our first construction theorem we need the following:

Let u, k and λ be positive integers. Suppose that $E = \{e_1, e_2, \ldots, e_k\}$ is a set of k residues mod u with the property that for any residue $a \not\equiv 0 \pmod u$ the congruence

$$e_i - e_j \equiv a \pmod u$$

has exactly λ solution pairs (e_i, e_j) with e_i and e_j in E. Such a residue set is called a *cyclic difference set*, denoted by (u, k, λ)-CDS. About cyclic difference sets see in L.D. Baumert [1]. It is known that the $u \times u$ $(+1, -1)$-incidence cyclic matrix R of E satisfies the matrix equation

$$RR^T = R^T R = 4(k-\lambda)I_u + (u - 4(k-\lambda))J_u. \tag{5}$$

Theorem 2. *Let for some $r \equiv 1 \pmod 2$ there exists a (u, k, λ)-CDS, where*

$$k = (u-c)/2, \lambda = (r-c)/2, c = (u(2r-u+2) - 2r)^{1/2}, \quad u \equiv 1 \pmod 2, \text{ or} \tag{6}$$

$$k = (u-c)/2, \lambda = (r+2-c)/2, c = (u(2r+u-6) - 2r - 4)^{1/2}, \quad u \equiv 1 \pmod 2, \text{ or} \tag{6a}$$

$$k = (u-c)/2, \lambda = (u+r-2c)/4, c = (u(r+1) - r)^{1/2}, \quad u \equiv 1 \pmod 2. \tag{7}$$

Then, there exists a matrix W satisfying (4a).

Proof: (i) Let R be the corresponding $u \times u$ $(+1, -1)$-incidence matrix of the (u, k, λ)-CDS whose parameters satisfy (6). Then, from (5) we obtain

$$R^T R = 2(u-r)I_u + (2r-u)J_u.$$

Since

$$R^T J_u = J_u R = (u-k)J_u \text{ and } R^T R + J_u^2 = 2(u-r)I_u + 2rJ_u$$

it follows that the matrix W of the form (4) with $W_1 = R$, $W_2 = J_u$ satisfies (4a).

(ii) Let S be the corresponding $u \times u$ $(+1, -1)$-incidence matrix of the (u, k, λ)-CDS, whose parameters satisfying (6a). Making use of (5) we get

$$S^T S = 2(u - r - 2)I_u - (u - 2r - 4)J_u.$$

Hence the matrix W of the form (4) with $W_1 = S$, $W_2 = -2I_u + J_u$ satisfies (4a).

(iii) Let Q be the corresponding $u \times u$ $(+1, -1)$-incidence matrix of the (u, k, λ)-CDS, whose parameters satisfying (7). Then again from (5) we get

$$Q^T Q = (u - r)I_u + rJ_u.$$

So, the matrix W of the form (4) with $W_1 = W_2 = Q$ satisfies also (4a). □

For the construction of cyclic difference sets of theorem 2 see in Raghavarao [13], Baumert [1] and Table 1 below.

Remark 1: The system of equations (6) or (7) has always a solution for $u = r+2$ or $u = r+4$ respectively (with $c = u-2$) and $R = Q = -2I_u + J_u$.

In the sequel we consider only $1 \le k_1 \le k_2$.

Lemma 3. *Let $u \equiv 1 \pmod 2$, $r \equiv 1 \pmod 2$, $u > r$. Assume that (i) in the decomposition of $u(r+1) - r$ into prime factors, a prime $p \equiv 3 \pmod 4$ appears to an odd power, or (ii) $r \le -3$. Then there do not exist supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$, where k_1, k_2, λ satisfy (4b), (4c).*

Proof: If there exist such supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$, then from (4b) we have (8)

$$2(u(r+1) - r) = \gamma_1^2 + \gamma_2^2 \tag{8}$$

where γ_1, γ_2 is the sum of the elements in each row of the corresponding cyclic incidence matrices. (i) It is known in number theory (Landau [12], p. 135) that the above Diophantine equation has no solution when $2(u(r+1) - r)$ has at least one prime factor $p \equiv 3 \pmod 4$ appearing to an odd power. (ii) For $r \le -3$ it is obvious that (8) has no solution for any $u > 1$. □

Remark 2: For the following values of $u \equiv 1 \pmod 2$, $r \equiv 1 \pmod 2$, $u < 50$, $r < 20$, $u > r \ge 3$ there do not exist supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$, where k_1, k_2, λ satisfy (4b), (4c) (i.e. (8) has no integer solutions):

$r=3, u=9, 15, 27, 33, 41, 45; r=5, u=23, 37, 43; r=7,$
$u=17, 21, 23, 27, 35, 41, 49; r=9, u=15, 17, 21, 31, 33, 35, 39, 47; r=11,$
$u=19, 33, 47; r=13, u=19, 23, 29, 41, 43, 47; r=15,$
$u=21, 25, 27, 33, 39, 45, 47; r=17, u=27, 37, 41, 47; r=19, u=29, 35, 37, 47.$

Next, as in Chadjipantelis and Kounias [4] we facilitate the construction problem of supplementary difference set 2-$\{u; k_1, k_2; \lambda\}$ if we know the existence of a multiplier, i.e. and integer t, $(t, u) = 1$ such that $tC = C + c$, $tD = D + d$ where

$$tC = \{tc_1, tc_2, \dots, tc_{k_1}\} \pmod u, tD = \{td_1, td_2, \dots, td_{k_2}\} \pmod u$$
$$C + c = \{c_1 + c, c_2 + c, \dots, c_{k_1} + c\} \pmod u,$$
$$D + d = \{d_1 + d, d_2 + d, \dots, d_{k_2} + d\} \pmod u.$$

Especially if the shifts $C_1 = (C+c) \pmod u$, $D_1 = (D+d) \pmod u$ are fixed by an integer t, i.e. $t(C+c) \pmod u = (C+c) \pmod u$, $t(D+d) \pmod u = (D+d) \pmod u$, then the supplementary difference sets C_1, D_1 are unions of sets $\{a, ta, \dots, t^{m-1}a\}$ where $t^m a \equiv a \pmod u$, $1 < t < u$, $(t, u) = 1$.

A slight modification of the algorithm given in the aforementioned paper by Chadjipantelis and Kounias [4] described below, is applied for the construction of supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$.

Algorithm 1.: Given r, n, $r \equiv 1 \pmod 2$, $n \equiv 2 \pmod 4$, $n > 2r$, $r \geq 3$

(i) Find non-negative integers k_1, k_2, λ satisfying:

$$(n/2 - 2k_1)^2 + (n/2 - 2k_2)^2 = n(r+1) - 2r$$
$$1 \leq k_1 \leq k_2 \leq (n-2)/4$$
$$\lambda = k_1 + k_2 - (n - 2r)/4$$

(ii) For an integer $t, 1 < t < u$, $u = n/2$, $(t, u) = 1$, form all the sets $\{a, ta, \dots, t^{m-1}a\}$ with

$$t^m a \equiv a \pmod u \text{ for all } a = 0, 1, \dots, (u-1).$$

(iii) Form two sets C, D with k_1, k_2 elements respectively as unions of sets found in step (ii).

(iv) Examine if C, D are supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$.

