A Combinatorial Queueing Model Related To The Ballot Problem

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ABSTRACT. Consider a queue of N customers waiting to purchase an item that costs 1 dollar. Of them, m customers have a 1-dollar bill and n customers have only a $(1+\mu)$ dollar bill, where μ is a positive integer. The latter need to get change in the amount of μ dollars. If at the time of their service, the cashier has less than μ 1-dollar bills, they have to wait for change according to some queue discipline. It is assumed that the cashier has no initial change, and that all the queue arrangements are equi-probable. Using transformations of lattice graphs, we derive the probability distribution of the number of customers who will have to wait for change under a queue discipline that corresponds to the ballot problem. Limiting results and other applications are also given.

1 Introduction

Suppose N customers are queued up in a line to purchase an item that costs 1 dollar. There are two types of customers in the queue: Type I customers have a 1-dollar bill and need no change; Type II customers have only a $(1+\mu)$-dollar bill, where μ is a positive integer. The latter will need to get change in the amount of μ dollars. Let m (n) denote the number of type I (type II) customers ($m+n=N$). The following assumptions are made:

(A1) The cashier initially has no change and its only source of change is the receipts from the type I customers.

(A2) All the arrangements of the $m+n$ customers in the queue are equi-probable.

If at the time when a type II customer approaches, the cashier has less than μ 1-dollar bills (and thus unable to give change), then this customer cannot be served and has to wait until there is change. Let $W(m,n,\mu)$ denote the number of type II customers who will have to wait for change. In this paper we are concerned with the probability distribution of $W(m,n,\mu)$.

To show the relationship to the ballot problem, let $B_1(i)$ ($B_2(i)$) denote the number of type I (type II) customers among the first i customers, $i = 1,2,\ldots,N$. Since $P[W(m,n,\mu) = 0] = P[B_1(i) \geq \mu B_2(i)\ \forall i]$, finding the probability that no customer will need to wait for change is analogous to the classical ballot problem (with $m \geq \mu n$). Generally, the ballot problem deals with the distribution of the random variable $\delta(m,n,\mu)$, denoting the number of indices i such that $B_1(i) < \mu B_2(i)$. The distributions of $\delta(m,n,\mu)$ and other similar random variables were derived by L. Takàcs [6], [7] and turned out to have important applications in nonparametric statistics. A comprehensive discussion of ballot theorems and their applications is given in Takàcs [8]. Various extensions and variations have also been studied. For example, Engelberg [2] derived the distribution of δ without conditioning on the values of m and n, Takàcs [9] studied various related distributions by using properties of exchangeable random variables, Srinivasan [5] obtained conditions for a uniform distribution of δ, and Chao et al. [1] derived the distribution of these random variables for any positive real μ. The queuing version has been mentioned by Gnedenko ([3], p. 43) but only for the simple case of $m = n$, and $\mu = 1$. Gnedenko uses the so-called 'reflection principle' to show that in this case P (no customer waits for change) $= 1/(n+1)$.

In this paper we obtain the entire distribution of $W(m,n,\mu)$ for any integer valued parameters $\{m,n,\mu\}$. Although $W(m,n,\mu)$ is related to the ballot random variables, it is not the same as $\delta(m,n,\mu)$. Clearly, W has a different range than δ and there does not exist any explicit functional relationship between them. In fact, $W(m,n,\mu)$ is not yet well defined, since it depends in general on the prevailing queue discipline. Thus, we make the following additional assumption:

(A3) The customers who need to wait for change, stand in a secondary queue with priority, namely, such a customer is served as soon as the change is available.

Other assumptions about the queue discipline would yield different distributions of W. Such an alternative assumption is mentioned in Section 4.

The method we use features an extension of the methods used by Srinivasan [5] and Chao et al. [1] for the ballot problem, and it is based on transformations of the queue arrangements. We also obtain a necessary and sufficient condition for a uniform distribution of W, and the limiting distribution of W/n.

