Existence of Room frames of type $2^n u^1$

G. Ge and L. Zhu*
Department of Mathematics
Suzhou University
Suzhou, 215006
P.R. China

ABSTRACT. It is shown that for any even integer $u \geq 20$ a Room frame of type $2^n u^1$ exists if and only if $n \geq u + 1$.

1. Introduction

Let S be a finite set, let ∞ be a "special" symbol not in S, and let $\mathbf{H}$ be a set of subsets of S. As defined in [23] a *holey Room square* (briefly HRS) having *hole set* $\mathbf{H}$ is an $|S| \times |S|$ array, F, indexed by S, which satisfies the following properties:

1. every cell of F either is empty or contains an unordered pair of symbols of $S \cup \{\infty\}$.
2. every symbol of $S \cup \{\infty\}$ occurs at most once in any row or column of F, and every unordered pair of symbols occurs in at most one cell of F.
3. the subarrays $H \times H$ are empty, for every $H \in \mathbf{H}$ (the subarrays are referred to as *holes*).
4. symbol $s \in S$ occurs in row or column t if and only if $(s,t) \in (S \times S) \setminus \cup_{H \in \mathbf{H}} H \times H$; and symbold ∞ occurs in row or column t if and only if $t \in S \setminus \cup_{H \in \mathbf{H}} H$.
5. the pair (s,t) occurs in F if and only if $(s,t) \in (S \times S) \setminus \cup_{H \in \mathbf{H}} H \times H$; the pair $\{\infty, t\}$ occurs in F if and only if $t \in S \setminus \cup_{H \in \mathbf{H}} H$.

*Research supported by NSFC grant 19231060–2.

The *order* of F is $|S|$. Note that ∞ does not occur in *any* cell of F if $\cup_{H \in \mathbf{H}} H = S$. If $\mathbf{H} = \emptyset$, then an HRS(H) is called a *Room square* of side $|S|$.

If $\mathbf{H} = \{S_1, S_2, \ldots, S_n\}$ is a partition of S, then an HRS(H) is called a *Room frame*. As is usually done in the literature, we shall refer to a Room frame simply as a *frame*. The *type* of the frame is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. We usually use an "exponential" notation to describe types: a type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \leq i \leq k$. We briefly denote a frame of type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ by $\mathrm{RF}(t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k})$.

If $\mathbf{H} = \{S_1, S_2, \ldots, S_n, H\}$, where $\{S_1, S_2, \ldots, S_n\}$ is a partition of S, then an HRS(H) is called an *incomplete frame* or an *I-frame*. The *type* of the I-frame is defined to be the multiset $\{(|S_i|, |S_i \cap H|) : 1 \leq i \leq n\}$. We may also use an "exponential" notation to describe types of I-frame: a type $(t_1, r_1)^{u_1} (t_2, r_2)^{u_2} \ldots (t_k, r_k)^{u_k}$ denotes u_i occurrences of (t_i, r_i), $1 \leq i \leq k$.

The first Room frame was used to prove the existence of a Room square of side 257 by Wallis [24]. Since then the notion of Room frame has played an important role in recursive constructions of various combinatorial designs, such as Room squares (see, for example, the recent survey [8]), incomplete Room squares [10], [23], cycle decomposition [13], [15], [16], and weakly 3-chromatic linear spaces [17], [18]. This paper is a continuation of [11] investigating the existence of Room frames with type $2^n u^1$. The known existence results for frames of related types can be summarized in the following theorem.

Theorem 1.1. *There exist frames of the following types:*

(1) [7], [12] t^4 *for all even integers* $t > 2$*, except possibly when* $t \in \{14, 22, 26, 34, 38, 46, 62, 74, 82, 86, 98, 122, 134, 146\}$.

(2) [5], [7], [12] t^5 *for any integer* $t > 1$.

(3) [7] t^u *for* $u \geq 6$ *and both* t *and* u *even.*

(4) [7] t^u *for all* t *and all odd* $u \geq 7$.

(5) [9] $2^a 4^{(s-a)}$ *for* $s \in \{6, 7, \ldots, 14, 31, 42, 43, 44\}$ *or* $s \geq 48$, $0 \leq a \leq s$.

(6) [11] $2^n 4^1$ *for any* $n \geq 5$.

(7) [11] $2^n u^1$ *for any even* $u > 4$ *and* $n \geq 5\lceil u/4 \rceil + 20$.

For the existence of frames of type $2^n u^1$ the following necessary condition is known.

Theorem 1.2. [9] *If a frame of type* $2^n u^1$ *exists for some integer* $u > 2$*, then* $u \equiv 0 \pmod 2$ *and* $n \geq u + 1$.

We shall describe direct and recursive constructions for frames in Section 2. Define

$$U = \{u \text{ even: there exists an RF}(2^n u^1) \text{ for all } n \geq u+1\}.$$

In Section 3, we shall show that $u \in U$ for any even u, $20 \leq u \leq 34$. With this interval we can show in Section 4 that $u \in U$ for any even $u \geq 428$. In Section 5 we shall deal with the values between 36 and 426. Therefore, the main result of this paper can be stated as follows.

Theorem 1.3. *For any even integer $u \geq 20$, there exists a Room frame of type $2^n u^1$ if and only if $n \geq u+1$.*

2. Constructions

The main direct construction used is the "starter-adder" construction and its modifications, see [8], [20]. Let G be an abelian group, written additively, and let H be a subgroup of G. Denote $g = |G|$, $h = |H|$ and suppose that $g-h$ is even. A *frame starter* in $G \backslash H$ is a set of unordered pairs $S = \{\{s_i, t_i\} : 1 \leq i \leq (g-h)/2\}$ satisfying

(1) $\cup_{1 \leq i \leq (g-h)/2}(\{s_i\} \cup \{t_i\}) = G \backslash H$, and

(2) $\cup_{1 \leq i \leq (g-h)/2}\{\pm(s_i - t_i)\} = G \backslash H$.

An *adder* for S is an injection $A : S \to G \backslash H$, such that

$$\cup_{1 \leq i \leq (g-h)/2}(\{s_i + a_i\} \cup \{t_i + a_i\}) = G \backslash H,$$

where $a_i = A(s_i, t_i)$, $1 \leq i \leq (g-h)/2$. We have the following construction for Room frames.

We have the following construction for Room frames.

Lemma 2.1. [20, Lemma 3.1] *Suppose there exists a frame starter S in $G \backslash H$, and an adder A for S. Then there exists a frame of type $h^{g/h}$, where $g = |G|$ and $h = |H|$.*

As above, let G be an abelian group of order g and let H be a subgroup of order h, where $g-h$ is even. A *$2k$-intransitive frame starter* in $G \backslash H$ is defined to be a triple (S, C, R), where

$$S = \{\{s_i, t_i\} : 1 \leq i \leq (g-h)/2 - 2k\} \cup \{\{u_i\} : 1 \leq i \leq 2k\},$$
$$C = \{\{p_i, q_i\} : 1 \leq i \leq k\}, \text{ and}$$
$$R = \{\{p_i', q_i'\} : 1 \leq i \leq k\},$$

satisfying

(1) $\{s_i\} \cup \{t_i\} \cup \{u_i\} \cup \{p_i\} \cup \{q_i\} = G \backslash H$,

(2) $\{\pm(s_i - t_i)\} \cup \{\pm(p_i - q_i)\} \cup \{\pm(p_i' - q_i')\} = G\backslash H$, and

(3) all $p_i - q_i$ and $p_i' - q_i'$ have even orders in G.