(v) If the answer in (iv) is negative, go to step (iii) and take another union of sets C, D.

(vi) Repeat step (v) until the answer in (iv) is positive or until all possible combinations of unions of sets C, D are examined.

(vii) If the answer in (vi) is still negative repeat steps (ii)–(vi) for another value of $t, 1 < t < u$, $(t, u) = 1$.

In the sequel we give a construction method of supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$ by using the *non-periodic autocorrelation function of two sequences.*

Let the sequences $A = \{a_1, a_2, \ldots, a_u\}$, $B = \{b_1, b_2, \ldots, b_u\}$ of length u, consisting of the elements of the first row of the $u \times u$ circulant incidence matrices W_1, W_2 respectively, with $a_i = \pm 1$, $b_i = \pm 1$, $i = 1, 2, \ldots, u$.

The non-periodic autocorrelation functions $N_A(s), N_B(s)$ are defined as

$$N_A(s) = \sum_{i=1}^{u-s} a_i a_{i+s},\ N_B(s) = \sum_{i=1}^{u-s} b_i b_{i+s},\ s = 0, 1, \ldots, u-1$$

and the *polynomials* associated with A and B (also called *generating functions*) are

$$A(x) = a_1 + a_2 x + \cdots + a_u x^{u-1},\ B(x) = b_1 + b_2 x + \cdots + b_u x^{u-1},$$

Then for every $x \neq 0$

$$A(x)A(x^{-1}) = \sum_{i=1}^{u}\sum_{j=1}^{u} a_i a_j x^{i-j},\ B(x)B(x^{-1}) = \sum_{i=1}^{u}\sum_{j=1}^{u} b_i b_j x^{i-j},$$

and

$$\begin{aligned} A(x)A(x^{-1}) + B(x)B(x^{-1}) &= N_A(0) + N_B(0) \\ &+ \sum_{s=1}^{u-1}[N_A(s) + N_B(s) + (N_A(u-s) + N_B(u-s))x^{-u}]x^s. \end{aligned} \tag{9}$$

Lemma 4. *Suppose there exist supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$, where k_1, k_2 satisfy (4b) and $\lambda = k_1 + k_2 - (u-r)/2$. If W_1, W_2 are the cyclic incidence matrices and A, B the corresponding sequences of length u, then*

$$A(x)A(x^{-1}) + B(x)B(x^{-1}) = 2(ur + u - r),\ \textit{if } x = 1 \tag{10}$$

$$A(x)A(x^{-1}) + B(x)B(x^{-1}) = 2(u - r),\ \textit{if } x^u = 1, x \neq 1. \tag{11}$$

Proof: Let there exist such two supplementary difference sets, i.e. the relation (3) is valid. Then (3) is equivalent to

$$N_A(0) + N_B(0) = 2u,\ s = 0 \tag{12a}$$

$$\begin{aligned} &N_A(s) + N_B(s) + N_A(u-s) \\ &\quad + N_B(u-s) = 2r,\ 1 \leq s \leq u-1. \end{aligned} \tag{12b}$$

Let $x = 1$. Then from (9) we have

$$\begin{aligned} A(x)A(x^{-1}) + B(x)B(x^{-1}) &= N_A(0) + N_B(0) \\ &+ \sum_{s=1}^{u-1}[N_A(s) + N_B(s) + N_A(u-s) + N_B(u-s)] \\ &= 2u + 2r(u-1) \text{ (from (12)).} \end{aligned}$$

Let $x^u = 1$, $x \neq 1$. Then again from (9) we get

$$\begin{aligned} A(x)A(x^{-1}) + B(x)B(x^{-1}) &= N_A(0) + N_B(0) \\ &+ \sum_{s=1}^{u-1}[N_A(s) + N_B(s) + N_A(u-s) + N_B(u-s)]x^s \\ &= 2u + 2r\sum_{s=1}^{u-1} x^s = 2(u-r) \text{ (from (12)).} \end{aligned}$$

since $1 + x + \cdots + x^{u-1} = 0$. □

For every given sequence $H = \{h_1, h_2, \ldots, h_u\}$ and for some $m \in \{2, 3, \ldots, u\}$ we define the m subsequences H_i, $i = 1, 2, \ldots, m$, where

$$H_i = \{h_i, h_{i+m}, \ldots, h_{i+s_i m}\},\ s_i = \text{int}((u-i)/m), i = 1, 2, \ldots, m$$

with associated polynomials

$$H_i(x) = \sum_{p=0}^{s_i} h_{i+pm}x^p, i = 1, 2, \ldots, m.$$

Then

$$H(x) = H_1(x^m) + xH_2(x^m) + \cdots + x^{m-1}H_m(x^m)$$

and hence for every $x \neq 0$

$$\begin{aligned} A(x)A(x^{-1}) + B(x)B(x^{-1}) &= \sum_{i=1}^{m}(A_i(x^m)A_i(x^{-m}) + B_i(x^m)B_i(x^{-m})) \\ &+ \sum_{j=1}^{m-1}\left\{\sum_{i=1}^{m-s}(A_i(x^m)A_{i+s}(x^{-m}) + B_i(x^m)B_{i+s}(x^{-m}))\right. \\ &\left. + x^m\sum_{i=1}^{s}(A_{i+m-s}(x^m)A_i(x^{-m}) + B_{i+m-s}(x^m)B_i(x^{-m}))\right\}x^{-j}, \end{aligned} \tag{13}$$

where A_i, B_i are the corresponding subsequences of A, B respectively.

Theorem 5. *Suppose there exist supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$, where k_1, k_2 satisfy (4b) and $\lambda = k_1 + k_2 - (u-r)/2$. Let W_1, W_2 be the corresponding cyclic incidence matrices and A, B be their sequences of length $u = mw \equiv 1 \pmod 2$, $m, w > 1$ and m is a prime. Let also*

(a) $k_{im} = \sum_{p \equiv i \mod m} a_p, r_{im} = \sum_{p \equiv i \mod m} b_p$

(b) $K_m = \{k_{1m}, \dots, k_{mm}\}$, $R_m = \{r_{1m}, \dots, r_{mm}\}$

(c) $N_{K_m}(s) = \sum_{i=1}^{m-s} k_{im}k_{i+s,m}$, $N_{R_m}(s) = \sum_{i=1}^{m-s} r_{im}r_{i+s,m}$.

Then, for the given m

$$\begin{aligned} N_{K_m}(0) + N_{R_m}(0) &= k_{1m}^2 + \cdots + k_{mm}^2 + r_{1m}^2 + \cdots + r_{mm}^2 \\ &= 2u + 2(w-1)r, w = u/m \end{aligned} \tag{14}$$

$$\begin{aligned} &N_{K_m}(s) + N_{K_m}(m-s) + N_{R_m}(s) + N_{R_m}(m-s) \\ &= 2wr, s = 1, 2, \dots, (m-1)/2. \end{aligned} \tag{15}$$

Proof: Let there exist such two supplementary difference sets. Then the relations (10) and (11) in Lemma 2 are valid.

