Total Colouring of Hypergraphs

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ABSTRACT. We provide two new upper bounds on the total chromatic number of all hypergraphs and give two conjectures related to both the total colouring conjecture for graphs and the Erdős, Faber, Lovász conjecture.

1 Introduction

A *hypergraph* H is a pair $(V(H), E(H))$, where $V(H)$ is a set of *vertices* and $E(H)$ is a family of non-empty subsets of $V(H)$ called *hyperedges* or just *edges*. H is *non-trivial* if $E(H)$ (and thus $V(H)$) is non-empty. H is *linear* if for all distinct E, $E' \in E(H)$, $|E \cap E'| \leq 1$. Distinct vertices v, $v' \in V(H)$ are *adjacent* if there is some hyperedge $E \in E(H)$ with v, $v' \in E$. Distinct hyperedges E, $E' \in E(H)$ are adjacent if $E \cap E' \neq \emptyset$. Vertex $v \in V(H)$ is *incident* with hyperedge $E \in E(H)$, and vice versa, if $v \in E$.

The *dual* of $H = (\{v_1, v_2, \ldots, v_n\}, [E_1, E_2, \ldots, E_m])$, H^*, is the hypergraph whose vertices $\{e_1, e_2, \ldots, e_m\}$ correspond to the hyperedges of H, and with hyperedges

$$V_i = \{e_j : v_i \in E_j \text{ in } H\} \qquad (i = 1, 2, \ldots, n).$$

The *rank* of H, *rank*(H), is the maximum cardinality of a hyperedge in $E(H)$. A hyperedge of rank one is a *loop*. The *degree* of a vertex $v \in V(H)$,

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$\deg_H(v)$, is the number of hyperedges containing v. The maximum degree among vertices of H is denoted $\Delta(H)$. The *2-section* of H, H_2, is the simple graph with vertex set $V(H)$ where distinct $x, y \in V(H)$ are adjacent in H_2 if and only if they are adjacent in H. The *line graph* of H, $L(H)$, is the simple graph with vertex set $E(H)$ where distinct E, $E' \in E(H)$ are adjacent in $L(H)$ if and only if E and E' are adjacent in H. Note that $L(H) \cong (H^*)_2$.

A *strong vertex colouring* of H is a mapping $C\colon V(H) \to \{1, 2, \ldots, k_s\}$ such that every pair of adjacent vertices receives different colours. Note that if $C(x) = i$, we say that x receives colour i. For each i the set of vertices coloured i is a *strongly stable set.* The smallest k_s for which such a colouring exists is the *strong chromatic number* $\chi^s(H)$. Note that a strong vertex colouring of H defines a vertex colouring of H_2 and vice versa, hence $\chi^s(H) = \chi(H_2)$. A *(hyper) edge colouring* of H is a mapping $C\colon E(H) \to \{1, 2, \ldots, k_e\}$ such that every pair of adjacent hyperedges receives different colours. For each i the set of edges coloured i form a *matching.* The smallest k_e for which such a colouring exists is the *edge chromatic number* $\chi_e(H)$. A *total colouring* of H is a mapping $C\colon (V(H) \cup E(H)) \to \{1, 2, \ldots, k_T\}$ such that every pair of adjacent vertices, every pair of adjacent hyperedges and every incident vertex and hyperedge receive different colours. The smallest k_T for which such a colouring exists is the *total chromatic* number $\chi_T(H)$. Note that a total colouring of H defines a total colouring of H^*, hence $\chi_T(H) = \chi_T(H^*)$. This "self-duality" is one of the most useful properties of total colourings of hypergraphs, which we will use repeatedly in this paper.

The study of the total chromatic number for hypergraphs and in particular linear hypergraphs, is motivated in part by the total colouring conjecture, posed independently by Behzad [1] and Vizing [19], which we now give.

Total colouring Conjecture (Behzad, Vizing): Let G be a simple graph. Then

$$\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2.$$

□

Evidence for this conjecture has been gathered in two ways, first by proving the conjecture true for a wide range of classes of graphs and secondly by bounding the total chromatic number for all graphs. In [7], [16] results are proven about total chromatic numbers of specific classes of hypergraphs. In this paper we present two bounds on the total chromatic number of all hypergraphs.

In the last section we propose two conjectures for total colouring of linear hypergraphs which are related to the total colouring conjecture for graphs. We show that both these conjectures are also related to the celebrated conjecture of Erdős, Faber and Lovász.