An *adder* for (S, C, R) is an injection $A : S \to G\backslash H$, satisfying

(4) $\{s_i + a_i\} \cup \{t_i + a_i\} \cup \{u_i + b_i\} \cup \{p_i', q_i'\} = G\backslash H$,

where

$$a_i = A(s_i, t_i), 1 \le i \le (g - h)/2 - 2k,$$
$$b_i = A(u_i), 1 \le i \le 2k.$$

The following result is known.

Lemma 2.2. [20, Lemma 3.3] *If there is a $2k$-intransitive frame starter and an adder in $G\backslash H$, where $g = |G|$ and $h = |H|$, then there is a Room frame of type $h^{g/h}(2k)^1$.*

The next known recursive construction is useful in this paper.

Construction 2.3. (Filling in holes): (1) If there exist frames of type $(2n_1)^1(2n_2)^1 \ldots (2n_k)^1 h^1$ and type $2^{n_i}v^1$ for $1 \le i \le k$, then there exists a frame of type $2^n u^1$ where $n = n_1 + n_2 + \cdots + n_k$, and $u = h + v$. (2) If there exist an I-frame of type $(2n_1 + u_1, u_1)^1(2n_2 + u_2, u_2)^1 \ldots (2n_k + u_k, u_k)^1$ and frames of type $2^{(d+n_i)}(u_i)^1$, where $d \in \{0, 1\}$ and $1 \le i \le k$, then there exists a frame of type $2^{(d+n)}u^1$ where $n = d + n_1 + n_2 + \cdots + n_k$ and $u = u_1 + u_2 + \cdots + u_k$.

The following recursive construction for frames uses group divisible designs. A *group divisible design* (or GDD), is a triple $(X, \mathbf{G}, \mathbf{A})$ which satisfies the following properties:

(1) $\mathbf{G}$ is a partition of X into subsets called *groups*,

(2) $\mathbf{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,

(3) every pair of points from distinct groups occurs in a unique block.

Construction 2.4. (Weighting): [20] Suppose $(X, \mathbf{G}, \mathbf{A})$ is a GDD and let $w : X \to \mathbf{Z}^+ \cup \{0\}$ (we say that w is a *weighting*). Suppose there exists a frame of type $\{w(x) : x \in A\}$ for every $A \in \mathbf{A}$. Then there exists a frame of type $\{\sum_{x \in G} w(x) : G \in \mathbf{G}\}$.

Suppose F is a Room frame with hole set $\{S_1, S_2, \ldots, S_n\}$, where $S = \cup S_i$. A *holey transversal* (with respect to hole S_i) is a set T of $|S\backslash S_i|$ filled cells in F such that every symbol of $S\backslash S_i$ is contained in exactly two cells of T. If the pairs in the cells of T are ordered so that every symbol of $S\backslash S_i$

occurs once as a first co-ordinate and once as a second co-ordinate in a cell of T. then T is said to be *orderered*. Note that any holey transversal T can always be ordered since the union of all edges from the pairs in T forms a disjoint union of cycles. If these cycles are arbitrarily oriented, then the direction of each edge provides an ordering for the holey transversal.

If $|S\backslash S_i|$ is even and the cells of T can be partitioned into two subsets T_1 and T_2 of $|S\backslash S_i|/2$ cells, so that every symbol of $S\backslash S_i$ is contained in one cell in each of T_1 and T_2, then T is said to be *partitioned*. A holey ordered partitioned (holey ordered, resp.) transversal will be referred to as an *HOP transversal* (*HO transversal* resp.). If $S_i = \emptyset$, an HOP or HO transversal will be called a COP or CO transversal, respectively, where "C" stands for complete.

We assume that the reader is familiar with Latin squares and their orthogonality, see [4]. A pair of mutually orthogonal Latin squares of order n is denoted by MOLS(n). If an MOLS(n) has a sub-MOLS(k) missing, then we call it an incomplete MOLS, denoted IMOLS($n; k$). If an MOLS(n) has two disjoint sub-MOLS missing, which are of orders k and t, then we denote it by IMOLS($n; k, t$).

Construction 2.5.: [23, Construction 3.6] Suppose there is a frame of type $t_1^g t_2^1$ having s disjoint HOP transversals with respect to the hole of size t_2. For $1 \leq i \leq s$, let $u_i \geq 0$ be an integer. Let m be a positive integer, $m \neq 2$ or 6, and suppose there exist IMOLS($m + u_i; u_i, 1$), for $1 \leq i \leq s$. Then there is an I-frame of type $(mt_1, t_1)^g(mt_2 + 2u, t_2)^1$, where $u = \sum u_i$.

There are direct and recursive constructions which provide the frames required in Construction 2.5.

Lemma 2.6. *Suppose G is a cyclic group of order $g = 2^s pq$ and H is a subgroup of even order $h = 2^t q$, where p and q are both odd. If there is a $2k$-intransitive frame starter and an adder in $G\backslash H$, then there is a Room frame of type $h^{g/h}(2k)^1$ having $((g-h)-(p-1)q-4k)/2$ disjoint HOP transversals with respect to the size $2k$ hole.*

Proof: Let $G = \{0, 1, \ldots, g-1\}$. The addition in G is taken module g. It is clear that there are altogether pq odd order elements in G. If an odd order element is in H, then it has the form $i(g/h)$, where $i \in \{0, 1, \ldots, h-1\}$, and $i(g/h) \equiv 0 \pmod{2^s}$. This equation is equivalent to requiring that $i \equiv 0 \pmod{2^t}$. So, there are altogether q elements in H each having odd order in G. Hence, we know that there are $((g-h)-pq+q)$ even order elements in $G\backslash H$. On the other hand, the differences $\pm(p_i - q_i)$ and $\pm(p_i' - q_j')$ in $2k$-intransitive starter are all even order elements in $G\backslash H$. Therefore, there are $((g-h)-(p-1)q-4k)$ even order elements among differences $\pm(s_i - t_i)$. Each pair $\{s_i, t_i\}$ with even order differences can provide an HOP transversal. The conclusion then follows. □

In the next construction we shall use SOLSSOM and partitionable IMOLS. A *self-orthogonal Latin square* is one that is orthogonal to its transpose. A Latin square is called *symmetric* if it is equal to its transpose. A self-orthogonal Latin square (SOLS) with a symmetric orthogonal mate (SOM) of order m will be denoted by SOLSSOM(m). If the main diagonal of the SOM is constant, then the SOM is termed unipotent.