Let $S(m,n,\mu;k)$ denote the set of queue arrangements in which k customers have to wait for change, and $|A|$ denote the cardinality of A. From Assumption (A2) we have

$$P[W(m,n,\mu)=k]=\frac{|S(m,n,\mu;k)|}{\binom{m+n}{m}},\quad k=0,1,\ldots,n \tag{1.1}$$

Therefore, the problem is to derive $|S(m,n,\mu;k)|$. For the case of $k=0$, the well known result from the ballot problem (see for instance Takàcs [7]) gives,

$$|S(m,n,\mu;0)|=\frac{m+1-\mu n}{m+1}\binom{m+n}{m}. \tag{1.2}$$

However, for $k\geq 1$, $|S(m,n,\mu;k)|$ cannot be directly deduced from the ballot problem results.

Define $B(0)=0$ and $B(i)=B_1(i)-\mu B_2(i)$, $i=1,\ldots,N$. The random process $B(i)$ represents the surplus or deficit of 1-dollar bills at the cashier after i customers have been served. It is therefore useful to view the process graphically by plotting the points $(i,B(i))$, $i=0,1,\ldots,N$, and connecting them by diagonal segments. Each realization of the queue is represented by a graph (a lattice path) that starts at (0, 0) and ends at $(m+n, m-\mu n)$. In this representation, the number of customers who have to wait for change is exactly the number of descending segments that lie partly or entirely below the x-axis. Thus, this is essentially a problem of counting a particular type of lattice paths. Lattice path counting is a familiar problem in combinatorics. For a good survey, we recommend Mohanty [4].

For the sake of brevity, the parameters m, n and μ will sometimes be omitted in our notation. For example, $S(m,n,\mu;k)$ is often replaced by $S(k)$.

Now, for a graph $u\in S(k)$, $k=0,1,\ldots,n-1$, define:

$$x_1=\min\{i>0\colon B(i)=\mu\}$$

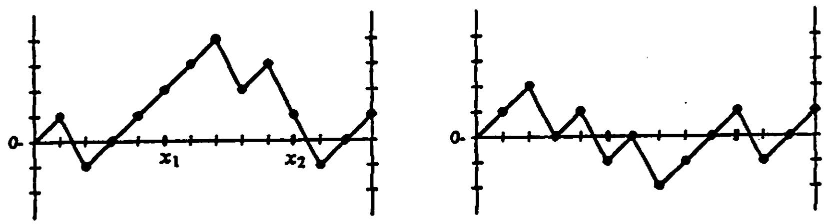
and

$$x_2=\min\{i>x_1\colon B(i)<\mu\}.$$

It is easy to see that for $k=0,1,\ldots,n-1$, x_1 always exists.

Let $T(k)\subset S(k)$ be the set of graphs in $S(k)$ for which x_2 also exists. For a graph $u\in T(k)$, define *segment 1* to be the segment of the graph on the interval $(0,x_1]$, and *segment 2* to be the segment of the graph on the interval $(x_1,x_2]$.

Let f be a mapping on $T(k)$ defined as follows: For a graph $u\in T(k)$, $f(u)$ is the graph obtained by interchanging segments 1 and 2, i.e., by connecting segment 2 to the origin, and connecting segment 1 to the end of segment 2. The remainder of the graph on the interval $(x_2, m+n]$ is left unchanged. An illustration of the mapping f is given in Figure 1.



Graph of u, $(k=2)$ Graph of $f(u)$, $(k=3)$

$m=9$, $n=4$, $\mu=2$

Figure 1

The following lemma states the key property of this mapping.

Lemma 1.1. *For $k=0,1,\ldots,n-1$, if $u \in T(k)$, then $f(u) \in S(k+1)$.*

The lemma follows from the fact that customer x_2 is a type II customer who does not wait for change in the graph u but waits in the graph $f(u)$, while all the other type II customers do not change their status under the mapping f.

For a graph $v \in S(k)$, $k=1,2,\ldots,n$, define:

$$z_1 = \min\{i>0\colon B(i)<0\}$$

and

$$z_2 = \min\{i>z_1\colon B(i)=\mu+B(z_1)\}.$$

Notice that for $k=1,2,\ldots,n$, z_1 always exists.

Let $R(k) \subset S(k)$ be the set of graphs in $S(k)$ for which z_2 also exists. For a graph $v \in R(k)$, we define *segment 1* and *segment 2* in the same manner as we did for $T(k)$, now using z_1 and z_2 instead of x_1 and x_2.

