Complete Sets Of Pairwise Orthogonal Latin Squares Of Order 9

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ABSTRACT. We define two complete sets $\mathcal{L}$ and $\mathcal{L}'$ of pairwise orthogonal 9×9 Latin squares to be equivalent if and only if $\mathcal{L}'$ can be obtained from $\mathcal{L}$ by some combination of (i) applying a permutation θ to the rows of each of the 8 squares in $\mathcal{L}$ (ii) applying a permutation ϕ to the columns of each square from $\mathcal{L}$ and (iii) permuting the symbols separately within each square from $\mathcal{L}$. We use known properties of the projective planes of order 9 to show that, under this equivalence relation, there are 19 equivalence classes of complete sets. For each equivalence class, we list the species and transformation sets of the 8 Latin squares in a complete set. As this information alone is not sufficient for determining the equivalence class of a given complete set, we provide a convenient method for doing this.

1 Introduction

A Latin square of order n is an $n \times n$ array the entries of which comprise n symbols (here taken to be $1, 2, \ldots, n$), with each symbol occurring exactly once in each row and exactly once in each column. Two Latin squares of order n are orthogonal to each other if the n^2 ordered pairs of corresponding entries are all different. A complete set of pairwise orthogonal Latin squares of order n consists of $n-1$ Latin squares, each orthogonal to each of the others. Such complete sets are known to exist for n equal to any prime power [3, Section 5.2].

By merely relabeling the symbols in each square of any complete set of pairwise orthogonal Latin squares, we can arrange that the entries in the first row of each square are in ascending order. We therefore restrict our attention to complete sets of pairwise orthogonal Latin squares with this property. For brevity, we call them simply *complete sets.*

All complete sets of order n are obtainable from projective planes of order n. For some values of n, of which 9 is the smallest, there is more than one projective plane. Moreover, a single projective plane may give rise to complete sets having different structures, as described below. Even for $n = 9$, no systematic account of the available complete sets seems to have been given. This paper fills the gap by collating and extending previous work. It also provides a solution to Problem 8.2 in [3, p.490], which asks for the number of non-equivalent complete sets of order 9.

Four projective planes of order 9 have been known for many years and there are no others [9]. In the notation of Room and Kirkpatrick [14], the four planes are the desarguesian plane Φ, the translation plane Ω, the dual Ω^D of the translation plane and the Hughes plane Ψ. To construct a complete set from a given projective plane we must first choose one of its lines, which we then call the line at infinity ℓ_∞; when this line and the points on it are deleted from the projective plane, the remaining lines and points constitute an affine plane. Two non-isomorphic affine planes are obtainable in this way from each of the planes Ω, Ω^D and Ψ, but only one from Φ. There are thus 7 affine planes to be considered, but as we need to refer to particular points on ℓ_∞ we use mostly the terminology of projective planes.

The next step in constructing a complete set is to choose two points E and A on ℓ_∞. We denote the other points on ℓ_∞ by E_k, $2 \leq k \leq n$ and reserve E_0 and E_1 as alternative symbols for E and A respectively, as in [11]. The complete set will consist of Latin squares L_k, one corresponding to each of the points E_k, $2 \leq k \leq n$. We label arbitrarily the lines through E and A, other than ℓ_∞, as e_i, $1 \leq i \leq n$, and a_j, $1 \leq j \leq n$, respectively. We then set the (i, j)th entry of L_k equal to the integer m, $1 \leq m \leq n$, such that $e_1 \cap a_m$ lies on the line joining E_k to $e_i \cap a_j$. Thus the rows and columns of the Latin squares correspond to the lines other than ℓ_∞ through E and A respectively.

For the projective planes Ω, Ω^D and Ψ, the outcome of the above construction depends importantly not only on the choice of ℓ_∞ but also on the choice of E and A. We distinguish between 5 outcomes for Ω, another 5 for Ω^D and 8 for Ψ. Together with the outcome for Φ we thus have a total of 19 possibilities, which we present as 19 equivalence classes. As 19 is also the number of 'completed planes' found by Hall, Swift and Killgrove [6] in their search for further projective planes of order 9, we emphasise that the completed planes do not correspond one-one with our equivalence classes (see below).