Suppose L_1 and L_2 are IMOLS($m+u;u$) on symbol set S and hole set $\mathbf{H} = \{H\}$. A *holey* row or column of L_1 or L_2 is one that meets the hole. A holey row (or column), T is said to be *partitionable* if the superposition of row (or column) T of L_1 and L_2 can be partitioned into two subsets T_1 and T_2 of $m/2$ cells, so that every symbol of $S \backslash H$ is contained in one cell in each of T_1 and T_2. An IMOLS($m+u;u$) is said to be *partitionable* if every holey row and column is partitionable.

Construction 2.7: [23, Construction 3.4] Suppose there is a frame of type t^g having s disjoint CO transversals. For $1 \leq i \leq s$ let $u_i \geq 0$ be an integer, and let m be an even positive integer. Suppose there exists an SOLSSOM(m) such that the SOM is unipotent. Suppose also that there exist partitionable IMOLS($m+u_i;u_i$), for $1 \leq i \leq s$. Let $k = |\{i : u_i = 0\}|$. Then there is a frame of type $(mt)^g(2u)^1$, where $u = \sum u_i$, having $k(m-1)$ HOP transversals with respect to the size $2u$ hole

Remark 2.8: In Construction 2.7, the $k(m-1)$ HOP transversals come from k CO transversals each providing $m-1$ of them. For certain i, $u_i \neq 0$, there is some possibility to get extra HOP transversals from the partitionable IMOLS($m+u_i;u_i$). m cells in an IMOLS($m+v;v$) is called a *holey partitionable transversal* with respect to the size v hole if (1) they occupy different rows and different columns and contain different elements in each square avoiding the size v hole, and (2) they can be divided into two parts each containing m different elements from the two squares. For each i, $u_i \neq 0$, let the partitionable IMOLS($m+u_i;u_i$) contain d_i disjoint holey partitionable transversals with respect to the size u_i hole, where d_i might be zero. Let $d = d_1 + d_2 + \cdots + d_{s-k}$. We observe that the resultant RF($(mt)^g(2u)^1$) in Construction 2.7 contains $d+k(m-1)$ HOP transversals with respect to the size $2u$ hole.

3. Even $u \in U$ for $20 \leq u \leq 34$

A *transversal design* TD(k,n) is a GDD with kn points, k groups of size n, and n^2 blocks of size k. It is well known that a TD(k,n) is equivalent to $k-2$ MOLS of order n. From [22] we have the following lemma.

Lemma 3.1. [1] [3] [22] *A TD(6,n) exists for $n \geq 5$ and $n \notin$ {6, 10, 14, 18, 22, 34, 42}.*

Lemma 3.2. *Let v be an even integer, and suppose there exists a TD(6,t).*

Suppose there exist $RF(2^{n_i}, v^1)$, where $1 \le i \le 5$ and $t \le n_i \le 2t$. Then there exists an $RF(2^n u^1)$, where u is even, $2t+v \le u \le 4t+v$, and $n = \sum n_i$.

Proof: Apply Construction 2.4 and give weight two or four to each point of the given TD$(6,t)$. We obtain a Room frame of type $\{2n_1, 2n_2, \ldots, 2n_6\}$, where $t \le n_i \le 2t$ for $1 \le i \le 6$. The required input designs RF$(2^a 4^{6-a})$ for $0 \le a \le 6$ all exist from [9]. Further apply Construction 2.3 to fill in the first five holes with RF$(2^{n_i} v^1)$ for $1 \le i \le 5$. The conclusion then follows by taking $u = 2n_6 + v$. □

Lemma 3.3. *Suppose there is a frame of type $t_1^g t_2^1$ having s disjoint HOP transversals with respect to the size t_2 hole. Suppose there exist frames of type $2^{(1+u+t_2)}(t_2)^1$ and $2^{(1+t_i)}(t_i)^1$ for $1 \le i \le g$ and $0 \le u \le s$. Then there is an $RF(2^n v^1)$ where $v = gt_1 + +t_2$ and $n = 1 + u + v$.*

Proof: Apply Construction 2.5 with $m = 3$ and $u_j = 0$ or 1 for $1 \le j \le s$. The input designs we need are IMOLS$(4;1,1)$ and IMOLS$(3;1)$, which are easy to construct. We obtain an I-frame of type $(3t_1, t_1)^g (3t_2 + 2u, t_2)^1$, where $0 \le u \le s$. Further applying Construction 2.3 (2) with frames of type $2^{(1+u+t_2)}(t_2)^1$ and $2^{(1+t_i)}(t_i)^1$ for $1 \le i \le g$, we get the desired frame. □

We are now in a position to prove the main results of this section.

Lemma 3.4. *If $u = 20$, 22 or 24 then $u \in U$.*

Proof: Apply Lemma 3.2 with $t = 5$ and $v = 4$, we get an RF$(2^n u^1)$ for $25 \le n \le 50$ and $14 \le u \le 24$. A TD$(6,5)$ exists from Lemma 3.1 and all the input designs exist from Theorem 1.1 (6). From Theorem 1.1 (7) we need further discuss the existence of frames of type $2^n 20^1$ for $21 \le n \le 24$ and $2^n 22^1$ for $23 \le n \le 24$.

From Theorem 1.1 (1) we have an RF(16^4). Filling in the first three holes with RF$(2^8 v^1)$ where $v = 4$ or 6, we obtain an RF$(2^{24} u^1)$ for $u = 20$ or 22.

Applying Lemma 3.3 with an RF$(4^4 6^1)$ and taking $s = 0$, we get an RF$(2^{23} 22^1)$. The starting RF$(4^4 6^1)$ comes from [20, Lemma 5.1] and the input frames RF$(2^5 4^1)$ and RF$(2^7 6^1)$ exist from Theorem 1.1 (6) and [21], respectively.

We construct an RF$(4^4 4^1)$ by the following 4-intransitive starter and adder:

$$G = \mathbf{Z}_{16}, H = \{0,4,8,12\}, S = \{\{2,7\},\{5,11\}\} \cup \{15,10,3,1\},$$
$$C = \{\{13,14\},\{6,9\}\}, R = \{\{11,13\},\{7,14\}\}, a_1 = 15, a_2 = 14,$$
$$A(15) = 11, A(10) = 5, A(3) = 2, A(1) = 1.$$

From Lemma 2.6 there are too disjoint HOP transversals in the RF$(4^4 4^1)$. We then apply Lemma 3.3 with $u = 0$, 1 or 2 to get an RF$(2^n 20^1)$ for $n = 21$, 22 or 23. The input frames come from Theorem 1.1 (6). This completes the proof. □

Lemma 3.5. *An RF(2^nu^1) exists for even u, $26 \le u \le 34$, and $n \ge 35$.*

Proof: Apply Lemma 3.2 with $(t,v) = (8,4)$ or $(7,6)$, we get an RF(2^nu^1) for

$$40 \le n \le 80 \text{ and } 26 \le u \le 36,$$
$$35 \le n \le 40 \text{ and } 26 \le u \le 34.$$

A TD($6,t$) exists from Lemma 3.1 and all the input designs exist from Theorem 1.1 (6) and the existence of RF(2^76^1) and RF(2^86^1), where the former comes from [21] and the latter from [11, Example 2.3]. Then the conclusion follows from Theorem 1.1 (7). □

Lemma 3.6. *$u \in U$ for even u, $26 \le u \le 34$.*

Proof: By Lemma 3.5 we need further consider the cases $26 \le u \le 32$ and $u+1 \le n \le 34$. Apply Lemma 3.3 with the following parameters:

$u = 26$	RF(4^62^1)	$s = 7$	$28 \le n \le 34$,
$u = 26$	RF(4^56^1)	$s = 0$	$n = 27$,
$u = 28$	RF(4^64^1)	$s = 5$	$29 \le n \le 34$,
$u = 30$	RF(4^72^1)	$s = 7$	$32 \le n \le 34$,
$u = 30$	RF(4^66^1)	$s = 0$	$n = 31$,
$u = 32$	RF(4^74^1)	$s = 5$	$33 \le n \le 34$.