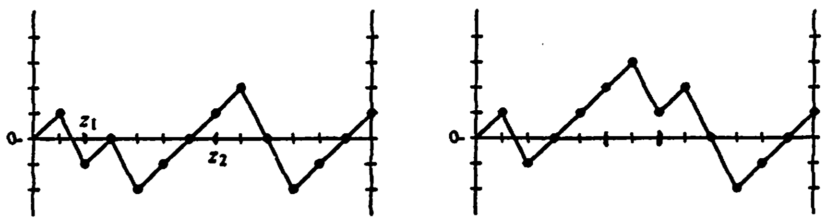
Let g be a mapping defined on $R(k)$ as follows: For $v \in R(k)$, $g(v)$ is obtained by interchanging segments 1 and 2 (as in the definition of f), and leaving the rest of the graph unchanged. An illustration of the mapping g is given in Figure 2.

Lemma 1.2.

(1) For $k=1,\ldots,n$, if $v \in R(k)$, then $g(v) \in S(k-1)$.

(2) For $k=1,\ldots,n$, if $v \in R(k)$, then $g(v) \in T(k-1)$ and $f(g(v))=v$.

(3) For $k=0,1,\ldots,n-1$, if $u \in T(k)$, then $f(u) \in R(k+1)$ and $g(f(u))=u$.



Graph of v, $(k = 3)$ Graph of $g(v)$, $(k = 2)$

$m = 9$, $n = 4$, $\mu = 2$

Figure 2

Part (1) can be proved by an argument similar to that used for Lemma 1.1. Parts (2) and (3) follow immediately from Part (1), Lemma 1.1, and the fact that f maps x_1 to z_2 and x_2 to z_1, and g maps z_1 to x_2 and z_2 to x_1 (Figures 1 and 2 illustrate these facts).

Since $T(k)$ and $R(k)$ are finite sets, Lemmas 1.1 and 1.2 yield the following proposition which plays a key role in deriving the distribution of W.

Proposition 1.1. *For $k = 0, 1, \ldots, n-1$, $|T(k)| = |R(k+1)|$, or alternatively, for $k = 1, \ldots, n$, $|T(k-1)| = |R(k)|$.*

Now, from the definition of x_2 and z_2, it is clear that for $k = 0, 1, \ldots, n-1$, if $m - \mu n \leq \mu - 1$, then x_2 exists (since $B(m+n) = m - \mu n < \mu$). Similarly, for $k = 1, \ldots, n$, if $m - \mu n \geq \mu - 1$, then z_2 exists (since $B(m+n) = m - \mu n \geq \mu - 1 \geq \mu + B(z_1)$).

It follows that for any set of parameters $\{m, n, \mu\}$, either x_2 or z_2 (or both) exist, which implies that either $T(k) = S(k)$ or $R(k) = S(k)$. This leads us to consider separately the three cases, $m - \mu n = \mu$, $m - \mu n > \mu$ and $m - \mu n < \mu$. In the third case we also need to distinguish between $m - \mu n \geq 0$ and $m - \mu n < 0$, hence four cases are to be considered. In the next section, the tools we developed so far are used to derive the distribution of W. Limiting results are derived in Section 3, and some extensions and open problems are mentioned in Section 4.

2 The distribution of $W(m, n, \mu)$

We start with a sufficient and necessary condition for the uniformity of W. Aside from being a nice counterpart to the uniform case of the ballot random variables, it is useful in the derivation of the distribution of W in the other cases.

Proposition 2.1. *$P[W(m,n,\mu)=k]=\frac{1}{n+1}$, $k=0,1,\ldots,n$, namely $W(m,n,\mu)$ has a discrete uniform distribution on $\{0,1,\ldots,n\}$ if and only if $m-\mu n=\mu-1$.*

Proof: If $m-\mu n=\mu-1$, then both x_2 and z_2 exist. Therefore, $T(k)=S(k)$, for $k=0,1,\ldots,n-1$ and $R(k)=S(k)$ for $k=1,\ldots,n$.