2 Two Bounds on χ_T

It is clear that

$$\chi_T(H) \le \chi_e(H) + \chi^s(H)$$

since we may "combine" a vertex and a hyperedge colouring to give a total colouring. Several bounds on the total chromatic number for graphs have been obtained by combining a vertex and an edge colouring in some non-trivial way. See for example [10], [18]. We can obtain a non-trivial bound on χ_T for all hypergraphs by fixing a strong vertex colouring and then applying edge colours in such a way that we reuse some vertex colours on edges. First we need a technical lemma.

Lemma 1. *Let H be a non-trivial hypergraph. Let $r = rank(H)$. Then*

$$\chi_e(H) \ge \frac{\chi^s(H)}{r}.$$

Proof:

$$\begin{aligned}\frac{\chi^s(H)}{r} &\le \frac{\Delta(H_2)+1}{r}\\ &\le \frac{(r-1)\Delta(H)+1}{r}\\ &\le \Delta(H)\\ &\le \chi_e(h).\end{aligned}$$

□

Theorem 1. *Let H be a non-trivial hypergraph. Let $r = rank(H)$. Then*

$$\chi_T(H) \le \chi_e(H) + \left\lfloor \frac{r-1}{r}\chi^s(H) \right\rfloor + 1.$$

Proof: Let $\chi^s = \chi^s(H)$, $\chi_e = \chi_e(H)$. Let $\chi^s = kr + l$ where $0 \le l < r$. Note that by lemma 1, $\chi_e \ge k$. Partition $V(H)$ into strongly stable sets $V_1, V_2, \ldots, V_{\chi^s}$. Partition $E(H)$ into matchings $M_1, M_2, \ldots, M_{\chi_e}$. We will demonstrate a total colouring $C\colon (V(H) \cup E(H)) \to \{1, 2, \ldots, \chi_e + \lfloor \frac{r-1}{r} \rfloor + 1\}$. We consider two cases.

1. $l > 0$.

   Let

$$\begin{array}{lll} C(v) = i & \text{for } v \in V_i & (i = 1, 2, \ldots, \chi^s)\\ C(E) = \chi^s + j - k & \text{for } E \in M_j & (j = k+1, k+2, \ldots, \chi_e)\end{array}$$

We will complete the colouring by colouring hyperedges in $M_1, M_2, \ldots, M_k$ using colours $1, 2, \ldots, \chi^s$. If hyperedge E is in M_i $(1 \leq i \leq k)$ then

$$C(E) = \begin{cases} \chi^s & \text{if } e \cap V_j \neq \emptyset \text{ for all } j \in \{(i-1)r+1, (i-1)r+2, \ldots, ir\} \\ j & \text{otherwise, for some } j \in \{(i-1)r+1, (i-1)r+2, \ldots, ir\}, \\ & \text{with } E \cap V_j = \emptyset. \end{cases}$$

Thus we can see that none of these hyperedges contains a vertex of the same colour as itself, and if two of these hyperedges intersect, they have different colours. In particular, two hyperedges both having colour χ^s cannot intersect. Thus we have a colouring of H using

$$\chi_e + \chi^s - k = \chi_e + \frac{r-1}{r}\chi^s + \frac{l}{r} \leq \chi_e + \left\lfloor \frac{r-1}{r}\chi^s \right\rfloor + 1$$

colours.

2. $l = 0$.

The proof here is identical to case 1 except that we cannot keep reusing colour χ^s to colour hyperedges in $M_1, M_2, \ldots, M_k$. We need to introduce an entirely new colour for this purpose. Thus we need

$$\chi_e + \chi^s - k + 1 = \chi_e + \frac{r-1}{r}\chi^s + 1$$

colours.

□

Using the self-duality of total colourings we obtain immediately

Corollary 1. *Let H be a non-trivial hypergraph. Let $\Delta = \Delta(H)$. Then*

$$\chi_T(H) \leq \chi^s(H) + \left\lfloor \frac{\Delta - 1}{\Delta}\chi_e(H) \right\rfloor + 1.$$

□

Our second bound on the total chromatic number of all hypergraphs is obtained by using probabilistic techniques similar to those used in [14],[15]. We prove a result which is of particular interest for hypergraphs of large rank.

Theorem 2. *Let H be a hypergraph. Let $\Delta = \Delta(H) \geq 2$, $r = rank(H)$. Let k be an integer, with*

$$\frac{k!}{(\Delta - 1)^{k-1}(\Delta + k - 1)} > |\{E \in E(H) : |E| \geq k\}|.$$