The required input RF(2^n4^1) and RF(2^n2^1) are from Theorem 1.1 and RF(2^76^1) is known [21]. The starting frames RF(4^56^1) and RF(4^66^1) are taken from [9]. The above values of s come from Lemma 2.6 by taking $2k$-intransitive starter and adder as follows.

RF(4^62^1) : $G = Z_{24}$, $H = \{0, 6, 12, 18\}$.
$S = \{\{2,10\},\{4,7\},\{11,13\},\{8,15\},\{5,14\},\{9,23\},\{3,16\},\{19,20\}\} \cup \{21,1\}$,
$C = \{17,22\}$, $R = \{15,19\}$,
$(a_1, a_2, \ldots, a_8) = (23,22,21,20,17,14,11,1)$, $A(21) = 19$, $A(1) = 16$.

RF(4^64^1) : $G = Z_{24}$, $H = \{0, 6, 12, 18\}$.
$S = \{\{2,10\},\{8,15\},\{5,14\},\{9,23\},\{3,16\},\{19,20\},\cup\{21,1,4,7\}$,
$C = \{\{17,22\},\{11,13\}\}$, $R = \{\{15,19\},\{2,5\}\}$,
$(a_1, a_2, \ldots, a_6) = (23,20,17,14,11,1)$, $A(21) = 19$, $A(1) = 16$, $A(4) = 4$, $A(7) = 3$.

RF(4^72^1) : $G = Z_{28}$, $H = \{0, 7, 14, 21\}$.
$S = \{\{2,10\},\{4,8\},\{11,13\},\{9,15\},\{17,27\},\{5,24\},\{1,12\},\{3,16\},\{6,18\},\{19,22\} \cup \{26,23\}$,
$C = \{20,25\}$, $R = \{17,18\}$,
$(a_1, a_2, \ldots, a_{10}) = (27,26,25,24,23,20,19,16,9,4)$, $A(26) = 15$, $A(23) = 1$.

$RF(4^7 4^1) : G = \mathbf{Z}_{28}$, $H = \{0, 7, 14, 21\}$.
$S = \{\{2, 10\}, \{4, 8\}, \{11, 13\}, \{17, 27\}, \{5, 24\}, \{1, 12\}, \{6, 18\}, \{19, 22\}$
$\cup\{15, 26, 9, 23\}$,
$C = \{\{20, 25\}, \{3, 16\}\}$, $R = \{\{17, 18\}, \{11, 5\}\}$,
$(a_1, a_2, \ldots, a_8) = (27, 26, 25, 23, 20, 19, 9, 4)$, $A(15) = 17$, $A(26) = 15$, $A(9) = 10$,
$A(23) = 1$.

□

Combining Lemmas 3.4 and 3.6 we get the main result of this section.

Theorem 3.7. *$u \in U$ for even u, $20 \leq u \leq 34$.*

4. $u \in U$ for even $u \geq 428$

We need the following known constructions, see Lemmas 4.1–4.3 in [23].

Lemma 4.1. *Suppose there exist a starter and adder in $\mathbf{Z}_g$. Then there exists a frame of type 1^g having $(g-1)/2$ disjoint CO transversals.*

Lemma 4.2. *Suppose there exist a starter and adder in $\mathbf{Z}_g$. Suppose $0 \leq u \leq 3(g-1)/2$. Then there exists a frame of type $8^g(2u)^1$ having $7((g-1)/2 - \lceil u/3 \rceil)$ disjoint HOP transversals with respect to the hole of size $2u$.*

Lemma 4.3. *Suppose there exist a starter and adder in $\mathbf{Z}_g$. Suppose $0 \leq u \leq 3(g-1)/2$ and $0 \leq k \leq 7((g-1)/2 - \lceil u/3 \rceil)$. Then there exists an I-frame of type $(24, 8)^g(6u + 2k, 2u)^1$.*

Lemma 4.4. *Suppose there exist a starter and adder in $\mathbf{Z}_g$. Suppose $0 \leq u \leq 3(g-1)/2$ and $0 \leq k \leq 7((g-1)/2 - \lceil u/3 \rceil)$. Further, suppose there is an $RF(2^{(2u+k+1)}(2u)^1)$. Then there exists an $RF(2^{(8g+2u+k+1)}(8g+2u)^1)$.*

Proof: Filling in the resultant I-frame in Lemma 4.3 with the given $RF(2^{(2u+k+1)}(2u)^1)$ and $RF(2^9 8^1)$ [21] gives the desired frame. □

Lemma 4.5. *Suppose $2u \in U$. If $g \geq (28\lceil u/3 \rceil + 2u + 101)/6$ and there exists a starter and adder in $\mathbf{Z}_g$, then $8g + 2u \in U$.*

Proof: Since $2u \in U$, we may apply Lemma 4.4 to get an $RF(2^n(8g+2u)^1)$ for $8g + 2u + 1 \leq n \leq 8g + 2u + 1 + 7((g-1)/2 - \lceil u/3 \rceil)$. The inequality $g \geq (28\lceil u/2 \rceil + 2u + 101)/6$ implies that $8g + 2u + 1 + 7((g-1)/2 - \lceil u/3 \rceil) \geq 5\lceil (8g + 2u)/4 \rceil + 19$, then the conclusion follows from Theorem 1.1 (7). □

Lemma 4.6. *$u \in U$ for even u, $428 \leq u \leq 8026$.*

Proof: For any even u, $428 \leq u \leq 8026$. we may write $u = 8g + 2v$ where $10 \leq v \leq 17$ and $51 \leq g \leq 999$ such that g is odd and $g \geq (28\lceil v/3 \rceil + 2v + 101)/6$. From [6] there is a starter and adder in $\mathbf{Z}_g$. By Theorem 3.7, $2v \in U$ for $10 \leq v \leq 17$. The conclusion then follows from Lemma 4.5. □

We can now prove the main result of this section.