Thus, it follows from Proposition 1.1 that $|S(k)|=|S(k+1)|$ for $k=0,1,\ldots,n-1$, which implies that $|S(k)|=|S(0)|\ \forall k=1,\ldots,n$. Hence, from (1.1) and (1.2)

$$P[W(m,n,\mu)=k]=\frac{|S(0)|}{\binom{m+n}{m}}=\frac{m+1-\mu n}{m+1},\quad k=0,1,\ldots,n. \tag{2.1}$$

The conclusion follows by substituting $m-\mu n=\mu-1$ in (2.1). The condition is necessary since $m-\mu n=\mu-1$ is the only case where both x_2 and z_2 exist for all k. □

Before we proceed with the distribution of W, we need the following lemma.

Lemma 2.1. *If $m-\mu n<\mu$, then*

$$|S(m,n,\mu;n)|=\binom{m+n}{n}-\sum_{i=0}^{\ell}\frac{1}{i+1}\binom{i(\mu+1)+\mu-1}{i}\cdot\binom{m+n-i-\mu(i+1)}{n-i}, \tag{2.2}$$

where $\ell=n-1-\lceil n-\frac{m}{\mu}\rceil$, and $\lceil x\rceil$ denotes the smallest integer greater than or equal to x.

Proof: Let $M(m,n,\mu)$ denote the set of graphs for which $B(i)=\mu$ for at least one $i>0$. Then

$$|S(m,n,\mu;n)|=\binom{m+n}{m}-|M(m,n,\mu)|, \tag{2.3}$$

Let $M(m,n,\mu;j)$ denote the subset of $M(m,n,\mu)$ corresponding to $j=\min\{i>0\colon B(i)=\mu\}$. Then $u\in M(m,n,\mu;j)$ if and only if (i) $B(i)\le\mu-1$ $\forall i\in(0,j-1]$ and (ii) $B(j-1)=\mu-1$. Let $m_1=B_1(j-1)$ and $n_1=B_2(j-1)$. From conditions (i) and (ii) it follows that $n_1=(j-\mu)/(\mu+1)$, $m_1=\mu(n_1+1)-1$, and also

$$|M(m,n,\mu;j)|=|S(m_1,n_1,\mu;n_1)|\cdot|S(m-m_1-1,n-n_1,\mu)|. \tag{2.4}$$

Since $m_1-\mu n_1=\mu-1$, we can apply Proposition 2.1 to determine $|S(m_1,n_1,\mu;n_1)|$. Also, j can only take the values $(i+1)\mu+i$, $i=0,1,\ldots,\ell$, where $\ell = n-1-\lceil n-m/\mu \rceil$. Thus, by substituting $j=(i+1)\mu+i$ in (2.4), using Proposition 2.1, and summing over i from 0 to ℓ, we get $|M(m,n,\mu)|$. The conclusion then follows from (2.3). □

We shall refer to the uniform case in Proposition 2.1 as case 1, and proceed by dividing the parameter space into three additional cases.

Case 2: $m-\mu n \geq \mu$

Lemma 2.2. *For* $k=1,\ldots,n$, $|S(k)| = |S(0)| - \sum_{i=0}^{k-1} |T^c(i)|$, *where* $T^c(i)$ *denotes the complement of* $T(i)$, *with respect to* $S(i)$.

Proof: If $m-\mu n \geq \mu$, then z_2 exists and $R(k)=S(k)$, $k=1,\ldots,n$. Therefore, by Proposition 1.1 we have $|S(k)|=|T(k-1)|$. But

$$S(k-1)=T(k-1)\cup T^c(k-1), \text{ hence,}$$
$$|S(k-1)|=|T(k-1)|+|T^c(k-1)|=|S(k)|+|T^c(k-1)|, \text{ or equivalently,}$$
$$|S(k-1)|-|S(k)| = |T^c(k-1)|. \tag{2.5}$$

Applying a summation on both sides of (2.5) gives the desired result. □

Now, $|S(0)|$ is given by (1.2), and $|T^c(k)|$ is given in the next lemma.