Then

$$\chi_T(H) \leq \max\{\chi^s(H), \chi_e(H)\} + (k-2)\Delta + 1.$$

Proof: Let $q = \max\{\chi^s(H), \chi_e(H)\}$. We define a (possibly improper) colouring $C\colon (V(H) \cup E(H)) \to \{1, 2, \ldots, q\}$. First colour vertices properly using colours $\{1, 2, \ldots, q\}$. Then colour hyperedges properly using the same set of colours. Choose a random permutation π from the probability space consisting of the $q!$ permutations on q elements, where each permutation occurs with probability $\frac{1}{q!}$. Permute the hyperedge colour classes according to π. We say that a vertex is "bad" if for some $E \in E(H)$ with $v \in E$, $C(v) = \pi(C(E))$. We show that with non-zero probability, each hyperedge of H contains at most $k-1$ bad vertices. Hence there exists some permutation π^* such that when we permute edge colour classes according to π^*, each hyperedge contains at most $k-1$ bad vertices. Now consider the subgraph G_{π^*} of H_2 induced on these bad vertices. Then $\Delta(G_{\pi^*}) \leq (k-2)\Delta$. Hence we can recolour these vertices greedily using $(k-2)\Delta+1$ new colours to complete the proof.

Thus it remains only to show that each hyperedge of H contains at most $k-1$ bad vertices with non-zero probability. If $r < k$ then this is clearly true with probability 1. So assume $r \geq k$ and consider a hyperedge $E \in E(H)$ with $|E| \geq k$. Let $W = \{(E_1, v_1), (E_2, v_2), \ldots, (E_k, v_k)\}$ where the E_is are distinct hyperedges which receive distinct colours under C, and the v_is are distinct vertices in E such that $v_i \in E \cap E_i$ for $i = 1, 2, \ldots, k$. Note that we may have $E_i = E$ for some i. Let $\mathcal{W}_E$ be the set of all such W. Let $A(W)$ be the event "for all $(E_i, v_i) \in W$, $\pi(C(E_i)) = C(v_i)$". So if $A(W)$ happens, then all of $v_1, v_2, \ldots, v_k$ are badly coloured. Then for fixed W

$$\mathbf{P}(A(W)) = \frac{1}{q}\frac{1}{q-1}\cdots\frac{1}{q-k+1} = \frac{(q-k)!}{q!}.$$

We also have

$$\begin{aligned} |\mathcal{W}_E| &\leq \binom{r}{k}(\Delta-1)^k + \binom{r}{k-1}(\Delta-1)^{k-1} \\ &\leq \binom{r}{k}(\Delta-1)^{k-1}(\Delta+k-1). \end{aligned}$$

Now if some k vertices from E are bad, then for some $W \in \mathcal{W}_E$, $A(W)$

must occur. We have

$$\begin{aligned}
&\mathbf{P}(k \text{ or more vertices of } E \text{ are badly coloured})\\
&\le \mathbf{P}(\text{for some } W,\ A(W) \text{ occurs})\\
&\le \mathbf{E}(\text{number of } W \text{ for which } A(W) \text{ occurs})\\
&= |W_E|\frac{(q-k)!}{q!}\\
&\le \binom{r}{k}(\Delta-1)^{k-1}(\Delta+k-1)\frac{(r-k)!}{r!}\\
&= \frac{(\Delta-1)^{k-1}(\Delta+k-1)}{k!}
\end{aligned}$$

So

$$\begin{aligned}
&\mathbf{P}(\text{for some hyperedge } F \in E(H),\ F \text{ has } k \text{ or more bad vertices})\\
&\le \mathbf{E}(\text{number of hyperedges with } k \text{ or more bad vertices})\\
&\le |\{E \in E(H)\colon |E| \ge k\}|\frac{(\Delta-1)^{k-1}(\Delta+k-1)}{k!}\\
&< 1.
\end{aligned}$$

□

Using self-duality of total colourings we obtain a corollary which is of particular interest for hypergraphs of large maximum degree.

Corollary 2. *Let H be a hypergraph. Let $r = rank(H) \ge 2$. Let k be an integer, with*

$$\frac{k!}{(r-1)^{k-1}(r+k-1)} > |\{v \in V(H)\colon \deg_H(v) \ge k\}|.$$

Then

$$\chi_T(H) \le \max\{\chi^s(H), \chi_e(H)\} + (k-2)r + 1.$$

□

This tells us, for example, using the result of [5], that a cyclic Steiner triple system on 501 vertices has total chromatic number at most 526. Applying corollary 2 to the case of multigraphs, we obtain

Corollary 3. *Let G be a multigraph, with more than one edge, which may have loops. Let k be an integer, with*

$$\frac{k!}{k+1} > |\{v \in V(H)\colon \deg_H(v) \ge k\}|.$$

Then

$$\chi_T(H) \le \chi_e(H) + 2k - 3.$$

This tells us, for example, that for multigraphs with less than 630 vertices of degree more than 6, the total chromatic number is bounded by the maximum degree plus 12.