2 Classification of the complete sets of order 9

We call two complete sets $\mathcal{L}$ and $\mathcal{L}'$ of order n *equivalent* if and only if there exist permutations θ and ϕ of the first n natural numbers such that the following transformation converts $\mathcal{L}$ into $\mathcal{L}'$:

($T.\theta.\phi$) Permute the rows of every square so that row i becomes row $i\theta$, $1 \leq i \leq n$. Permute the columns of every square so that column j becomes column $j\phi$, $1 \leq j \leq n$. Then permute the symbols, in each square separately, so that the new first rows are finally in natural order.

This definition differs from the definition of equivalence given in [3, p.276] and from the conflicting definition in [3, p.168].

The transformation $T.\theta.\phi$ has the same effect as has successive application of transformations $T1.\theta$ and $T2.\phi$ defined in [11]. Since $T.\theta.\phi$ may be interpreted geometrically as a mere renumbering of the lines through E and A, complete sets obtained from the same plane and the same points E and A are equivalent. Equivalence of complete sets is clearly an equivalence relation.

We show below that the four projective planes of order 9 yield exactly 19 equivalence classes of complete sets. We denote the equivalence classes by $\mathbf{L}_p$, $1 \leq p \leq 19$, and any complete set in $\mathbf{L}_p$ by $\mathcal{L}_p$. All the required properties of the four planes are given in [14].

The desarguesian plane Φ has the property:

(P1) Φ is transitive on proper quadrangles.

By P1, only one affine plane can be obtained from Φ, to within isomorphism. Moreover, all choices of E and A are equivalent, that is, yield equivalent complete sets. Let $\mathbf{L}_1$ denote the class of complete sets obtained from Φ.

Every collineation of the translation plane Ω preserves a special line, called the translation line t, and preserves a special partitioning of the set of points on t into five pairs [14, Theorem 4.3.12 and p.129(2)]. Denote the point paired with any point P on t by P'. The plane Ω is:

(P2a) transitive on pairs of points not on t,

(P2b) (V, ℓ')-transitive for all points V on t and lines ℓ' through V'.

For proofs, see [14, Theorems 4.2.4, 4.3.6]. By P2a, the plane Ω is transitive on lines other than t so it yields exactly two affine planes, one by taking $\ell_\infty = t$ and the other by taking any other line as ℓ_∞.

First, take $\ell_\infty = t$. By P2a, the plane Ω is transitive on points of t so all choices of E on ℓ_∞ are equivalent. Let $\mathbf{L}_2$ be the class of complete sets which corresponds to the choice $A = E'$. By P2b, with $V = E$, all other choices of A are equivalent; let $\mathbf{L}_3$ be the corresponding class of complete sets.

Now take $\ell_\infty \neq t$. By P2a, when $E = \ell_\infty \cap t$ all choices of A are equivalent, when $A = \ell_\infty \cap t$ all choices of E are equivalent and when neither E nor A is on t all choices of E and A are equivalent. Let the corresponding classes of complete sets be $\mathbf{L}_4$, $\mathbf{L}_5$ and $\mathbf{L}_6$ respectively.