Theorem 4.7. *$u \in U$ for even $u \geq 428$.*

Proof: The proof is by induction on u. To start the induction it is sufficient to observe that $u \in U$ if u is even and $428 \leq u \leq 4178$ (Lemma 4.6). Hence, assume $u \geq 4180$. We can write $u = 8g + 2v$, where $428 \leq 2v \leq 474$, and $g \equiv 1 \pmod 6$, $g \geq 469$. Then there is a starter and adder in $\mathbf{Z}_g$ [14], and $g \geq (28\lceil v/3 \rceil + 2v + 101)/6$ since $(28\lceil 237/3 \rceil + 474 + 101)/6 = 464.5$. Apply Lemma 4.5. □

5. Even $u \in U$ for $36 \leq u \leq 426$

Lemma 5.1. *There exists an RF($2^n 16^1$) for $n \geq 17$ and $n \neq 19, 21, 22, 23$.*

Proof: From the proof of Lemma 3.4 there is an RF($2^n 16^1$) for $25 \leq n \leq 50$. An RF($2^{24} 16^1$) comes from an RF(16^4) by filling in the first three holes with an RF(2^8). An RF($2^{20} 16^1$) is known [11, Appendix]. By Theorem 1.1 (7) we need discuss the existence of RF($2^{17} 16^1$) and RF($2^{18} 16^1$). The fist comes from Lemma 3.3 with $s = 0$ and a starting frame of type 4^4. Finally, take an RF(12^4) (Theorem 1.1 (1)) and fill in holes with RF($2^6 4^1$), we get the required RF($2^{18} 16^1$). □

Lemma 5.2. *Suppose for odd g there exist a starter and adder in $\mathbf{Z}_g$. If $u = mg + v$, $m \in \{4, 8\}$, $v \in \{2, 4\}$, then there exists an RF($2^n u^1$) for $u + 2 - \lfloor v/4 \rfloor \leq n \leq u + 1 + (m-1)((g-1)/2 - \lceil v/(m-2) \rceil)$.*

Proof: Apply Construction 2.7 with a starting frame RF(1^g) having $(g-1)/2$ disjoint CO transversals. This frame comes from Lemma 4.1. From [25] we have an SOLSSOM(m) such that the SOM is unipotent. The existence of partitionable designs IMOLS($5, 1$) and IMOLS($8 + i, i$) for $1 \leq n \leq 3$ are from [2] and [23]. We then get an RF($m^g v^1$) having $(m-1)((g-1)/2 - \lceil v/(m-2) \rceil)$ disjoint HOP transversals with respect to the size v hole. Further, apply Lemma 3.3. We need input designs RF($2^{(1+m)} m^1$) and RF($2^{(1+v+k)} v^1$) for $0 \leq k \leq (m-1)((g-1)/2 - \lceil v/(m-2) \rceil)$, which all exist from Theorem 1.1 and [21] except for $k = 0$ and $v = 2$ (an RF(2^4) does not exist). This completes the proof. □

Lemma 5.3. *Suppose for odd g there exist a starter and adder in $\mathbf{Z}_g$. If $u = 16g + v$, $v \in \{2, 4\}$, then there exists an RF($2^n u^1$) for $u + 2 - \lfloor v/4 \rfloor \leq n \leq u + 1 + 15((g-1)/2 - \lceil v/6 \rceil)$.*

Proof: The proof is similar to that of Lemma 5.2. Here, we use partitionable IMOLS($16 + i, i$) for $1 \leq i \leq 3$. These IMOLS can be constructed by using a generalized product construction for MOLS, see [26] for example. We start with an MOLS(4) having three disjoint symmetric transversals avoiding the main diagonal. Using a partitionable IMOLS(5, 1) as input designs gives the desired partitionable IMOLS($16 + i, i$) for $1 \leq i \leq 3$. We also need input designs RF($2^{17}16^1$) and RF($2^{(1+v+k)}v^1$) for $0 \leq k \leq 15((g-1)/2 - \lceil v/6 \rceil)$, which all exist from Lemma 5.1 and Theorem 1.1. This completes the proof. □

In the following lemma the number g is not necessarily odd.

Lemma 5.4. *Suppose there exist a starter and adder in $Z_{2g} \backslash \{0, g\}$. If $u = 8g + v$, $v \in \{2, 4\}$, then there exists an RF($2^n u^1$) for $u + 2 - \lfloor v/4 \rfloor \leq n \leq u + 1 + 3(g - 1 - v/2)$.*

Proof: Apply Lemma 2.1 with the given starter and adder, we get an RF(2^g), which contains $g - 1$ disjoint CO transversals since each of the $g - 1$ pairs in the starter gives rise to a CO transversal. We then apply Construction 2.7 with the RF(2^g) as starting frame. If we take $m = 4$, the remaining part of the proof is similar to that of Lemma 5.2. This completes the proof. □

To deal with the cases RF($2^{(1+u)}u^1$) missing in Lemmas 5.2, 5.3 and 5.4, we have the following two lemmas.

Lemma 5.5. *If there exist an RF($m^s v^t$) for $m \in \{4, 8\}$ and $v \in \{6, 8.10, 12\}$, then there exists an RF($2^{(1+u)}u^1$) for $u = ms + vt$.*

Proof: Apply Lemma 3.3 with $s = 0$. We need input designs RF($2^{(1+u)}u^1$) for $u = 4$, 6, 8, 10, 12, which are from Theorem 1.1 or [21]. Then, we obtain the desired frame. □

Lemma 5.6. *Suppose for odd g there exist a starter and adder in Z_g. If $u = mg + v$, $m \in \{4, 8, 16\}$, $v \in \{6, 8, 10, 12\}$ and $g \geq 2\lceil v/(m-2) \rceil + 1$ (for $m = 4, 8$) or $g \geq 2\lceil v/6 \rceil + 1$ (for $m = 16$), then there exists an RF($2^{(1+u)}u^1$).*

Proof: First, apply Construction 2.7 as in Lemma 5.2 or 5.3, we get an RF($m^g v^1$). Then, the conclusion follows from Lemma 5.5. □

We are now ready to prove the main result of this section.

Lemma 5.7. $36 \in U$.

Proof: In the proof of Lemma 3.5 it is shown that an RF($2^n 36^1$) exists for $40 \leq n \leq 80$. By Theorem 1.1 (7) we need deal with the cases $n = 37$, 38 and 39. We apply Lemma 3 3 with a starting frame RF($4^8 4^1$) constructed from a 4-intransitive starter and adder as follows:

RF($4^8 4^1$) : $G = Z_{32}$, $H = \{0, 8, 16, 24\}$.
$S = \{\{2, 15\}, \{4, 19\}, \{6, 18\}, \{10, 17\}, \{1, 12\}, \{3, 29\}, \{5, 28\}, \{13, 27\}, \{20, 30\}, \{21, 26\}\} \cup \{25, 11, 14, 31\}$,
$C = \{\{7, 9\}, \{22, 23\}\}$, $R = \{\{5, 9\}, \{19, 22\}\}$,
$(a_1, a_2, \ldots, a_{10}) = (31, 30, 29, 28, 27, 26, 25, 23, 22, 5)$, $A(25) = 19$, $A(11) = 14$, $A(14) = 13$, $A(31) = 12$.

