Six MOLS of Order 76

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ABSTRACT. A PBD construction for six MOLS of order 76 is given.

1 The Construction

See [1] for definitions and relevant background material.

Let $D = \{0, 1, 3, 9, 27, 81, 61, 49, 56, 77\}$. Then $\{D + i : 0 \leq i \leq 90\}$ is a projective plane of order 9 (a (91,10,1)–design) $(V, \mathcal{B})$, where $D + i$ is obtained by adding i to each entry of D, and reducing modulo 91.

Let $A = \{2, 4, 5, 12, 24\}$. Delete the points of $D \cup A$ from V, and shorten the blocks in $\mathcal{B}$ accordingly (also removing the empty block), to obtain a PBD $(V \setminus (D \cup A), \mathcal{B}')$. Now deleting the elements of D has the effect of removing all points from the block $D + 0$, and removing exactly one point from each other block. It is easy to check that A is a 5–arc in the resulting affine plane (every line meets at most two points of A), so that $(V \setminus (D \cup A), \mathcal{B}')$ is a $(76, \{7, 8, 9\})$–PBD. What is of importance to us is the structure of the blocks of size 7; naturally, there must be ten of them, and we list them explicitly below, having deleted the points of $D \cup A$.

D+1	10 28 82 62 $\boxed{50}$ $\underline{57}$ 78
D+2	11 29 $\underline{83}$ $\boxed{63}$ 51 58 79
D+3	6 30 84 $\boxed{64}$ $\underline{52}$ 59 80
D+4	7 $\underline{13}$ $\boxed{31}$ 85 65 53 60
D+12	$\boxed{13}$ $\underline{15}$ 21 39 73 68 89
D+15	$\boxed{15}$ 16 18 42 76 $\underline{64}$ 71
D+34	34 35 $\underline{37}$ 43 $\boxed{83}$ 90 20
D+47	47 48 $\underline{50}$ 74 $\boxed{37}$ 17 33
D+54	54 55 $\boxed{57}$ $\underline{63}$ 44 19 40
D+66	66 67 69 75 36 $\underline{31}$ $\boxed{52}$

It is easy to check that every element occurs 0, 1 or 2 times in these blocks of size 7; and that every element appearing twice has been underlined once, and boxed once, in the table.

The existence of this structure on the blocks of size 7 permits us to prove the following:

Theorem 1.1 *There are six MOLS of order 76. In fact, a* $TD(8,76) - 16\,TD(8,1)$ *exists.*

Proof: We use a simple variant of the standard PBD construction for MOLS. $TD(8,8) - 8TD(8,1)$ and $TD(8,9) - 9TD(8,1)$ both exist, so it suffices to treat the blocks of size 7. A $TD(8,7)$ exists, and this is equivalent to a $TD(8,7) - TD(8,1)$ by deleting an arbitrary block. Place the $TD(8,7) - TD(8,1)$ so that the hole of size one, for each block of size 7 in the PBD, coincides with the 8 copies of the "boxed" element in the list above. It is then easy to verify that the result is a $TD(8,76) - 16TD(8,1)$; filling the 16 holes of order 1 yields a $TD(8,76)$, which is equivalent to six MOLS of order 76. □

This construction can be stated more generally; see, for example, [2].

The order 76 has held a special place for 6 MOLS for sixteen years; indeed, 76 was the largest order for which 6 MOLS were unknown until this time [1].

References

[1] T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, Cambridge University Press, 1986.

[2] C.J. Colbourn and J.H. Dinitz, "Making the MOLS table", preprint.