Equivalence class of complete sets	Projective plane	Line at infinity	Choice of E and A	Transformation sets
$\mathbf{L}_1$	Φ	any	any	$8a$
$\mathbf{L}_2$	Ω	$\ell_\infty = t$	$A = E'$	$8a$
$\mathbf{L}_3$			$A \neq E'$	$8a$
$\mathbf{L}_4$		$\ell_\infty \neq t$	$E\,I\,t$	$8b_R$
$\mathbf{L}_5$			$A\,I\,t$	$8b_c$
$\mathbf{L}_6$			$E, A \not I\, t$	$a, b_S, 6c$
$\mathbf{L}_7$	Ω^D	$\ell_\infty\,I\,T$	$E = T$	$8a$
$\mathbf{L}_8$			$A = T$	$8a$
$\mathbf{L}_9$			$E, A \neq T$	$2a, 6d$
$\mathbf{L}_{10}$		$\ell_\infty \not I\, T$	$AT = (ET)'$	$8b_S$
$\mathbf{L}_{11}$			$AT \neq (ET)'$	$b_R, b_C, 6e$
$\mathbf{L}_{12}$	Ψ	real	E, A real	$2a, 6f_S$
$\mathbf{L}_{13}$			E real, A complex	$3f_C, 2g_R, 3h_R$
$\mathbf{L}_{14}$			E complex, A real	$3f_R, 2g_C, 3h_C$
$\mathbf{L}_{15}$			E, A complex	$a, 3d, 4g_S$
$\mathbf{L}_{16}$			E, A complex	$4d, 4h_S$
$\mathbf{L}_{17}$		complex	E real	$8i_R$
$\mathbf{L}_{18}$			A real	$8i_C$
$\mathbf{L}_{19}$			E, A complex	$i_S, 6j, k$

Table 1

Thus, Ω yields 5 equivalence classes of complete sets; of these, 2 are obtained from one affine plane and 3 from the other.

The first six rows of Table 1 summarize our results so far. In the table, the relation of incidence between a point and a line is denoted by I and its negation by $\not{I}$. The last column of Table 1 will be explained later.

Every collineation of Ω^D preserves a special point, called the translation point T, and preserves a special partitioning of the set of lines through T into five pairs. Denote the line paired with any line ℓ through T by ℓ'. The plane Ω^D is:

(P3a) transitive on pairs of lines not passing through T,

(P3b) (V', ℓ)-transitive for all lines ℓ through T and points V' on ℓ'.

These are the properties dual to P2a and P2b. The plane Ω^D yields two affine planes and 5 equivalence classes of complete sets, $\mathbf{L}_p$, $7 \leq p \leq 11$. The proofs are similar, but not exactly dual, to those for Ω. The results are in Table 1.

The points and lines of the Hughes plane Ψ may be classified as real or complex in such a way that the real points and lines form a special subplane of order 3 which is preserved under all collineations [14]. The collineation group $G(\Psi)$ is:

(P4a) transitive on each of the four types of flag (P, ℓ), namely those with each of P and ℓ either real or complex,

(P4b) transitive on real (ordered) quadrangles.

For proofs, see [14, Theorem 5.1.6]. By P4a and since ℓ_∞ can be either real or complex, Ψ yields exactly two affine planes.

First, let ℓ_∞ be real. By P4b, all choices of real E and A are equivalent, so there is just one corresponding class of complete sets, $\mathbf{L}_{12}$. By P4a, all choices of real E are equivalent. Also, given the point E, all choices of a complex A on ℓ_∞ are equivalent [14, Theorem 5.5.1]. Hence there is just one class $\mathbf{L}_{13}$ corresponding to real E and complex A and, similarly, just one class $\mathbf{L}_{14}$ corresponding to real A and complex E. However, there are two classes $\mathbf{L}_{15}$ and $\mathbf{L}_{16}$ corresponding to complex E and A (see below).

Now let ℓ_∞ be complex. Exactly one point of ℓ_∞ is real. If E is this real point then, by P4a, all choices of A are equivalent, so there is just one corresponding class $\mathbf{L}_{17}$. Similarly, there is just one class $\mathbf{L}_{18}$ corresponding to real A and complex E. Finally, by P4a, all choices of a complex E on ℓ_∞ are equivalent. Also (see below), given a complex E, all choices of a second complex point A on ℓ_∞ are equivalent. Hence there is just one class of complete sets, $\mathbf{L}_{19}$, corresponding to complex ℓ_∞, E and A.

Thus, the plane Ψ has yielded 8 classes of complete sets, 5 from one of the affine planes and 3 from the other. Our enumeration of equivalence classes of complete sets is now complete and we have found $1 + 5 + 5 + 8 = 19$ of them.