Let $x^m = 1$, $x \neq 1$ for the given m. Then $x^u = 1$ and since $A_i(1) = K_{im}$, $B_i(1) = r_{im}$, $i = 1, \dots, m, x^{-j} = x^{m-j}$, the relation (11) with the help of (9) becomes

$$\sum_{i=1}^{m}(k_{im}^2 + r_{im}^2) + \sum_{j=1}^{m-1}\left\{\sum_{i=1}^{m-s}(k_{im}k_{i+s,m} + r_{im}r_{i+s,m}) + \sum_{i=1}^{s}(k_{im}k_{i+m-s,m} \right.$$
$$\left. + r_{im}r_{i+m-s,m})\right\} x^{m-j} = 2(u-r), s = 1, 2, \dots, (m-1).$$

or

$$N_{K_m}(0) + N_{R_m}(0) + \sum_{j=1}^{m-1}(N_{K_m}(s) + N_{K_m}(m-s) + N_{R_m}(s)$$
$$+ N_{R_m}(m-s))x^{m-j} = 2(u-r)$$

and since $x^m = 1$ we get

$$\begin{aligned} N_{K_m}(0) + N_{R_m}(0) - (N_{K_m}(s) + N_{K_m}(m-s) \\ + N_{R_m}(s) + N_{R_m}(m-s)) = 2(u-r) \end{aligned} \tag{16}$$

for $s = 1, 2, \dots, (m-1)$. Let $x = 1$. Then, the relation (1) with the help of (9) deduces to

$$\sum_{i=1}^{m}(k_{im}^2 + r_{im}^2) + \sum_{j=1}^{m-1}\left\{\sum_{i=1}^{m-s}(k_{im}k_{i+s,m} + r_{im}r_{i+s,m}) \right.$$
$$\left. + \sum_{i=1}^{s}(k_{im}k_{i+m-s,m} + r_{im}r_{i+m-s,m})\right\} = 2(ur + u - r),$$
$$s = 1, 2, \dots, (m-1)$$

or

$$N_{K_m}(0) + N_{R_m}(0) + (m-1)((N_{K_m}(s) + N_{K_m}(m-s) + N_{R_m}(s) + N_{R_m}(m-s)) = 2(ur+u-r) \tag{17}$$

for $s = 1, 2, \ldots, (m-1)$.

Therefore the relations (14), (15) can be easily obtained from (16), (17). By taking complex conjugates in (9), (10) and (11) we see that it is enough to take $s = 1, 2, \ldots, \mathrm{int}(m/2)$, i.e. $s = 1, 2, \ldots, (m-1)/2$ since m is a prime. □

Algorithm 2: Since it is difficult to find directly the values $a_1, \ldots, a_u$; $b_1, \ldots, b_u$ of the sequences A, B respectively, we find the values of integers $k_{1m}, \ldots, k_{mm}$; $r_{1m}, \ldots, r_{mm}$ as defined from relations (a) in Theorem 3.

Step 1: Given r and u find all the integers solutions of the system (18):

$$\begin{gathered} 2(ur+u-r) = k_{11}^2 + r_{11}^2 \\ k_{11} \geq r_{11} > 0, k_{11} \equiv 1 \pmod 2, r_{11} \equiv 1 \pmod 2 \\ u \equiv 1 \pmod 2, r \equiv 1 \pmod 2, u > r \geq 3. \end{gathered} \tag{18}$$

Step 2: For every solution k_{11}, r_{11} of the system (18) and a given prime $m > 1$, with $u = mw$ find $k_{1m}, \ldots, k_{mm}$; $r_{1m}, \ldots, r_{mm}$ satisfying

(i) $k_{11} = k_{1m} + \cdots + k_{mm}$; $r_{11} = r_{1m} + \cdots + r_{mm}$

(ii) $k_{1m} \geq k_{2m} \geq k_{3m}, \ldots, k_{mm}$; $r_{1m} \geq r_{2m}, \ldots, r_{mm}$

(iii) $k_{1m}, \ldots, k_{mm}$; $r_{1m}, \ldots, r_{mm}$ are all odd (even) if $\mathrm{int}((u-i)/m) + 1$ is odd (even), $i = 1, \ldots, m$

(iv) $|k_{im}| \leq \mathrm{int}((u-i)/m) + 1$, $|r_{im}| \leq \mathrm{int}((u-i/m) + 1$ $i = 1, \ldots, m$

(v) $k_{1m}^2 + \cdots + k_{mm}^2 + r_{1m}^2 + \cdots + r_{mm}^2 = 2u + 2(w-1)r$, $w = u/m$

(vi) $N_{K_m}(s) + N_{K_m}(m-s) + N_{R_m}(s) + N_{R_m}(m-s) = 2wr$, $s = 1, 2, \ldots, (m-1)/2$, where

$$N_{K_m}(s) = \sum_{i=1}^{m-s} k_{im} k_{i+s,m}, \quad N_{R_m}(s) = \sum_{i=1}^{m-s} r_{im} r_{i+s,m}.$$

Step 3: (i) For every set $k_{1m}, \ldots, k_{mm}$; $r_{1m}, \ldots, r_{mm}$ found in step 2, find

$$k_{1,wm}, \ldots, k_{wm,wm}; r_{1,wm}, \ldots, r_{wm,wm}$$

satisfying (ii),(iii) and (iv) in step 2. (ii) Go to step 2 (v),(vi), setting wm instead of m.

Step 4: Stop when $m = u$ and examine if

$$N_{K_m}(0) + N_{R_m}(0) = 2u$$
$$N_{K_m}(s) + N_{K_m}(m-s) + N_{R_m}(s) + N_{R_m}(m-s) = 2r,$$
$$s = 1, 2, \ldots, (m-1).$$

□

3. D-Optimal complex linear designs for $rn \equiv 2 \pmod 4$, $r \equiv 1 \pmod 2$

Let $B_1, B_2, \ldots, B_r$ be $n \times q$ matrices with entries in some closed interval and let $D(r,n)$ be the set of all $rn \times (n+q)$ matrices X, where

$$X = \begin{bmatrix} I_n & I_n & \ldots & I_n \\ B_1^T & B_2^T & \ldots & B_r^T \end{bmatrix}^T .$$

Such matrices, called *design matrices*, arise in *linear equireplicated covariates models*, as defined by J.L. Troya [15], for comparing $n \geq 2$ treatments (or one treatment at n levels), making r observations (the common replication number) per treatment level, and it is known that for each subject entering the study, q continuous covariates in a completely randomized statistical set-up can be observed, each of them assuming values on a closed interval. Since these models involve both discrete (ANOVA type) and continuous (regression type) factors of influence, they are also called "complex linear models". For more information regarding covariates models see Harville [8], J.L. Troya [15], Kurotschka [11], Wierich [21, 22], Chadjiconstantinidis and Moyssiadis [3], Chadjiconstantinidis and Chadjipadelis [2]. Our primary interest is in the joint estimation of regression coefficients (covariates) in addition to the estimation of treatment contrasts under the *D-optimality criterion*, which is related to the maximization of the determinant of the *information matrix* $M = X^TX$, when $\text{rank}(X) = n+q$. Any such design matrix $X^* \in D(r,n)$ maximizing the determinant of X^TX over all $X \in D(r,n)$ is called "D-optimal complex linear design". It is known (J.L. Troya [15], Lemma 2.1) that $\det(X^TX)$ attains its maximum value for a design matrix X such that B_j has all its entries in $\{+1,-1\}$ for all $j = 1, 2, \ldots, r$. Also for $r \equiv 1 \pmod 2$ (J.L. Troya [15], Theorem 3.1) it holds

$$\max \det(X^TX) = r^n(N - n/r)^q, N = rn. \tag{19}$$

For $N \equiv 2 \pmod 4$, $q \geq 3$, J.L. Troya [15] proved that a necessary and sufficient condition for the existence of D-optimal complex linear designs is $n \geq 2r$. In the same paper, see Theorem 3.10, she constructed D-optimal complex linear designs for $n = 2r$, $q \leq n$ or $n > 2r$, $q \leq 8n/(n-2r)$. Chadjiconstantinidis and Moyssiadis [3] constructed D-optimal complex linear designs when $q \leq n - 2r$ (Theorem 3.3) or $q \leq 2r$ (Theorem 3.5). In this section we give a construction of D-optimal complex linear designs when the maximum number of covariates $q = n$ for $n > 2r$.