By Lemma 2.6 we know that the frame has 10 disjoint HOP transversals with respect to the size four hole. The conclusion then follows. □

Lemma 5.8. $38, 40 \in U$.

Proof: Apply Lemma 3.2 with $(t, v) = (9, 4)$, we get an $\mathrm{RF}(2^n u^1)$ for $45 \le n \le 90$ and $u = 38$, 40. A TD(6, 9) exists from Lemma 3.1. All input frames are from Theorem 1.1 (6). By Theorem 1.1 (7) we need deal with the cases $u + 1 \le n \le 44$. Apply Lemma 5.2 with $(m, g, v) = (4, 9, 2)$ or $(4, 9, 4)$, we need only deal with the existence of $\mathrm{RF}(2^{39}38^1)$. From [9] there exists an $\mathrm{RF}(4^2 6^5)$, from which we apply Lemma 5.5 to get the desired $\mathrm{RF}(2^{39}38^1)$. □

Lemma 5.9. $u \in U$ *for even* u, $42 \le u \le 48$.

Proof: Apply Lemma 3.2 with $(t, v) = (11, 4)$, we get an $\mathrm{RF}(2^n u^1)$ for $55 \le n \le 110$ and $42 \le u \le 48$. A TD(6, 11) exists from Lemma 3.1. All input frames are from Theorem 1.1 (6). By Theorem 1.1 (7) we need deal with the cases $u + 1 \le n \le 54$.

For $u = 42, 44$, we apply Lemma 3.3 with a starting frame $\mathrm{RF}(4^{10} v^1)$ for $v = 2, 4$, constructed from a v-intransitive starter and adder as follows:

$\mathrm{RF}(4^{10}2^1) : G = \mathbb{Z}_{40}$, $\quad H = \{0, 10, 20, 30\}$.
$S = \{\{2,15\},\{4,19\},\{22,36\},\{1,12\},\{17,35\},\{23,31\},\{18,37\},\{16,32\},$
$\{6,9\},\{29,38\},\{33,39\},\{25,21\},\{14,26\},\{7,24\},\{27,34\},\{3,5\}\}$
$\cup\{28,11\}$,
$C = \{8,13\}$, $\quad R = \{26,27\}$,
$(a_1, a_2, \ldots, a_{16}) = (39,38,37,35,34,33,31,29,28,24,19,18,9,8,4,3)$,
$A(28) = 16$, $A(11) = 14$.

$\mathrm{RF}(4^{10}4^1) : G = \mathbb{Z}_{40}$, $\quad H = \{0, 10, 20, 30\}$.
$S = \{\{2,15\},\{4,19\},\{22,36\},\{1,12\},\{17,35\},\{23,31\},\{18,37\},\{16,32\},$
$\{29,38\},\{33,39\},\{25,21\},\{14,26\},\{27,34\},\{3,5\}\} \cup \{9,28,6,11\}$,
$C = \{\{8,13\},\{7,24\}\}$, $\quad R = \{\{26,27\},\{34,37\}\}$,
$(a_1, a_2, \ldots, a_{14}) = (39,38,37,35,34,33,31,24,19,18,9,8,4,3)$, $A(9) = 6$,
$A(28) = 16$, $A(6) = 26$, $A(11) = 14$.

This leaves the existence of an $\mathrm{RF}(2^{43}42^1)$, which can be done by Lemma 5.6 with $(m, 9, v) = (4, 9, 6)$.

For $u = 46, 48$, we apply Lemma 5.2 with $(m, g, v) = (4, 11, 2)$ or $(4, 11, 4)$. This leaves the existence of an $\mathrm{RF}(2^{47}46^1)$. Apply Lemma 5.5 with an $\mathrm{RF}(4^{10}6^1)$ which can be constructed by a 6-intransitive starter and adder as follows:

$\mathrm{RF}(4^{10}6^1) : G = \mathbb{Z}_{40}$, $\quad H = \{0, 10, 20, 30\}$.
$S = \{\{4,19\},\{22,36\},\{1,12\},\{17,35\},\{23,31\},\{16,32\},\{29,38\},\{33,39\},$
$\{25,21\},\{14,26\},\{27,34\},\{3,5\}\} \cup \{9,18,37,28,6,11\}$,
$C = \{\{2,15\},\{8,13\},\{7,24\}\}$. $\quad R = \{\{9,28\},\{26,27\},\{34,37\}\}$,

$(a_1, a_2, \ldots, a_{12}) = (38, 37, 35, 34, 33, 29, 24, 19, 18, 9, 4, 3)$, $A(9) = 6$, $A(18) = 23$, $A(37) = 17$, $A(28) = 16$, $A(6) = 26$, $A(11) = 14$.

Then, we get the desired $\mathrm{RF}(2^{47}46^1)$ and the proof is complete. □

Lemma 5.10. *$u \in U$ for even u, $50 \leq u \leq 56$.*

Proof: Apply Lemma 3.2 with $(t, v) = (13, 4)$, we get an $\mathrm{RF}(2^n u^1)$ for $65 \leq n \leq 130$ and $50 \leq u \leq 56$. A TD(6, 13) exists from Lemma 3.1. All input frames are from Theorem 1.1 (6). By Theorem 1.1 (7) we need deal with the cases $u + 1 \leq n \leq 64$.

For $u = 50, 52$, we apply Lemma 3.3 with a starting frame $\mathrm{RF}(4^{12}v^1)$ for $v = 2, 4$, constructed from a v-intransitive starter and adder as follows:

$\mathrm{RF}(4^{12}2^1) : G = Z_{48}$, $H = \{0, 12, 24, 36\}$,
$S = \{\{2, 15\}, \{4, 19\}, \{23, 43\}, \{44, 3\}, \{21, 39\}, \{18, 37\}, \{26, 42\}, \{28, 38\}, \{9, 11\}, \{20, 41\}, \{25, 47\}, \{22, 45\}, \{8, 14\}, \{35, 46\}, \{7, 16\}, \{30, 31\}, \{5, 10\}, \{13, 27\}, \{17, 34\}, \{32, 40\}\} \cup \{1, 6\}$
$C = \{29, 33\}$, $R = \{3, 6\}$
$(a_1, a_2, \ldots, a_{20}) = (47, 46, 45, 41, 40, 39, 38, 35, 34, 33, 31, 30, 27, 23, 22, 16, 13, 6, 5, 2)$, $A(1) = 10$, $A(6) = 1$.

$\mathrm{RF}(4^{12}4^1) : G = Z_{48}$, $H = \{0, 12, 24, 36\}$,
$S = \{\{2, 15\}, \{4, 19\}, \{23, 43\}, \{44, 3\}, \{21, 39\}, \{18, 37\}, \{26, 42\}, .\{28, 38\}, \{25, 47\}, \{22, 45\}, \{8, 14\}, \{35, 46\}, \{7, 16\}, \{30, 31\}, \{5, 10\}, \{13, 27\}, \{17, 34\}, \{32, 40\}\} \cup \{11, 9, 1, 6\}$
$C = \{\{29, 33\}, \{20, 41\}\}$, $R = \{\{3, 6\}, \{45, 43\}\}$
$(a_1, a_2, \ldots, a_{18}) = (47, 46, 45, 41, 40, 39, 38, 35, 31, 30, 27, 23, 22, 16, 13, 6, 5, 2)$, $A(11) = 42$, $A(9) = 17$, $A(1) = 10$, $A(6) = 1$.