3 Species and transformation sets of Latin squares

To describe the structure of a complete set and to distinguish between non-equivalent complete sets we use some concepts due to Norton [10] (see also [3]).

A Transformation of a Latin square consists of three permutations, in general all different, applied to the row labels, column labels and symbols. Two Latin squares are isotopic if one may be converted into the other by a Transformation. Isotopy is an equivalence relation and the equivalence classes are called transformation sets (or isotopy classes).

A Latin square has three constraints, namely rows, columns and symbols. Two Latin squares are conjugate to one another (or parastrophic) if one may be obtained from the other by a permutation of the three constraints. A species (or main class) of Latin squares consists of all members of a transformation set together with all their conjugates. A species comprises 1, 2, 3 or 6 transformation sets [3, Theorem 4.2.1].

An intercalate of a Latin square of order n is a Latin subsquare of order two. Latin squares of the same species all have the same number of intercalates. Counting intercalates and noting the patterns of their occurrence in Latin squares are useful techniques for distinguishing between species. It may also be useful to consider Latin subsquares of higher order.

The 19 equivalence classes of complete sets of order 9 involve a total of 11 species of Latin squares, which we denote by the letters a to k inclusive. Table 2 shows the numbers of transformation sets in these species and also the numbers of intercalates and of Latin subsquares of order 3 in Latin squares of these species. Species j was discussed by Parker and Killgrove [13].

No species with 2 or 6 transformation sets are involved in complete sets of order 9. Where a species has 3 transformation sets, two of the three constraints are equivalent in the sense that their interchange does not cause a change of transformation set. It is then convenient to distinguish the three transformation sets by a subscript R, C or S (standing for rows, columns or symbols) to indicate the constraint which is not equivalent to either of the others. This notation is used in the final column of Table 1. For example, the entry "$2a, 6f_S$" indicates that two of the Latin squares in a complete set $\mathcal{L}_{12}$ belong to species a and the other six belong to the particular transformation set in species f which consists of Latin squares isotopic to their transposes.

Species	No. of Transformation sets	No. of intercalates	No. of latin subsquares of order three
a	1	0	36
b	3	48	12
c	1	32	0
d	1	72	0
e	1	24	4
f	3	36	18
g	3	0	9
h	3	0	12
i	3	24	6
j	1	24	3
k	1	0	18

Table 2

4 Transformations between complete sets

Several non-equivalent complete sets of order 9 are familiar from the literature. Those given in [1, 4,15], [5], [12] and [2] belong to the classes $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_{11}$ and $\mathbf{L}_{12}$ respectively. Seven complete sets, one corresponding to each of the affine planes of order 9, are given by Kamber [8] and reproduced in [7]. These complete sets belong to $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_5$, $\mathbf{L}_7$, $\mathbf{L}_{10}$, $\mathbf{L}_{12}$ and $\mathbf{L}_{18}$; not all of them are distinct from those given in the other references cited above.

Hall, Swift and Killgrove [6] examined complete sets each of which includes at least one Latin square of species a. However, 10 of our 19 equivalence classes $\mathbf{L}_i$ do not involve species a (see Table 1, last column), so the 19 'completed planes' of [6] correspond to just 9 of the equivalence classes.

This section shows how every complete set of order 9 may be derived, by means of certain transformations defined in [11], from as few as three given complete sets. These three must include complete sets corresponding to Φ, Ω (or Ω^D) and Ψ so, for instance, those given in [4], [5] and [2] would be suitable. Our definition of equivalence indicates how to convert a given complete set into any equivalent complete set, so our task is to show how to obtain one complete set in each class $\mathbf{L}_p$. To make the present paper self-contained we repeat the definitions of the required transformations T3, T4.r and T5. Proofs are in [11].

Let $\mathcal{L} = \{L_k : 2 \le k \le n\}$ be a complete set of order n. The transformation T3 is as follows:

(T3) For $2 \le k \le n$, replace L_k by its transpose (in the matrix sense) and then permute the symbols in each square so that the new first rows are finally in natural order.

Geometrically, T3 corresponds to interchanging the points E and A.