Let $n = 2u$, $u \equiv 1 \pmod 2$ and $q = n$. In the case $q < n$, the D-optimal complex linear designs can be obtained by deleting appropriate columns of the D-optimal complex linear designs for $q = n$.

Theorem 6. *Suppose there exist $n \times n(+1,-1)$-matrices U and W such that*

$$U^T U = 2[(u+1)\mathbf{I}_u - \mathbf{J}_u] \otimes \mathbf{I}_2 \tag{20}$$

$$W^T W = 2[(u-r)\mathbf{I}_u + r\mathbf{J}_u] \otimes \mathbf{I}_2. \tag{21}$$

Then the design matrix $X^* \in D(r,n)$ given by

$$B_j^* = (-1)^{j+1}U, j = 1,2,\ldots,(r-1)$$

$$B_r^* = W$$

is a D-optimal complex linear design.

Proof: By definition of matrices B_j^* we find that

$$M_{12}^* = \sum_{j=1}^{r} B_j^* = W, M_{22}^* = \sum_{j=1}^{r} B_j^{*^T} B_j^* = 2[(ru-1)\mathbf{I}_u + J_u] \otimes \mathbf{I}_2.$$

Thus

$$\det(M^*) = \det(X^{*^T} X^*) = r^n \det(M_{22}^* - 1/r M_{12}^{*^T} M_{12}^*)$$
$$= r^n (N - n/r)^n$$

and making use of (19) the result follows. □

Hence the construction of D-optimal complex linear designs when $N = rn \equiv 2 \pmod 4$, $u = n/2 \equiv 1 \pmod 2$, $q = n$ is equivalent to the construction of two matrices U, W satisfying (20), (21) respectively.

The problem of construction of W is examined in the previous section.

Let U_1, U_2 are $u \times u(+1,-1)$-commuting matrices such that

$$U_1^T U_1 + U_2^T U_2 = 2(u+1)\mathbf{I}_u - 2\mathbf{J}_u. \tag{21}$$

Then the $n \times n$ matrix U, where

$$U = \begin{bmatrix} U_1^T & U_2 \\ U_2^T & -U_1 \end{bmatrix}$$

satisfies the relation (20).

Let U_1, U_2 both be circulant. Then from Theorem 1 it follows that their existence is equivalent to the existence of two supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$ where k_1, k_2 and λ are found from (setting $r = -1$ in (4b), (4c))

$$(u - 2k_1)^2 + (u - 2k_2)^2 = 2, \lambda = k_1 + k_2 - (u+1)/2$$

i.e. from

$$\begin{array}{lll} k_1 = k_2 = (u-1)/2, & \lambda = (u-3)/2, \text{ or} & \\ k_1 = k_2 = (u+1)/2, & \lambda = (u+1)/2, \text{ or} & \\ k_1 = (u \pm 1)/2, & k_2 = (u \mp 1)/2, & \lambda = (u-1)/2. \end{array}$$

Theorem 7. *Let $u \equiv 1 \pmod 2$ be a positive integer. Then there exist supplementary difference sets*

(i) 2-$\{u; (u-1)/2, (u+1)/2; (u-1)/2\}$, *if* $u \equiv 1 \pmod 4$ *is a prime*

(ii) 2-$\{u; (u-1)/2, (u-1)/2; (u-3)/2\}$, *if* $u \equiv 3 \pmod 4$ *is a prime or* $u = p(p+2)$, p *is prime (twin primes).*

Proof: When u is a prime, consider the Galois Field of u elements, GF(u), and define the matrix $Q = [q_{ij}]$ where $q_{ij} = \chi(a_j - a_i)$, $0 \le i, j \le u-1$ and χ is the Legendre symbol (or quadratic character) defined on the elements of GF(u). It is known (cf. Wallis, Street and Wallis [19], Lemma 1.19) that Q is circulant, has diagonal zero and Q is symmetric if $u \equiv 1 \pmod 4$ and is skew-symmetric if $u \equiv 3 \pmod 4$.

Thus, if $q_1, q_2, \ldots, q_{(u-1)/2}$ are the non-zero quadratic residues $\pmod u$ of the GF(u), the required two supplementary difference sets C and D are given by:

(i) $C = \{q_1 + 1, q_2 + 1, \ldots, q_{(u-1)/2} + 1\}$, $D = \{1\} \cup C$, if $u \equiv 1 \pmod 4$ with corresponding incidence matrices $U_1 = Q + \mathbf{I}_u$, $U_2 = Q - \mathbf{I}_u$

(ii) $C = D = \{q_1, q_2, \ldots, q_{(u-1)/2}\}$, if $u \equiv 3 \pmod 4$ with corresponding incidence matrices $U_1 = U_2 = Q + \mathbf{I}_u$.

When $u = p(p+2)$, p is a prime, then the set C of powers of a generator of the Galois domain plus the multiples of $p+2$ is a $(u, (u-1)/2, (u-3)/4)$-CDS. So, the required two supplementary difference sets are the sets C and $D = C$. □

Remark 3: Let $u \equiv 1 \pmod 4$ be a prime. The set of non-zero quadratic residues $\pmod u$ and the set of the non-quadratic residues $\pmod u$ of the GF(u) constitute supplementary difference sets 2-$\{u; (u-1)/2, (u-$

$1)/2; (u-3)/2\}$, which are equivalent to those given in Theorem 7(i) by definition (see also in Street and Wallis [14], p.194). Note that for $u = 9$ and $u = 27$ there exist supplementary difference sets 2-$\{9; 4, 4; 3\}$ and 2-$\{27; 13, 13; 12\}$ given respectively by

$$C = \{1, 2, 3, 5\}, D = \{1, 2, 5, 7\} \text{ and}$$
$$C = D = \{x^{2a} : a = 0, 1, \ldots, 12\}$$

where x is a primitive element of the field GF(3^3).

Also for $u = 39$, with the help of Algorithm 1, we construct for the first time the two supplementary difference sets 2-$\{39; 19, 19; 18\}$ (see in Table 1).

4. Tables of 2-$\{u; k_1, k_2; \lambda\}$ SDS

In Table 1 are indicated the values of the parameters of all 201 supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$ and also the method of their construction for all $u \equiv 1 \pmod 2$, $u < 50$. The symbol $*$ is referred to known supplementary difference sets corresponding to D-optimal first order designs (see in [4], [5], [6], [7], [9], [10], [16], [20], [23–26]).