This leaves the existence of an $\mathrm{RF}(2^{51}50^1)$, which can be done by Lemma 5.6 with $(m, g, v) = (4, 11, 6)$.

For $u = 54, 56$, we apply Lemma 5.2 with $(m, g, v) = (4, 13, 2)$ or $(4, 13, 4)$. This leaves the existence of an $\mathrm{RF}(2^{55}54^1)$. Applying Lemma 5.6 with $(m, g, v) = (4, 11, 10)$. we get the desired $\mathrm{RF}(2^{55}54^1)$ and the proof is complete. □

Lemma 5.11. *$u \in U$ for even u, $58 \leq u \leq 138$.*

Proof: Apply Lemma 3.2 with appropriate parameters t and v, we get $\mathrm{RF}(2^n u^1)$ for some values u and n, listed below. We also list the input frames (type and authority). All TD(6, t) exist from Lemma 3.1. By Theorem 1.1 (7) we then know that there is a bound $b(u)$ for $58 \leq u \leq 138$ such that an $RF(2^n u^1)$ exists if $n \geq b(u)$.

t	v	type (authority)	u	n	$b(u)$
15	4	$2^m 4^1$ Th. 1.1 (6)	$58 \leq u \leq 64$	$75 \leq n \leq 150$	75
16	4	$2^m 4^1$ Th. 1.1 (6)	$66 \leq u \leq 68$	$80 \leq n \leq 160$	80
17	16	$2^m 16^1$ Th. 5.1	$70 \leq u \leq 84$	$85 \leq n \leq 170$	85
21	20	$2^m 20^1$ Th. 3.7	$86 \leq u \leq 104$	$105 \leq n \leq 210$	105
23	22	$2^m 22^1$ Th. 3.7	$106 \leq u \leq 114$	$115 \leq n \leq 230$	115
25	24	$2^m 24^1$ Th. 3.7	$116 \leq u \leq 124$	$125 \leq n \leq 250$	125
26	24	$2^m 24^1$ Th. 3.7	$126 \leq u \leq 128$	$130 \leq n \leq 260$	130
27	26	$2^m 26^1$ Th. 3.7	$130 \leq u \leq 134$	$135 \leq n \leq 270$	135
29	28	$2^m 28^1$ Th. 3.7	$136 \leq u \leq 138$	$145 \leq n \leq 290$	145

Next, we shall discuss the interval $u + 2 \leq n \leq b(u) - 1$. For each $u = mg + v$ we shall use Lemmas 5.2–5.4. The parameters are listed below.

u	m	g	v	Lemma	starter-adder	n
58	8	7	2	5.2	1^7 [19]	$60 \leq n \leq 73$
60	8	7	4	5.2	1^7 [19]	$62 \leq n \leq 74$
62	4	15	2	5.2	1^{15} [19]	$64 \leq n \leq 74$
64	4	15	4	5.2	1^{15} [19]	$66 \leq n \leq 74$
66	8	8	2	5.4	2^8 [7]	$68 \leq n \leq 79$
68	8	8	4	5.4	2^8 [7]	$70 \leq n \leq 79$
70	4	17	2	5.2	1^{17} [19]	$72 \leq n \leq 84$
72	4	17	4	5.2	1^{17} [19]	$74 \leq n \leq 84$
74	8	9	2	5.2	1^9 [19]	$76 \leq n \leq 84$
76	8	9	4	5.2	1^9 [19]	$78 \leq n \leq 84$
78	4	19	2	5.2	1^{19} [19]	$80 \leq n \leq 84$
80	4	19	4	5.2	1^{19} [19]	$82 \leq n \leq 84$
86	4	21	2	5.2	1^{21} [19]	$88 \leq n \leq 104$
88	4	21	4	5.2	1^{21} [19]	$90 \leq n \leq 104$
90	8	11	2	5.2	1^{11} [19]	$92 \leq n \leq 104$
92	8	11	4	5.2	1^{11} [19]	$94 \leq n \leq 104$
94	4	23	2	5.2	1^{23} [19]	$96 \leq n \leq 104$
96	4	23	4	5.2	1^{23} [19]	$98 \leq n \leq 104$
98	8	12	2	5.4	2^{12} [7]	$100 \leq n \leq 104$
100	8	12	4	5.4	2^{12} [7]	$102 \leq n \leq 104$
102	4	25	2	5.2	1^{25} [19]	$104 \leq n \leq 104$
106	8	13	2	5.2	1^{13} [19]	$108 \leq n \leq 114$
108	8	13	4	5.2	1^{13} [19]	$110 \leq n \leq 114$
110	4	27	2	5.2	1^{27} [19]	$112 \leq n \leq 114$
112	4	27	4	5.2	1^{27} [19]	$114 \leq n \leq 114$
116	16	7	4	5.3	1^7 [19]	$118 \leq n \leq 124$

u	m	g	v	Lemma	starter-adder	n
118	4	29	2	5.2	1^{29} [19]	$120 \le n \le 124$
120	4	29	4	5.2	1^{29} [19]	$122 \le n \le 124$
122	8	15	2	5.2	1^{15} [19]	$124 \le n \le 124$
126	4	31	2	5.2	1^{31} [19]	$128 \le n \le 129$
130	8	16	2	5.4	1^{16} [7]	$132 \le n \le 134$
132	8	16	4	5.4	1^{16} [7]	$134 \le n \le 134$
136	4	33	4	5.2	1^{33} [19]	$138 \le n \le 144$
138	8	17	2	5.2	1^{17} [19]	$140 \le n \le 144$

Notice that the above list does not provide an RF($2^{74}58^1$), which can be done by applying Construction2.7 with $t = 1$, $g = 7$ and $m = 8$. By Lemma 4.1 we have three CO transversals in RF(1^7). Taking $u = 1$ gives an RF($8^7 2^1$). Here, an IMOLS(9; 1) is used. Observe that the cells (2,3), (3,4), (6,7), (7,8), (1,2), (4,5), (5,6) and (8,1) in [23, Fig. 1] forms a hole partitionable transversal which can be divided into two parts, the first four cells and the last four cells, each containing elements $1, 2, \ldots, 8$. From Remark 2.8 we know that the RF($8^7 2^1$) contains at least 15 HOP transversals with respect to the size two hole. Applying Lemma 3.3 gives an RF($2^{74}58^1$) as desired.

The above list does not provide an RF($2^{84}82^1$) either. Applying Construction 2.7 with $t = 1$, $g = 19$, $m = 4$ and $u = 3$, we obtain an RF($4^{19}6^1$) having at least one HOP transversal with respect to the size six hole. Further, apply Lemma 3.3 with RF($2^5 4^1$) and RF($2^8 6^1$) ([11]) as input designs, we get the desired RF($2^{84}82^1$).