For the other two transformations we need further notation. Let L_1 denote the row-Latin, but non column-Latin, square [3, p.104] whose rows are all in natural order and which is related to the point $E_1(= A)$ exactly as L_k is related to E_k, $2 \leq k \leq n$. Let $R = (\rho_{ik})$ be the $n \times n$ array whose (i, k)th entry ρ_{ik} is the permutation that converts the first row of L_k into the ith row. We call R the *representational array* of $\mathcal{L}$. In a representational array, every entry in the first row and in the column that represents the row-Latin square (in R, the first column) is ϵ, the identity permutation. Now we can define T4.r:

(T4.r) Choose r, $2 \leq r \leq n$. Replace R by $S = (\sigma_{ik})$, where $\sigma_{ik} = \rho_{ir}^{-1}\rho_{ik}$ for all (i, k).

Geometrically, T4.r corresponds to moving A from E_1 to E_r with no change in ℓ_∞ or E. In S it is the rth column whose entries are ϵ and the other $n - 1$ columns that specify the new complete set. The first column of S specifies a Latin square whose row-permutations are merely the inverses of those of L_r. This square is of the same species as L_r but with two of the constraints, columns and symbols, interchanged.

The transformation T5 is as follows:

(T5) Replace R by $R^D = (\rho_{ik}^D)$, where $\rho_{ik}^D = \rho_{ki}^{-1}$. In other words, transpose R and then replace every entry by the inverse permutation.

Geometrically, T5 corresponds to replacing the plane Π, which is represented by R, by the dual plane Π^D. The points and lines E, $A(= E_1)$, ℓ_∞ and e_1 of Π correspond to the lines and points ℓ_∞^D, e_1, E^D and A^D of Π^D, respectively. Which complete set is obtained by the use of T5 depends on the numbering of the Latin squares in $\mathcal{L}$. A change of numbering results in a different, but equivalent, complete set.

By using the geometrical interpretations of these transformations and the information from Table 1 concerning the choice of ℓ_∞, E and A, it can be seen that representative complete sets $\mathcal{L}_p$ in the classes $\mathbf{L}_p$ corresponding to the projective planes Ω and Ω^D may be derived from one another as shown in Figure 1. Because T3 and T5 coincide with their inverses, no arrows are placed on the corresponding lines of the diagram. The abbreviation of T4.r to T4 indicates that, when the transformation acts in the sense indicated by the arrows, the value of r is arbitrary. The inverse transformations are also of type T4.r, but now the choice of r is important. For instance, if $\mathcal{L}_{11}$ is taken to be the complete set due to Paige and Wexler [12] then, to obtain a complete set in $\mathbf{L}_{10}$, the integer r must be the number given to the unique Latin square in $\mathcal{L}_{11}$ that is in the transformation set b_C. This is the square denoted by L_1 in [12]. Any other choice of r leads merely to another complete set in $\mathbf{L}_{11}$.

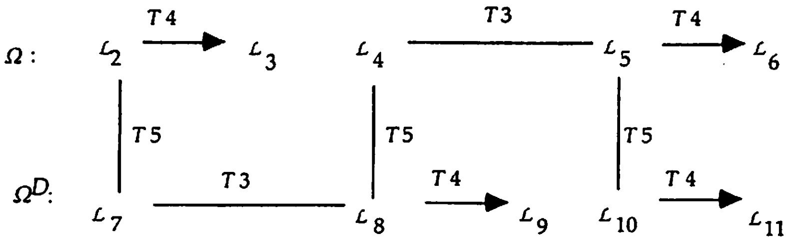

Figure 1

Table 3 gives an example of a complete set in $\mathbf{L}_8$. If this complete set is taken as $\mathcal{L}_8$ and the transformations T3 and then T5 are applied, the resulting complete set $\mathcal{L}_2$ coincides with that of Fisher and Yates [5] although the squares will be differently numbered.