Table 1
Supplementary difference sets 2-$\{u; k_1, k_2; \lambda\}$
with k_1, k_2, λ satisfying (4b), (4c)

$u = 3; k_1 = 1, k_2 = 1; \lambda = 0, r = -1$	Th.7
$u = 5; k_1 = 1, k_2 = 1; \lambda = 0, r = 1$	Trivial
$u = 5; k_1 = 2, k_2 = 2; \lambda = 1, r = -1$	Th.7
$u = 7; k_1 = 1, k_2 = 1; \lambda = 0, r = 3$	Trivial
$u = 7; k_1 = 1, k_2 = 3; \lambda = 1, r = 1$	$*$; Th.2
$C = \{1\}, D = \{1, 2, 4\}$	
$u = 7; k_1 = 3, k_2 = 3; \lambda = 2, r = -1$	Th.7
$u = 9; k_1 = 1, k_2 = 1; \lambda = 0, r = 5$	Trivial
$u = 9; k_1 = 2, k_2 = 3; \lambda = 1, r = 1$	$*$
$u = 9; k_1 = 4, k_2 = 4; \lambda = 3, r = -1$	Remark 3; ALG1
$C = \{1, 2, 4, 9\}, D = \{1, 4, 6, 9\}$	
$u = 11; k_1 = 1, k_2 = 1; \lambda = 0, r = 7$	Trivial
$u = 11; k_1 = 1, k_2 = 5; \lambda = 2, r = 3$	Th.2
$C = \{1\}, D = \{1, 3, 4, 5, 9\}$	
$u = 11; k_1 = 5, k_2 = 5; \lambda = 4, r = -1$	Th.2; Th.7
$C = D = \{1, 3, 4, 5, 9\}$	

Table 1 (continued)

$u=13; k_1=1, k_2=1; \lambda=0, r=9$	Trivial
$u=13; k_1=1, k_2=4; \lambda=1, r=5$	Th.2
$C=\{1\}, D=\{1,3,9,13\}$	
$u=13; k_1=3, k_2=3; \lambda=1, r=3$	ALG1
$C=\{1,3,9\}, D=\{2,5,6\}$	
$u=13; k_1=3, k_2=6; \lambda=3, r=1$	*
$u=13; k_1=4, k_2=4; \lambda=2, r=1$	*
$u=13; k_1=6, k_2=6; \lambda=5, r=-1$	Th.7
$u=15; k_1=1, k_2=1; \lambda=0, r=11$	Trivial
$u=15; k_1=1, k_2=7; \lambda=3, r=5$	Th.2; ALG1, ALG2
$C=\{1\}, D=\{1,2,4,5,8,10,15\}$	
$u=15; k_1=2, k_2=4; \lambda=1, r=5$	ALG1, ALG2
$C=\{1,15\}, D=\{2,5,9,15\}$	
$u=15; k_1=4, k_2=6; \lambda=3, r=1$	*
$u=15; k_1=7, k_2=7; \lambda=6, r=-1$	Th.7
$u=17; k_1=1, k_2=1; \lambda=0, r=13$	Trivial
$u=17; k_1=2, k_2=6; \lambda=2, r=5$	ALG1, no solution
$u=17; k_1=3, k_2=7; \lambda=3, r=3$	ALG1, no solution
$u=17; k_1=4, k_2=5; \lambda=2, r=3$	ALG1
$C=\{1,4,6,17\}, D=\{1,4,9,11,17\}$	
$u=17; k_1=8, k_2=8; \lambda=7, r=-1$	Th.7
$u=19; k_1=1, k_2=1; \lambda=0, r=15$	Trivial
$u=19; k_1=1, k_2=9; \lambda=4, r=7$	Th.2
$C=\{1\}, D=\{1,4,5,6,7,9,11,16,17\}$	
$u=19; k_1=3, k_2=4; \lambda=1, r=7$	ALG1
$C=\{2,5,19\}, D=\{1,7,11,19\}$	
$u=19; k_1=3, k_2=6; \lambda=2, r=5$	ALG1
$C=\{1,7,11\}, D=\{2,3,4,6,9,11\}$	
$u=19; k_1=4, k_2=7; \lambda=3, r=3$	ALG1
$C=\{1,7,11,19\}, D=\{1,4,6,7,9,11,19\}$	
$u=19; k_1=6, k_2=7; \lambda=4, r=1$	*
$u=19; k_1=9, k_2=9; \lambda=8, r=-1$	Th.7
$u=21; k_1=1, k_2=1; \lambda=0, r=17$	Trivial
$u=21; k_1=1, k_2=5; \lambda=1, r=11$	Th.2
$C=\{1\}, D=\{3,6,7,12,14\}$	
$u=21; k_1=5, k_2=5; \lambda=2, r=5$	Th.2
$C=D=\{3,6,7,12,14\}$	
$u=21; k_1=6, k_2=6; \lambda=3, r=3$	ALG2, no solution
$u=21; k_1=6, k_2=10; \lambda=6, r=1$	*
$u=21; k_1=10, k_2=10; \lambda=9, r=-1$	?

Table 1 (continued)

$u = 23; k_1 = 1, k_2 = 1; \lambda = 0, r = 19$	Trivial
$u = 23; k_1 = 1, k_2 = 11; \lambda = 5, r = 9$	Th.2
$C = \{1\}, D = \{1,2,3,4,6,8,9,12,13,16,18\}$	
$u = 23; k_1 = 2, k_2 = 5; \lambda = 1, r = 11$	ALG1
$C = \{1,23\}, D = \{2,7,10,19,23\}$	
$u = 23; k_1 = 2, k_2 = 7; \lambda = 2, r = 9$	ALG1, no solution
$u = 23; k_1 = 5, k_2 = 10; \lambda = 5, r = 3$	?
$u = 23; k_1 = 7, k_2 = 10; \lambda = 6, r = 1$	*
$u = 23; k_1 = 11, k_2 = 11; \lambda = 10, r = -1$	Th.7
$u = 25; k_1 = 1, k_2 = 1; \lambda = 0, r = 21$	Trivial
$u = 25; k_1 = 1, k_2 = 9; \lambda = 3, r = 11$	ALG2, no solution
$u = 25; k_1 = 3, k_2 = 7; \lambda = 2, r = 9$	ALG2, no solution
$u = 25; k_1 = 3, k_2 = 10; \lambda = 4, r = 7$	ALG2, no solution
$u = 25; k_1 = 4, k_2 = 4; \lambda = 1, r = 11$	ALG2, no solution
$u = 25; k_1 = 4, k_2 = 12; \lambda = 6, r = 5$	ALG2
$C = \{10,20,23,24\}$,	
$D = \{2,4,9,10,14,17,18,20,21,23\}$	
$u = 25; k_1 = 6, k_2 = 7; \lambda = 3, r = 5$	ALG2
$C = \{5,10,18,20,23,24\}$,	
$D = \{4,13,15,21,22,24,25\}$	
$u = 25; k_1 = 6, k_2 = 10; \lambda = 5, r = 3$	ALG2
$C = \{13,15,20,23,24,25\}$,	
$D = \{2,4,9,10,14,17,18,20,21,23\}$	
$u = 25; k_1 = 9, k_2 = 9; \lambda = 6, r = 1$	*
$u = 25; k_1 = 12, k_2 = 12; \lambda = 11, r = -1$	?
$u = 27; k_1 = 1, k_2 = 1; \lambda = 0, r = 23$	Trivial
$u = 27; k_1 = 1, k_2 = 13; \lambda = 6, r = 11$	ALG1, ALG2, no solution
$u = 27; k_1 = 3, k_2 = 5; \lambda = 1, r = 13$	ALG1, ALG2
$C = \{1,8,27\}, D = \{2,5,11,15\}$	
$u = 27; k_1 = 3, k_2 = 9; \lambda = 3, r = 9$	ALG1, ALG2, no solution
$u = 27; k_1 = 5, k_2 = 11; \lambda = 5, r = 5$	ALG2
$C = \{15,17,23,24,27\}$,	
$D = \{3,6,9,11,13,20,21,22,25,26,27\}$	
$u = 27; k_1 = 9, k_2 = 11; \lambda = 7, r = 1$	no solution
$u = 27; k_1 = 13, k_2 = 13; \lambda = 12, r = -1$	Remark 3
$u = 29; k_1 = 1, k_2 = 1; \lambda = 0, r = 25$	Trivial
$u = 29; k_1 = 1, k_2 = 8; \lambda = 2, r = 15$	no solution
$u = 29; k_1 = 2, k_2 = 11; \lambda = 4, r = 11$	?