Finally, we deal with the case $n = u + 1$. For any even u, $58 \le u \le 138$, we may write $u = 4g + v$, where g is odd, $13 \le g \le 33$, v = 6, 8, 10, 12. From [19] there exist a starter and adder in Z_g. Then, apply Lemma 5.6.□

Lemma 5.12. *For even u, $140 \le u \le 426$, there exists an RF($2^n u^1$), where $u + 1 \le n \le u + 8$.*

Proof: For any even u, $140 \le u \le 426$, we may write $u = 8g + 2v$, where $10 \le v \le 17$ and g is odd, $15 \le g \le 49$. From [19] there exist a starter and adder in Z_g. Apply Construction 2.7 with $t = 1$, $m = 8$, we get an RF($8^g(2v)^1$) having at least 7 HOP transversals with respect to the size $2v$ hole. Further, apply Lemma 3.3 with input frames RF($2^9 8^1$) and RF($2^n(2v)^1$) for $10 \le v \le 17$ and $2v + 1 \le n \le 2v + 8$, which all exist from [21] and Theorem 3.7. The proof is complete. □

Lemma 5.13. *For even u, $140 \le u \le 426$, there exists an RF($2^n u^1$), where $n \ge u + 9$.*

Proof: For any even u, $140 \le u \le 426$, we may write $u = 10t + s$, where s = 6, 8, 10, 12, 14 and $13 \le t \le 42$. By Lemma 3.1 there is a

TD$(6, 2t+3)$. Apply Lemma 3.2 with $v = 2t+2$. From Theorem 3.7 and Thereoms 5.7–5.11 we have the required input frames RF$(2^n v^1)$ for $v+1 \le n \le 2v+2$. Thus, we obtain an RF$(2^n u^1)$ for $10t+15 \le n \le 20t+30$. Since $u+9 \ge 10t+15$ and $5\lceil u/4 \rceil + 20 \le 20t+30$, the conclusion follows from Theorem 1.1 (7). □

Lemma 5.14. *$u \in U$ for even u, $140 \le u \le 426$.*

Proof: Combine Lemmas 5.12–5.13. □

Combining Lemmas 5.7–5.11 and Lemma 5.14, we have the main result of this section.

Theorem 5.15. *$u \in U$ for even u, $36 \le u \le 426$.*

6. Concluding remarks

Combining the results in Sections 3, 4 and 5, we know that $u \in U$ for any even $u \ge 20$ (Theorem 1.3). For even $u < 20$, an RF$(2^n u^1)$ exists if $n \ge 5\lceil u/4 \rceil + 20$ (Theorem 1.1 (7)). So, there are now finite pairs (n, u) for which the existence is undecided. We believe that these pairs will eventually disappear, that is, we conjecture that for any even $u \ge 4$ there exists an RF$(2^n u^1)$ if and only if $n \ge u+1$. The results on the existence of RF$(1^n u^1)$ and RF$(2^n u^1)$ will serve as a starting point in solving the existence of Room frames of arbitrary hole sizes.

Note added in proof (June 14, 1994) Recent work shows that for any even $u \ge 4$ there exists an RF$(2^n u^1)$ if and only if $n \ge u+1$ except possibly when $n = 19$ and $u = 18$. [D.R. Stinson, L. Zhu and J.H. Dinitz, On the spectra of certain classes of Room frames, preprint.]

References

[1] R.J.R. Abel and D.T. Todorov, *Four MOLS of orders 20, 30, 38 and 44*, J. Combin. Theory (A), to appear.

[2] K. Chen and L. Zhu, *On the existence of skew Room frames of type t^u*, Ars. Combinatoria, to appear.

[3] C.J. Colbourn, *Four MOLS of order 26*, JCMCC **17** (1995) 147–148.

[4] J. Dénes and A.D. Keedwell, "Latin Squares and Their Applications", Academic Press, New York and London, 1974.

[5] J.H. Dinitz and E.R. Lamken, *Uniform Room frames with five holes*, J. Combinatorial Designs **1** (1993) 323–328.

[6] J.H. Dinitz and D.R. Stinson, *A note on Howell designs of odd side*, Utilitas Math. **18** (1980) 207–216.

[7] J.H. Dinitz and D.R. Stinson, *Further results on frames*, Ars Combinatoria **10** (1981) 275–288.

[8] J.H. Dinitz and D.R. Stinson, *Room squares and related designs* in: "Contemporary Design Theory", Wiley, 1992, 137–204.

[9] J.H. Dinitz and D.R. Stinson, *A few more Room frames*, in "Graphs Matrices and Designs", R. Rees ed., Marcel Dekker, 1993, 133–146.

[10] B. Du and L. Zhu, *The existence of incomplete Room squares*, JCMCC, **14** (1993) 183–192.

[11] G. Ge, *On the existence of Room frames of type* $2^n u^1$, preprint.

[12] G. Ge and L. Zhu, *On the existence of Room frames of type* t^u *for* $u = 4$ *and 5*, J. Combinatorial Designs **1** (1993) 183–191.

[13] K. Heinrich, C.C. Lindner and C.A. Rodger, *Almost resolvable decompositions of* $2k_n$ *into cycles of odd length*, J. Combinatorial Theory (A) **49** (1988) 218–232.

[14] J. D. Horton, *Orthogonal starters in finite abelian groups*, Discrete Math. **79** (1989/1990) 265–278.

[15] C.C. Lindner, C.A. Rodger and D.R. Stinson, *Nesting of cycle systems of odd length*, Discrete Math. **77** (1989) 191–203.

[16] C.A. Rodger, *Graph decompositions*, Le Mathematiche XLV (1990) 119–140.

[17] C.A. Rodger, *Linear spaces with many small lines*, Discrete Math., to appear.

[18] C.A. Rodger, E.B. Wantland, K. Chen and L. Zhu, *Existence of certain skew Room frames with application to weakly 3-chromatic linear spaces*, J. Combinatorial Designs, to appear.

[19] R.G. Stanton and R.C. Mullen, *Construction of Room squares*, Ann. Math. Stat. **39** (1968) 1540–1548.

[20] D.R. Stinson, *Some constructions for frames, Room squares, and subsquares*, Ars Combinatoria **12** (1981) 229–267.

[21] D.R. Stinson, Private communication.

[22] D.R. Stinson and L. Zhu, *On the existence of three MOLS with equal-sized holes*, Austrialasian J. Combinatorics **4** (1991) 33–47.

[23] D.R. Stinson and L. Zhu, *Towards the spectrum of Room squares with subsquares*, J. Combinatorial Theory (A) **63** (1993) 129–142.

[24] W.D. Wallis, *Solution of the Room square existence problem*, J. Combinatorial Theory (A) **17** (1974) 379–383.

[25] S.M.P. Wang, *On self-orthogonal Latin squares and partial transversal of Latin squares*, Ph. D. thesis, Ohio State University, Columbus, Ohio (1978).

[26] L. Zhu, *Orthogonal Latin squares with subsquares*, Discrete Math. **48** (1984) 315–321.