L_2	L_3	L_4	L_5
123456789	123456789	123456789	123456789
231564897	312645978	765189432	946817325
312645978	231564897	498732156	587293641
456789123	789123456	231975648	875341962
564897231	978312645	657248913	231689574
645978312	897231564	984613275	469725138
789123456	456789123	312867594	694538217
897231564	645978312	576394821	758162493
978312645	564897231	849521367	312974856

L_6	L_7	L_8	L_9
123456789	123456789	123456789	123456789
854971263	498732156	679328514	587293641
679328514	765189432	854971263	946817325
548692371	312867594	967214835	694538217
796135428	849521367	485763192	312974856
231847956	576394821	312589647	758162493
967214835	231975648	548692371	875341962
312589647	984613275	231847956	469725138
485763192	657248913	796135428	231689574

Table 3 A complete set $\mathcal{L}_8$.

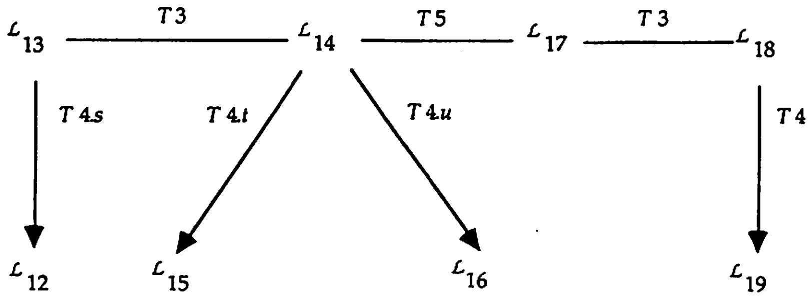

Figure 2

Figure 2 shows how representatives of the eight classes of complete sets corresponding to the Hughes plane Ψ may be derived from one another. Table 4 gives an example of a complete set $\mathcal{L}_{14}$ in $\mathbf{L}_{14}$. In this complete set, L_2 and L_3 belong to the transformation set g_C, while L_4, L_5 and L_6 belong to h_C and L_7, L_8 and L_9 belong to f_R. Geometrically, three of the real points on ℓ_∞ correspond to L_7, L_8 and L_9 and the other real point is A. If T3 is applied to $\mathcal{L}_{14}$, followed by T4.s with $s = 7$, 8 or 9, the resulting complete set $\mathcal{L}_{12}$ is the same as that of Bose and Nair [2], except that our notation is slightly different and the squares will be differently numbered.

L_2	L_3	L_4	L_5
123456789	123456789	123456789	123456789
312564897	231645978	756189423	964817235
231645978	312564897	489723156	578392641
546978213	879213645	231597864	785124396
654789132	798321564	567834291	231968457
465897321	987132456	894261537	649573812
987321564	654897321	312678945	496285173
798213645	546789132	675942318	857631924
879132456	465978213	948315672	312749568

L_6	L_7	L_8	L_9
123456789	123456789	123456789	123456789
845971362	498732156	679328514	587293641
697238514	765189432	854971263	946817325
458369127	312645978	967832451	694781532
976512843	849273615	485197326	312645978
231784695	576918243	312645978	758329164
769143258	231564897	548719632	875932416
312895476	984327561	231564897	469178253
584627931	657891324	796283145	231564897

Table 4 A complete set $\mathcal{L}_{14}$

Now take $\mathcal{L}_{14}$, as given in Table 4, and apply two particular cases of the transformation $T.\theta.\phi$ defined earlier. First, take $\theta = (23)(56)(89)$ and $\phi = (47)(58)(69)$. This leads to a complete set composed of the same eight squares as $\mathcal{L}_{18}$ but reordered according to the permutation $(L_2L_3)(L_5L_6)(L_8L_9)$. Since the permutation just interchanges L_2 and L_3, they are in the same transformation set (as stated above) and also the transformations T4.2 and T4.3 convert $\mathcal{L}_{14}$ into equivalent complete sets. Thus, in Figure 2, $t = 2$ or 3. Secondly, take $\theta = \phi = (456)(798)$. Again, the transformation leads to the same set of eight squares in a different order, given here by the permutation $(L_4L_5L_6)(L_7L_9L_8)$. Hence the transformations T4.u with $u = 4$, 5 or 6 convert $\mathcal{L}_{14}$ into equivalent sets. Thus we obtain only two non-equivalent complete sets from $\mathcal{L}_{14}$ by moving A to a complex point; these are members of $\mathbf{L}_{15}$ and $\mathbf{L}_{16}$.