Table 1 (continued)

$u = 29; k_1 = 4, k_2 = 9; \lambda = 3, r = 9$	?
$u = 29; k_1 = 4, k_2 = 13; \lambda = 6, r = 7$	?
$u = 29; k_1 = 6, k_2 = 11; \lambda = 5, r = 5$	?
$u = 29; k_1 = 7, k_2 = 7; \lambda = 3, r = 7$	ALG1
$C = \{1, 7, 16, 20, 23, 24, 25\}, D = \{2, 3, 11, 14, 17, 19, 21\}$	
$u = 29; k_1 = 7, k_2 = 14; \lambda = 8, r = 3$	ALG1
$C = \{1, 7, 16, 20, 23, 24, 25\},$	
$D = \{1, 4, 5, 6, 7, 9, 13, 16, 20, 22, 23, 24, 25, 28\}$	
$u = 29; k_1 = 8, k_2 = 8; \lambda = 4, r = 5$	ALG1
$C = \{1, 7, 16, 20, 23, 24, 25, 29\},$	
$D = \{2, 3, 11, 14, 17, 19, 21, 29\}$	
$u = 29; k_1 = 14, k_2 = 14; \lambda = 13, r = -1$	Th.7
$u = 31; k_1 = 1, k_2 = 1; \lambda = 0, r = 27$	Trivial
$u = 31; k_1 = 1, k_2 = 6; \lambda = 1, r = 16$	Th.2
$C = \{1\}, D = \{1, 5, 11, 24, 25, 27\}$	
$u = 31; k_1 = 1, k_2 = 10; \lambda = 3, r = 15$	no solution
$u = 31; k_1 = 1, k_2 = 15; \lambda = 7, r = 13$	Th.2
$C = \{1\}, D = \{1, 2, 3, 4, 6, 8, 12, 15, 16, 17, 23, 24, 27, 29, 30\}$	
$u = 31; k_1 = 6, k_2 = 6; \lambda = 2, r = 11$	Th.2
$C = D = \{1, 5, 11, 24, 25, 27\}$	
$u = 31; k_1 = 6, k_2 = 10; \lambda = 4, r = 7$	ALG1
$C = \{1, 2, 4, 8, 16, 31\},$	
$D = \{3, 6, 7, 12, 14, 17, 19, 24, 25, 28\}$	
$u = 31; k_1 = 6, k_2 = 15; \lambda = 8, r = 5$	ALG1
$C = \{2, 3, 10, 13, 15, 19\},$	
$D = \{1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28\}$	
$u = 31; k_1 = 10, k_2 = 10; \lambda = 6, r = 3$	ALG1
$C = \{1, 2, 3, 4, 6, 8, 12, 16, 17, 24\},$	
$D = \{1, 2, 4, 7, 8, 14, 16, 19, 25, 28\}$	
$u = 31; k_1 = 10, k_2 = 15; \lambda = 10, r = 1$	*
$u = 31; k_1 = 15, k_2 = 15; \lambda = 14, r = -1$	Th.7
$u = 33; k_1 = 1, k_2 = 1; \lambda = 0, r = 29$	Trivial
$u = 33; k_1 = 2, k_2 = 6; \lambda = 1, r = 19$	?
$u = 33; k_1 = 3, k_2 = 10; \lambda = 3, r = 13$	ALG2
$C = \{11, 30, 32\}, D = \{8, 11, 17, 21, 25, 26, 28, 31, 32, 33\}$	
$u = 33; k_1 = 4, k_2 = 5; \lambda = 1, r = 17$	?
$u = 33; k_1 = 7, k_2 = 14; \lambda = 7, r = 5$	ALG2
$C = \{9, 10, 11, 22, 28, 29, 33\},$	
$D = \{6, 10, 14, 15, 17, 18, 20, 22, 24, 25, 27, 30, 31, 33\}$	

Table 1 (continued)

$u = 33; k_1 = 8, k_2 = 9; \lambda = 4, r = 7$	ALG2, no solution
$u = 33; k_1 = 11, k_2 = 15; \lambda = 10, r = 1$	*
$u = 33; k_1 = 12, k_2 = 13; \lambda = 9, r = 1$	*
$u = 33; k_1 = 16, k_2 = 16; \lambda = 15, r = -1$	?
$u = 35; k_1 = 1, k_2 = 1; \lambda = 0, r = 31$	Trivial
$u = 35; k_1 = 1, k_2 = 17; \lambda = 8, r = 15$	Th.2
$C = \{1\}, D = \{1,3,4,7,9,11,12,13,14,16,$ $17,21,27,28,29,33,35\}$	
$u = 35; k_1 = 4, k_2 = 8; \lambda = 2, r = 15$	?
$u = 35; k_1 = 4, k_2 = 10; \lambda = 3, r = 13$	?
$u = 35; k_1 = 6, k_2 = 9; \lambda = 3, r = 11$	?
$u = 35; k_1 = 8, k_2 = 14; \lambda = 7, r = 5$	ALG2, no solution
$u = 35; k_1 = 9, k_2 = 12; \lambda = 6, r = 5$	ALG2
$C = \{12,13,17,21,24,30,31,32,35\}$,	
$D = \{4,5,10,18,20,21,24,26,28,30,31,33\}$	
$u = 35; k_1 = 10, k_2 = 14; \lambda = 8, r = 3$	ALG2, no solution
$u = 35; k_1 = 17, k_2 = 17; \lambda = 16, r = -1$	Th.7
$u = 37; k_1 = 1, k_2 = 1; \lambda = 0, r = 33$	Trivial
$u = 37; k_1 = 1, k_2 = 9; \lambda = 2, r = 21$	Th.2
$C = \{1\}, D = \{1,7,9,10,12,16,26,33,34\}$	
$u = 37; k_1 = 3, k_2 = 6; \lambda = 1, r = 21$	ALG1
$C = \{1,10,26\}, D = \{2,5,13,15,19,20\}$	
$u = 37; k_1 = 3, k_2 = 15; \lambda = 6, r = 13$	ALG1
$C = \{1,10,26\}, D = \{3,4,5,13,17,18,$ $19,21,22,24,25,28,30,32,35\}$	
$u = 37; k_1 = 4, k_2 = 12; \lambda = 4, r = 13$	?
$u = 37; k_1 = 4, k_2 = 16; \lambda = 7, r = 11$	ALG1
$C = \{1,10,26,37\}, D = \{1,5,7,10,11,13,$ $14,19,26,27,29,31,33,34,36,37\}$	
$u = 37; k_1 = 6, k_2 = 7; \lambda = 2, r = 15$	ALG1
$C = \{1,3,4,10,26,30\}$,	
$D = \{2,15,20,21,25,28,37\}$	
$u = 37; k_1 = 7, k_2 = 15; \lambda = 7, r = 7$	ALG1
$C = \{3,4,5,13,19,30,37\}, D = \{2,6,8,11,15,$ $17,20,21,22,23,25,27,28,35,36\}$	
$u = 37; k_1 = 9, k_2 = 9; \lambda = 4, r = 9$	Th.2
$C = D = \{1,7,9,10,12,16,26,33,34\}$	
$u = 37; k_1 = 10, k_2 = 10; \lambda = 5, r = 7$	ALG1
$C = \{1,3,4,5,10,13,19,26,30,37\}$,	
$D = \{1,2,6,8,10,15,20,23,26,37\}$	