3 The Total Colouring Conjecture for Hypergraphs

The study of the total chromatic number for graphs was motivated largely by the total colouring conjecture posed independently by Behzad and Vizing. It seems only natural that we should attempt to find a related conjecture for hypergraphs, at least for hypergraphs with the "graph like" property of linearity, especially given the numerous conjectures which generalize Vizing's theorem to linear hypergraphs (see [12]). We will first give a conjecture which is substantially weaker than the total colouring conjecture, but which is of independent interest since it is a corollary of the Erdös, Faber, Lovász conjecture. First we need a proposition relating strong vertex and total colouring numbers of linear hypergraphs.

Proposition 1. *Let H be a linear hypergraph on n vertices with $\Delta(H) \leq n$. Then there exists an $(n+1)$-uniform linear hypergraph H' with $(n+1)$ hyperedges, such that $\chi_T(H) \leq \chi^s(H')$.*

Proof: By adding loops to H if necessary, we may obtain an n-regular linear hypergraph H_n on n vertices. The dual H_n^* of H_n is then an n-uniform linear hypergraph on n hyperedges. Add n extra vertices of degree 1 to H_n^*, one to each hyperedge. Then add a hyperedge containing all of the added vertices, plus one new vertex, to form H'. Then H' is an $(n+1)$-uniform linear hypergraph on $(n+1)$ hyperedges. Given a strong vertex colouring of H' we may obtain a total colouring of H_n^* and thus H. Colour each vertex of H_n^* using the colour of the corresponding vertex of H'. Colour each hyperedge of H_n^* using the colour of the vertex which was added to that hyperedge in the construction of H'. □

Using proposition 1 and the Erdös, Faber, Lovász conjecture we obtain a corresponding conjecture for total colouring of hypergraphs. First we give the

Erdös, Faber, Lovász conjecture: Let H be an n-uniform linear hypergraph on n hyperedges. Then $\chi^s(H) \leq n$. □

This is a slight restatement of the original conjecture, which was given for graphs. Erdös offers \$500 for its resolution. We can now give a total colouring conjecture for hypergraphs.

Conjecture 1: Let H be a linear hypergraph on n vertices, with $\Delta(H) \leq n$. Then

$$\chi_T(H) \leq n+1.$$

□

Using Proposition 1 we can see that the Erdös, Faber, Lovász conjecture implies this total colouring conjecture. The conjecture is true for hypergraphs with at most 10 vertices, by a result of Hindman (see [11]). Using Proposition 1 and partial results on the Erdös, Faber, Lovász conjecture, we can obtain additional partial results for Conjecture 1. Let H be a linear hypergraph on n vertices, with $\Delta(H) \leq n$.

Proposition 2.

$$\chi_T(H) \leq n + o(n).$$

Proof: This uses the result of [13] that $\chi_e(H) \leq n + o(n)$. □

Proposition 3.

$$\chi_T(H) \leq \left\lceil \frac{3}{2}n - \frac{1}{2} \right\rceil.$$

Proof: This uses the result of [3] that $\chi_e(H) \leq \lceil \frac{3}{2}n - 2 \rceil$. □

If $rank(H) = 2$, i.e. H is a multigraph, then Conjecture 1 is trivially true. In this case Vizing's theorem gives a much stronger result for the edge chromatic number. This has prompted several authors, including Meyniel, Berge [2] and Füredi [9] to conjecture the following strengthening of Vizing's theorem.

Hyperedge Colouring Conjecture: Let H be a linear hypergraph without loops, then

$$\chi_e(H) \leq \Delta(H_2) + 1.$$

□

We propose an analogous conjecture for total colouring of linear hypergraphs.

Conjecture 2: Let H be a linear hypergraph without loops or vertices of degree one, then

$$\chi_T(H) \leq \min\{\Delta(H_2), \Delta(L(H))\} + 2.$$

□

Computer search has shown that Conjecture 2 is true for all hypergraphs on at most 8 hyperedges, or equivalently all hypergraphs on at most 8 vertices. This conjecture has the obvious advantage over Conjecture 1 that it implies the total colour conjecture for graphs. However, evidence for the Hyperedge Colouring Conjecture is scarce, and Conjecture 2 seems even more difficult. For example, whereas the Hyperedge Colouring Conjecture is known to be true for intersecting hypergraphs, the problem of deciding whether Conjecture 2 is true in this case contains several difficult problems, most notably that of bounding the block-chromatic number of a Steiner 2-design, since the hypergraph dual of a Steiner 2-design is an intersecting linear hypergraph. The complexity of the problem of determining the block-chromatic number of a Steiner 2-design is unknown, although it appears to be difficult. No bounds which approach those which would be given by conjectures 1 and 2 are known for general Steiner 2-designs. See [17] for further discussion of block colourings.

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