Let $\mathcal{L}_{18}$ be the complete set obtained from the set $\mathcal{L}_{14}$ in Table 4 as indicated in Figure 2. If $T.\theta.\phi$, with $\theta = (24983756)$ and $\phi = (1798)(23)(46)$, is applied to $\mathcal{L}_{18}$ the new complete set consists of the same eight squares reordered by the cyclic permutation $(L_2L_4L_5L_8L_3L_7L_9L_6)$. This shows that all eight squares are in the same transformation set and also that the eight complete sets obtained by applying T4.r, $2 \le r \le 9$, to $\mathcal{L}_{18}$ are all equivalent; they belong to $\mathbf{L}_{19}$. The abbreviated notation T4 can thus be used for the transformation from $\mathcal{L}_{18}$ to $\mathcal{L}_{19}$ in Figure 2.

5 Identification of the class of a complete set

We now describe a "Do It Yourself" method for identifying the particular class $\mathbf{L}_p$ to which a given complete set $\mathcal{L}$ of order 9 belongs. Of course, many alternative procedures could be devised.

Let $\mathcal{L} = \{L_k \colon 2 \le k \le 9\}$. Count the numbers of intercalates in all the Latin squares L_k. Knowledge of these numbers is sufficient to fix p if it is 6, 9, 11, 12, 15, 16 or 19. Otherwise, decide which of the following four cases applies and use the corresponding method.

Case 1. Every square L_k has 0 intercalates.

Here, $p = 1$, 2, 3, 7 or 8 (and the squares L_k are all of species a). If the squares do not all have the same set of nine rows then, by [11, Theorem], the corresponding projective plane is not (E, ℓ_∞)-transitive and it follows that $p = 8$. The complete set given in Table 3 is of this type. If the squares all have the same set of rows, apply the transformation T3 to $\mathcal{L}$ and denote the new complete set by $\mathcal{L}'$. If the squares in $\mathcal{L}'$ do not all have the same set of rows, then $\mathcal{L}' \in \mathbf{L}_8$ and hence $p = 7$ (see Figure 1).

Now suppose that, in both $\mathcal{L}$ and $\mathcal{L}'$, all eight Latin squares have the same set of nine rows. Take the representational array R of $\mathcal{L}$ and delete all entries equal to ϵ, the identity permutation, leaving a Latin square of order 8. Permute rows 2 to 8 of this Latin square to obtain a Latin square R^* whose first column is in the same order as its first row. If R^* is symmetric in the matrix sense, then $p = 1$ (and R^*, bordered with the elements of its first row and column, is the multiplication table of the cyclic group C_8). If R^* is not symmetric but still, suitably bordered, forms the multiplication table of a group, then $p = 2$ (and the group is the quaternion group). Finally, if R^* fails to satisfy the quadrangle criterion [3, Theorem 1.2.1], then $p = 3$.

Case 2. Every square L_k has 48 intercalates.

Here, $p = 4$, 5 or 10. If the squares L_k do not all have the same set of nine rows, then $p = 4$ (and pairs of the squares have the same set of rows). Otherwise, apply T3 to $\mathcal{L}$ and denote the new complete set by $\mathcal{L}'$. If the squares in $\mathcal{L}'$ do not all have the same set of nine rows, then $\mathcal{L}' \in \mathbf{L}_4$ and hence $p = 5$ (see Figure 1). Otherwise, $p = 10$.

Case 3. Three of the squares L_k have 36 intercalates and the rest have none.