Table 1 (continued)

$u = 37; k_1 = 10, k_2 = 18; \lambda = 11, r = 3$	ALG1
$C = \{1,3,4,5,10,13,19,26,30,37\}, D = \{1,3,4,7, 9,10,11,12,16,21,25,26,27,28,30,33,34,36\}$	
$u = 37; k_1 = 12, k_2 = 13; \lambda = 8, r = 3$	ALG1
$C = \{2,3,4,5,11,13,15,19,20,27,30,36\}$, $D = \{1,2,5,6,8,10,13,15,19,20,23,26,37\}$	
$u = 37; k_1 = 13, k_2 = 16; \lambda = 11, r = 1$	*
$u = 37; k_1 = 18, k_2 = 18; \lambda = 17, r = -1$	Th.7
$u = 39; k_1 = 1, k_2 = 1; \lambda = 0, r = 35$	Trivial
$u = 39; k_1 = 1, k_2 = 19; \lambda = 9, r = 17$	no solution
$u = 39; k_1 = 5, k_2 = 8; \lambda = 2, r = 17$	ALG1
$C = \{1,16,22,26,39\}, D = \{2,4,5,10,13,25,32,39\}$	
$u = 39; k_1 = 5, k_2 = 12; \lambda = 4, r = 13$	?
$u = 39; k_1 = 7, k_2 = 9; \lambda = 3, r = 13$	ALG1
$C = \{1,2,5,16,22,32,39\}$, $D = \{4,6,10,13,15,18,25,26,39\}$	
$u = 39; k_1 = 7, k_2 = 11; \lambda = 4, r = 11$	?
$u = 39; k_1 = 8, k_2 = 15; \lambda = 7, r = 7$	ALG1
$C = \{1,3,9,13,16,22,27,39\}$, $D = \{3,4,5,7,8,13,14,15,19,22,24,26,36,38,39\}$	
$u = 39; k_1 = 9, k_2 = 13; \lambda = 6, r = 7$	ALG1
$C = \{1,2,3,5,9,16,22,27,32\}$, $D = \{1,2,5,6,7,13,15,16,18,22,32,34,37\}$	
$u = 39; k_1 = 11, k_2 = 13; \lambda = 7, r = 5$	ALG1
$C = \{1,2,3,5,9,13,16,22,27,32,39\}$, $D = \{3,4,7,9,10,13,14,25,27,29,34,35,37\}$	
$u = 39; k_1 = 12, k_2 = 15; \lambda = 9, r = 3$	ALG1
$C = \{1,2,4,5,6,10,15,16,18,22,25,32\}$, $D = \{1,2,3,4,5,9,10,12,16,22,25,27,30,32,36\}$	
$u = 39; k_1 = 19, k_2 = 19; \lambda = 18, r = -1$	ALG1
$C = \{1,3,4,6,7,8,9,10,11,15,16,18,20,22,25, 26,27,34,37\}, D = \{2,5,6,7,8,11,12,15,18, 19,20,26,28,30,31,32,34,36,37\}$	
$u = 41; k_1 = 1, k_2 = 1; \lambda = 0, r = 37$	Trivial
$u = 41; k_1 = 1, k_2 = 16; \lambda = 6, r = 19$	no solution
$u = 41; k_1 = 5, k_2 = 5; \lambda = 1, r = 23$	ALG1
$C = \{1,10,16,18,37\}, D = \{2,20,32,33,36\}$	
$u = 41; k_1 = 5, k_2 = 20; \lambda = 10, r = 11$	?
$u = 41; k_1 = 6, k_2 = 10; \lambda = 3, r = 15$	?
$u = 41; k_1 = 6, k_2 = 15; \lambda = 6, r = 11$	?

Table 1 (continued)

$u = 41; k_1 = 10, k_2 = 11; \lambda = 5, r = 9$	ALG1
$C = \{1,3,7,10,13,16,18,29,30,37\}$,	
$D = \{1,6,10,14,16,17,18,19,26,37,41\}$	
$u = 41; k_1 = 11, k_2 = 15; \lambda = 8, r = 5$	ALG1
$C = \{1,3,7,10,13,16,18,29,30,37,41\}$,	
$D = \{1,6,10,11,12,14,16,17,18,19,26,28,34,37,38\}$	
$u = 41; k_1 = 16, k_2 = 16; \lambda = 12, r = 1$	*
$u = 41; k_1 = 20, k_2 = 20; \lambda = 19, r = -1$	Th.7
$u = 43; k_1 = 1, k_2 = 1; \lambda = 0, r = 39$	Trivial
$u = 43; k_1 = 1, k_2 = 7; \lambda = 1, r = 29$	no solution
$u = 43; k_1 = 1, k_2 = 15; \lambda = 5, r = 21$	no solution
$u = 43; k_1 = 1, k_2 = 21; \lambda = 10, r = 19$	Th.2
$C = \{1\}, D = \{1,2,3,4,5,8,11,12,16,19,20,$	
$21,22,27,32,33,35,37,39,41,42\}$	
$u = 43; k_1 = 4, k_2 = 6; \lambda = 1, r = 25$	?
$u = 43; k_1 = 4, k_2 = 9; \lambda = 2, r = 21$	?
$u = 43; k_1 = 4, k_2 = 13; \lambda = 4, r = 17$	?
$u = 43; k_1 = 4, k_2 = 16; \lambda = 6, r = 15$	ALG1
$C = \{1,6,36,43\}, D = \{4,7,10,14,15,16,17,$	
$19,24,28,31,37,39,41,42,43\}$	
$u = 43; k_1 = 6, k_2 = 18; \lambda = 8, r = 11$	ALG1
$C = \{4,7,15,24,37,42\}, D = \{2,3,10,12,14,16,17,$	
$18,19,21,22,25,28,29,31,39,40,41\}$	
$u = 43; k_1 = 7, k_2 = 7; \lambda = 2, r = 19$	ALG1
$u = 43; k_1 = 7, k_2 = 15; \lambda = 6, r = 11$	ALG1
$C = \{1,4,11,16,21,35,41\}$,	
$D = \{1,3,4,5,11,12,16,19,20,21,33,35,37,41,43\}$	
$u = 43; k_1 = 7, k_2 = 21; \lambda = 11, r = 9$	ALG1
$C = \{3,4,15,18,22,24,43\}, D = \{1,2,4,6,7,9,11,12,$	
$13,15,19,23,24,28,29,35,36,37,38,39,42\}$	
$u = 43; k_1 = 9, k_2 = 18; \lambda = 9, r = 7$	ALG1
$C = \{1,3,4,6,15,18,22,24,36\}, D = \{2,3,5,$	
$7,8,9,11,12,13,18,22,23,29,30,35,37,38,42\}$	
$u = 43; k_1 = 13, k_2 = 18; \lambda = 11, r = 6$	ALG1
$C = \{2,3,4,5,8,12,15,18,22,24,29,30,43\}, D = \{2,3,4,$	
$7,9,11,12,13,15,18,22,23,24,29,35,37,38,42\}$	
$u = 43; k_1 = 15, k_2 = 15; \lambda = 10, r = 3$	ALG1
$C = \{1,2,3,4,6,7,12,15,18,22,24,29,36,37,42\}$,	
$D = \{4,7,9,11,13,14,15,23,24,31,35,37,38,41,42\}$	

