On Unitary Cayley Graphs

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ABSTRACT. We deal with a family of undirected Cayley graphs X_n which are unions of disjoint Hamilton cycles, and some of their properties, where n runs over the positive integers. It is proved that X_n is a bipartite graph when n is even. If n is an odd number, we count the number of different colored triangles in X_n.

1. Introduction

The graphs considered in this paper are undirected, simple and without loops. Given a graph X_n, we denote its vertex set and edge set by $V(X)$ and $E(X)$, respectively. Given a positive integer n and an element x of the additive cyclic group Z_n of integer residue classes mod n, there is only one integer representative y of x such that $0 \leq y < n$; we use the same symbol x to denote y and define $|x| = x$ if $x \leq n/2$ and $|x| = n - x$ if $x > n/2$.

The group Z_n possesses the subset $U_n = U(Z_n)$ of units modulo n, constituted by those x in Z_n with integer representatives relatively prime to n. It is known that U_n is a multiplicative group. Let $W_n = U_n/Z_2$ be given by the classes $\{x, -x\}$, where x varies in U_n. Notice that W_n inherites a group structure from U_n. We represent each $\{x, -x\} \in W_n$ again by x, where $0 < x \leq [n/2]$, unless confusion arises.

We deal with a class $\{X_n\}$ of undirected Cayley graphs X_n, that we call "Unitary Cayley graphs", defined as follows: $V(X_n) = Z_n$ and any two vertices v_1 and v_2 are adjacent if and only if $|v_1 - v_2| \in U_n$. Each edge

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of X_n is attributed a color from the set $\{1,2,\ldots,k\}$, where $k = [n/2]$, according to the following rule: If e is an edge with endvertices v_1 and v_2 and if $v_1 - v_2 \equiv \pm i \pmod n$, $i \in \{1,2,\ldots,k\}$, then we color the edge e with i. According to the definition of X_n, we have that $i \in U_n$. Therefore, X_n is the undirected Cayley graph of Z_n with generator set W_n. Written Cay $[Z_n, W_n]$, hence, we may say that X_n has an edge coloring with one half of the unit element of Z_n. For intance, if $n = 9$ then $U_n = \{1,2,4,5,7,8\}$ and $W_n = \{1,2,4\}$. Thus, $X_9 =$ Cay $[Z_9, W_9]$ is represented in the Figure 1.1.

In [1] Dejter dealt with Cayley graphs of the form Cay $[Z_n, I_n]$, where I_n is the generator set $\{1,2,\ldots,k\}$, n is odd and $k = (n-1)/2$, i.e., the complete graphs K_n edge-colored in a symmetric fashion, and studied some induced subgraphs K_r of these Cayley graphs that were called totally multicolored (TMC) subgraphs, motivated by a question of Erdös, Pyber and Tuza [2].

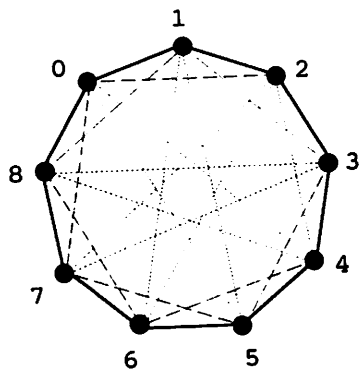

Figure 1.1

In the present paper, we deal with some properties of the X_n. In particular, besides proving that if n is even, then X_n is bipartite we count the number of different colored triangles of X_n, when n is odd.

2. Basic Properties of Unitary Cayley Graphs

We now state some properties of the Unitary Cayley graphs. The proofs are easy and left to the reader.

Proposition 2.1. *X_n is a regular graph of degree $\phi(n)$, where $\phi(n)$ is the Euler ϕ-function, and is a union of $\phi(n)$ Hamilton cycles of length n, for any positive integer n.*

Proposition 2.2. *X_n is isomorphic to a complete graph with n vertices, if n is a prime number and isomorphic to a complete bipartite graph $K(2^{t-1}, 2^{t-1})$ if $n = 2^t$.*

The graphs X_n for $n = 2^t$ are interesting from the chromatic point of view. They are regular of degree one half of the number of vertices and the number of edges turns out to be the square of the degree of the graph. These graphs are a special type studied in [3]; they are chromatically unique.

Unitary Cayley graphs have different properties according to the parity of n, as will be seen in the next proposition.