Lemma 2.3. *For* $k=0,1,\ldots,n-1$,

$$|T^c(k)| = \frac{1}{k+1}\cdot\frac{m+1-\mu(n+1)}{m+1-\mu(k+1)}\cdot\binom{(\mu+1)k+\mu-1}{k} \cdot\binom{m+n-\mu(k+1)-k}{n-k}. \tag{2.6}$$

Proof: It can be easily verified that a graph u belongs to $T^c(k)$ if and only if (i) $B(i)\leq \mu-1\ \forall i\in(0,x_1-1]$; (ii) $B(x_1)=B(x_1-1)+1=\mu$; (iii) $B(i)\geq\mu\forall i\in(x_1,m+n]$. Let $m_1=B_1(x_1-1)$ and $n_1=B_2(x_1-1)$. Then from (i)-(iii) we have

$$\begin{aligned}|T^c(k)| &= |S(m_1,n_1,\mu;n_1)|\cdot|S(m-m_1-1,n-n_1,\mu)|\\ &= |S(\mu(k+1)-1,k,\mu;k)|\cdot|S(m-\mu(k+1),n-k,\mu;0)|.\end{aligned}$$

The conclusion now follows by an application of (1.2) and Lemma 2.1. □

Applying lemmas 2.2 and 2.3 and substituting (1.2) for $|S(0)|$, gives the distribution of $W(m,n,\mu)$ in case 2, as stated in the next proposition.

Proposition 2.2. *If* $m-\mu n\geq\mu$, *then*

$$P[W(m,n,\mu)=k] = \begin{cases}\frac{m+1-\mu n}{m+1} & \text{for } k=0\\ \frac{m+1-\mu n}{m+1}-\binom{m+n}{m}^{-1}\cdot\sum_{i=0}^{k-1}|T^c(i)|, & k=1,\ldots,n\end{cases} \tag{2.7}$$

where $|T^c(i)|$ *is given in (2.6).*

Case 3: $0 \leq m - \mu n < \mu - 1$

Lemma 2.4. *For* $k = 1, \dots, n$, $|S(k)| = |S(0)| + \sum_{i=1}^{k} |R^c(i)|$, *where* $R^c(i)$ *denotes the complement of* $R(i)$, *with respect to* $S(i)$.

Proof: In this case x_2 exists and $T(k) = S(k)$, for $k = 0, \dots, n-1$. Therefore, by Proposition 1.1, $|S(k)| = |R(k+1)|$, and since $S = R \cup R^c$, we have

$$|S(k+1)| - |S(k)| = |R^c(k+1)|. \tag{2.8}$$

The assertion follows by summing both sides of (2.8). □

An expression for $|R^c(k)|$ is given in the next lemma.

Lemma 2.5. *For* $k = 1, \dots, n$,

$$\begin{aligned} |R^c(k)| = &\left\{ \binom{(k-1)(\mu+1)}{k-1} - \sum_{i=0}^{k-1} \frac{1}{i+1} \binom{i(\mu+1)+\mu-1}{i} \right. \\ &\left. \cdot \binom{(\mu+1)(k-1) - i - \mu(i+1)}{k-1-i} \right\} \\ &\cdot \sum_{b=1}^{\mu(n+1)-(m+1)} \frac{\mu - b + 1}{\mu(n-k+1) - b + 1} \binom{(n-k)(\mu+1) + \mu - b}{n-k}. \end{aligned} \tag{2.9}$$

Proof: Let $b = -B(z_1)$. Then b can take the values $1, 2, \dots, \mu$, and a graph v belongs to $R^c(k)$ if and only if (i) $B(i) \geq 0$ $\forall i \in (0, z_1 - 1]$, (ii) $B(z_1 - 1) = \mu - b$, (iii) $B(i) < \mu - b$ $\forall i \in (z_1, m+n]$. Let $R^c(k\colon b)$ denote the subset of $R^c(k)$ for a given value of b. Then (i)-(iii) imply that

$$|R^c(k\colon b)| = |S(\mu(n-k+1) - b, n-k, \mu; 0)| \cdot |S(\mu(k-1), k-1, \mu; k-1)|.$$

Equation (2.9) is obtained by applying (1.2) and (2.2), and summing over b. Finally, note that if $b \geq \mu(n+1) - m$, then $m - \mu n \geq \mu - b$, and hence z_2 exists. Therefore, the summation over b is only from 1 to $\mu(n+1)-(m+1)$. □

We can now write the distribution of $W(m, n, \mu)$ for this case.