Table 1 (continued)

$u=43; k_1=15, k_2=21; \lambda=15, r=1$	*
$u=43; k_1=16, k_2=18; \lambda=13, r=1$	*
$u=43; k_1=21, k_2=21; \lambda=20, r=-1$	Th.7
$u=45; k_1=1, k_2=1; \lambda=0, r=41$	Trivial
$u=45; k_1=1, k_2=12; \lambda=3, r=25$	no solution
$u=45; k_1=2, k_2=7; \lambda=1, r=29$	?
$u=45; k_1=2, k_2=18; \lambda=7, r=19$	?
$u=45; k_1=5, k_2=13; \lambda=4, r=17$	?
$u=45; k_1=5, k_2=21; \lambda=10, r=13$	?
$u=45; k_1=7, k_2=10; \lambda=3, r=17$	?
$u=45; k_1=10, k_2=18; \lambda=9, r=7$	?
$u=45; k_1=11, k_2=11; \lambda=5, r=11$	?
$u=45; k_1=11, k_2=22; \lambda=13, r=5$	ALG2, no solution
$u=45; k_1=12, k_2=12; \lambda=6, r=9$	?
$u=45; k_1=13, k_2=16; \lambda=9, r=5$	?
$u=45; k_1=16, k_2=21; \lambda=15, r=1$	*
$u=45; k_1=22, k_2=22; \lambda=21, r=-1$	?
$u=47; k_1=1, k_2=1; \lambda=0, r=43$	Trivial
$u=47; k_1=1, k_2=23; \lambda=11, r=21$	Th.2
$C=\{1\}, D=\{1,2,3,4,6,7,8,12,14,16,17,18,$ $21,24,25,27,28,32,34,36,37,42\}$	
$u=47; k_1=2, k_2=10; \lambda=2, r=27$	?
$u=47; k_1=2, k_2=14; \lambda=4, r=23$	?
$u=47; k_1=3, k_2=12; \lambda=3, r=23$	?
$u=47; k_1=5, k_2=9; \lambda=2, r=23$	?
$u=47; k_1=5, k_2=15; \lambda=5, r=17$	?
$u=47; k_1=9, k_2=19; \lambda=9, r=9$	?
$u=47; k_1=10, k_2=22; \lambda=12, r=7$	?
$u=47; k_1=12, k_2=21; \lambda=12, r=5$	?
$u=47; k_1=14, k_2=22; \lambda=14, r=3$	?
$u=47; k_1=15, k_2=19; \lambda=12, r=3$	?
$u=47; k_1=23, k_2=23; \lambda=22, r=-1$	Th.7
$u=49; k_1=1, k_2=1; \lambda=0, r=45$	Trivial
$u=49; k_1=1, k_2=16; \lambda=5, r=25$	no solution
$u=49; k_1=3, k_2=7; \lambda=1, r=31$	ALG1
$C=\{1,18,30\}$, $D=\{4,9,15,22,23,25,49\}$	
$u=49; k_1=3, k_2=10; \lambda=2, r=27$	?
$u=49; k_1=4, k_2=12; \lambda=3, r=23$	ALG1
$C=\{1,18,30,49\}$, $D=\{1,2,6,8,10,11,18,30,33,36,44,46\}$	

Table 1 (continued)

$u = 49; k_1 = 4, k_2 = 21; \lambda = 9, r = 17$	?
$u = 49; k_1 = 6, k_2 = 15; \lambda = 5, r = 17$	ALG1
$C = \{1, 2, 11, 18, 30, 36\}$,	
$D = \{2, 4, 6, 7, 8, 10, 11, 14, 22, 23, 28, 33, 36, 44, 46\}$	
$u = 49; k_1 = 6, k_2 = 18; \lambda = 7, r = 15$	?
$u = 49; k_1 = 7, k_2 = 19; \lambda = 8, r = 13$	?
$u = 49; k_1 = 9, k_2 = 9; \lambda = 3, r = 19$	?
$u = 49; k_1 = 9, k_2 = 24; \lambda = 13, r = 9$	ALG1
$C = \{1, 2, 3, 5, 11, 18, 30, 36, 41\}, D = \{4, 7, 8, 12, 14, 16, 17, 20, 21, 22, 23, 24, 28, 29, 32, 34, 35, 37, 39, 40, 42, 43, 44, 46\}$	
$u = 49; k_1 = 10, k_2 = 19; \lambda = 9, r = 9$	?
$u = 49; k_1 = 12, k_2 = 13; \lambda = 6, r = 11$	ALG1
$C = \{1, 2, 4, 6, 10, 11, 18, 22, 23, 30, 33, 36\}$,	
$D = \{1, 2, 3, 5, 9, 11, 15, 18, 25, 30, 36, 41, 49\}$	
$u = 49; k_1 = 13, k_2 = 21; \lambda = 12, r = 5$	ALG1
$C = \{1, 2, 3, 5, 7, 11, 14, 18, 28, 30, 36, 41\}, D = \{4, 7, 10, 14, 15, 16, 17, 19, 24, 28, 31, 37, 39, 41, 42, 49\}$	
$u = 49; k_1 = 15, k_2 = 22; \lambda = 14, r = 3$	?
$u = 49; k_1 = 16, k_2 = 16; \lambda = 5, r = 5$	?
$u = 49; k_1 = 18, k_2 = 22; \lambda = 16, r = 1$	*
$u = 49; k_1 = 24, k_2 = 24; \lambda = 23, r = -1$	?

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