Proposition 2.3. *X_n is a bipartite graph if n is an even number.*

Proof: Since n is even, units of Z_n must be odd. Thus, no two even labeled vertices are adjacent. This implies that the even labeled vertices and the odd labeled vertices form a bipartition of the vertex set. □

Corollary 2.3.1. *If n is even, then X_n has no odd length cycles. In particular, X_n is triangle-free.*

3. Number of Triangles in X_n

Since X_n is triangle-free for n even, we will consider n to be an odd number in this section. Let us denote by $\{a, b, c\}$ a triangle in X_n with vertices a, b and c; therefore $\pm(b-a)$, $\pm(c-b)$, $\pm(a-c)$ are elements in U_n. Without loss of generality we may assume that our triangles have vertices $\{0, 1, u\}$, $u \in U_n$. If we denote by T_{01} the set of all the triangles having the common vertices 0 and 1 i.e., $T_{01} = \{\{0, 1, u\} \mid u \in U_n\}$, then the cardinality $|T_{01}| = \{u \in U_n \mid (u-1) \in U_n\}$. Using elementary counting procedures it can be proved that

$$|T_{01}| = n \prod_{p/n} \left(1 - \frac{2}{n}\right).$$

To count the number of triangles in X_n let us consider the action of the group $G = U_n \times Z_n$ on the set of triangles of X_n i.e., if $(v, x) \in G$, then $(v, x)\{0, 1, u\} = \{vx, v(1 + x), v(u + x)\}$. Each orbit of the triangles corresponding to the pair (v, x) may have at most six different elements $(0, 1, u)$, $(0, u^{-1}, 1)$, $(1-u)^{-1}, 0, 1)$, $(1, (u-1)u^{-1}, 0)$, $(u(u-1)^{-1}, 1, 0)$ in T_{01}. It can happen that some orbits have exactly 3, 2 or even 1 elements. We have $|T_{01}| = \sum_{d/6} dh_d$, where h_d is the number of orbits with exactly d elements in T_{01}. The orbits with d different elements have a number of triangles given by $|Oh_d| = \frac{n\phi(n)}{6/d} = \frac{d(n)\phi(n)}{6}$. Thus, if T stands for the number of triangles of X_n, then

$$T = \sum_{d/6} h_d |O_d| = h_6 n\phi(n) + h_3 \frac{n\phi(n)}{2} + h_2 \frac{n\phi(n)}{3} + h_1 \frac{n\phi(n)}{6},$$

$$T = n\phi(n) \left(h_6 + \frac{h_3}{2} + \frac{h_2}{3} + \frac{h_1}{6}\right) = n\phi(n) \frac{|T_{01}|}{6} = n\phi(n) \frac{\phi_2(n)}{6},$$

where $\phi_2(n) = n\sum_{p/n}\left(1 - \frac{2}{n}\right)$ by its resemblance to the Euler ϕ-function. Note that $\phi_2(n)$ is the number of consecutive units modulo n.

Finally, if we denote by IT the number of triangles in X_n that have two sides painted with equal colors, let us call these "isosceles-color-triangles" and denote by ET the corresponding number of triangles with their three sides painted in different colors, called totally multicolored triangles. With obvious meaning of "scalene-color-triangles" we obtain the following result.

Proposition 3.1. *If X_n is a Unitary Cayley graph with n an odd number then*

$$IT = \frac{n\phi(n)}{2} \text{ and}$$
$$ET = \frac{n\phi(n)}{6}(\phi_2(n) - 3).$$

Proof: Since n is odd $\{0, 1, -1\}$ is a isosceles color triangle; its orbit under the group action gives all the others. It can easily be verified that $d = 3$. Thus the formula for IT follows.

For the scalene-color-triangles let us observe that there is no "equilateral-color-triangles", i.e. triangles with its three sides painted with the same color. Therefore, $ET = T - IT$. □

Remark: X_9 has 27 triangles and all of them are isosceles-color-triangles. The first graph having scalene-color-triangles for n different from a prime number is X_{21}. It has 126 isosceles-color-triangles, 84 scalene-color-triangles and a total of 210 triangles.

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References

[1] I.J. Dejter, *Totally Multicolored Subgraphs of Complete Cayley Graphs*, Congressus Num. **70** (1990). pp. 53–64.

[2] P. Erdös, *Problems and Results in Combinatorial Analysis and Combinatorial Number Theory*, in "Graph Theory, Combinatorics and Applications", (ed. Y. Alavi), J. Wiley 1991, vol. 1 pp. 397–406.

[3] R.E. Giudici and E. Lima de Sá, *Chromatic Uniqueness of Certain Bipartite Graphs*, Congressus Numerantium, **76** (1990), pp. 69–75.