Here, $p = 13$ or 14. Let $\mathcal{L}'$ denote the complete set obtained from $\mathcal{L}$ by the transformation T5. If every square in $\mathcal{L}'$ has 24 intercalates, then $\mathcal{L}' \in \mathbf{L}_{17}$ and hence $p = 14$ (see Figure 2). If no square in $\mathcal{L}'$ has 24 intercalates then $\mathcal{L}' \in \mathbf{L}_{13}$ or $\mathbf{L}_{12}$ and hence $p = 13$. To apply this test, only one square in $\mathcal{L}'$ need be obtained.

Case 4. Every square L_k has 24 intercalates.

Here, $p = 17$ or 18. If pairs of the squares L_k have the same set of nine rows, then $p = 18$. Otherwise, $p = 17$.

Finally, eight of the eleven species of Latin squares that occur in complete sets of order 9 are represented in Tables 3 and 4 or in the references cited earlier. Table 5 gives sample Latin squares of the other three species c, d and k.

c	d	k
123456789	123456789	123456789
987234561	576918243	231645897
546918372	984327561	312564978
678321945	648231975	456789132
351789624	391765824	564897321
792645138	752849136	645978213
864597213	867594312	798123456
435162897	439182657	879231564
219873456	215673498	987312645

Table 5. Sample latin squares of species c, d and k.

6 Concluding remarks

The main results of this paper are in Tables 1 and 2. With their help we have shown how to identify the equivalence class of a given complete set of order 9. There are various questions about complete sets of order 9 whose answers can be read off at once from Tables 1 and 2. For instance:

(1) How many non-equivalent complete sets have at least one member from species a? (Answer: 9.)

(2) How many non-equivalent complete sets have all 8 members from species a? (Answer: 5.)

(3) Given a Latin square from a species with 6 different transformation sets, can it be a member of a complete set? (Answer: No.)

(4) In a complete set of order 9, what is the greatest number of Latin squares with no Latin subsquares of order 3? (Answer: 6.)

References

[1] R.C. Bose, On the application of the properties of Galois fields to the construction of hyper-Graeco-Latin squares, *Sankhya* 3(1938), 323–338.

[2] R.C. Bose and K.R. Nair, On complete sets of Latin squares, *Sankhya* 5(1941), 361–382.

[3] J. Dénes and A.D. Keedwell, "Latin Squares and their Applications", Akadémiai Kiadó, Budapest; English Universities Press, London; Academic Press, New York, 1974.

[4] R.A. Fisher, "The Design of Experiments", Oliver and Boyd, Edinburgh, 8th edition, 1966.

[5] R.A. Fisher and F. Yates, "Statistical Tables for Biological, Agricultural and Medical Research", Oliver and Boyd, Edinburgh, 6th edition, 1963.

[6] M. Hall, Jr., J.D. Swift and R. Killgrove, On projective planes of order nine, *MTAC* 13(1959), 233–246.

[7] C.A.J. Hurkens and J.J. Seidel, Conference matrices from projective planes of order 9, *Europ. J. Combinatorics* 6(1985), 49–57.

[8] R. Kamber, "Untersuchungen über affine Ebenen der Ordnung 9", Thesis, ETH Zürich (1976).

[9] C.W.H. Lam, G. Kolesova and L. Thiel, A computer search for finite projective planes of order 9, *Discrete Math.* 92(1991), 187–195.

[10] H.W. Norton, The 7 × 7 squares, *Ann. Eugenics* 9(1939), 269–307.

[11] P.J. Owens, Complete sets of pairwise orthogonal Latin squares and the corresponding projective planes, *J. Combin. Theory Ser. A* 59(1992), 240–252.

[12] L.J. Paige and C. Wexler, A canonical form for incidence matrices of finite projective planes and their associated Latin squares, *Portugaliae Math.* 12(1953), 105–112.

[13] E.T. Parker and R.B. Killgrove, A planar Latin square lacking automorphism, *J. Combin. Theory Ser. A* 27(1979), 404–406.

[14] T.G. Room and P.B. Kirkpatrick, "Miniquaternion Geometry", Cambridge University Press, Cambridge, 1971.

[15] W.L. Stevens, The completely orthogonalized Latin square, *Ann. Eugenics* 9(1939), 82–93.