Proposition 2.3. *If* $0 \leq m - \mu n < \mu - 1$, *then*

$$\begin{aligned} &P[W(m,n,\mu) = k] \\ &\quad = \begin{cases} \frac{m+1-\mu n}{m+1} & \text{for } k = 0 \\ \frac{m+1-\mu n}{m+1} + \binom{m+n}{m}^{-1} \cdot \sum_{i=1}^{k} |R^c(i)|, & k = 1, \dots, n. \end{cases} \end{aligned} \tag{2.10}$$

where $|R^c(i)|$ *is given in (2.9).*

Case 4: $m - \mu n < 0$

In this case there will not be enough change for all the type II customers and the number of customers who will never get change is $d = \lceil n - m/\mu \rceil$. Hence, $W(m,n,\mu)$ only takes values that are greater than or equal to d. By the same arguments as in the proof of Lemma 2.4, we have that for $k = d+1, \ldots, n-1$,

$$|S(k)| = |S(d)| + \sum_{i=d+1}^{k} |R^c(i)|. \tag{2.11}$$

The formula for $|R^c(k)|$ in this case is derived in a similar way as in Lemma 2.5. It is given in the next lemma without proof.

Lemma 2.6. *For* $k = d+1, \ldots, n$,

$$|R^c(k)| = \sum_{j=1}^{\mu} \left\{ \frac{\mu - j + 1}{\mu(n-k+1) - j + 1} \binom{(n-k)(\mu+1) + \mu - j}{n-k} \right.$$
$$\cdot \left\{ \binom{(m+j+k) - \mu(n-k+1) - 1}{k-1} - \sum_{i=0}^{\ell} \frac{1}{i+1} \binom{i(\mu+1) + \mu - 1}{i} \right.$$
$$\left. \left. \cdot \binom{m+j+k-1-i-\mu(n-k+2+i)}{k-1-i} \right\} \right\},$$

where $\ell = k - 2 - \lceil n - (m+j)/\mu \rceil$.

An expression for $|S(d)|$ is easily determined from the fact that if exactly d customers wait for change, these must be the last d customers. Therefore, we have

$$\begin{aligned} |S(m,n,\mu;d)| &= |S(m,n-d,\mu;0)| \\ &= \frac{m+1-\mu(n-d)}{m+1} \binom{m+n-d}{n-d}. \end{aligned} \tag{2.12}$$

We can now present the distribution of $W(m,n,\mu)$ in case 4.

Proposition 2.4. *If* $m - \mu n < 0$, *then*

$$P[W(m,n,\mu) = k] = \begin{cases} \frac{m+1-\mu(n-d)}{m+1} & \text{for } k = d \\ \frac{m+1-\mu(n-d)}{m+1} + \binom{m+n}{m}^{-1} \cdot \sum_{i=d+1}^{k} |R^c(i)|, & d < k \le n. \end{cases} \tag{2.13}$$

where $R|^c(i)|$ *is given in Lemma 2.6.*

Proposition 2.1, Lemmas 2.2 and 2.4 and (2.11) also show that the distribution of $W(m,n,\mu)$ is decreasing in k if $m - \mu n > \mu - 1$, uniform if $m - \mu n = \mu - 1$, and increasing in k if $m - \mu n < \mu - 1$.

3 A limiting distribution

Let $X(m,n,\mu) = W(m,n,\mu)/n$, i.e., the proportion of type II customers who need to wait for change. We consider $\lim_{n\to\infty} P[X(m,n,\mu) \leq x]$, $0 \leq x \leq 1$, i.e. the limiting distribution of $X(m,n,\mu)$ as the size of the queue tends to infinity, and the ratio m/n is held approximately constant (up to order of $1/n$).

We denote $r = m/n$, and express $X(m,n,\mu)$ simply as X_n. In Propositions 3.1 - 3.3, we give the limiting distributions of X_n corresponding to the first three cases mentioned in Section 2. It turns out that these limiting distributions are either uniform or degenerate.

Proposition 3.1. *If $m-\mu n = \mu - 1$ $\forall n \in \mathbb{N}$, then $\lim_{n\to\infty} P(X_n \leq x) = x$, $0 \leq x \leq 1$, i.e., X_n converges in distribution to the uniform random variable on $[0,1]$.*

Proof: From Proposition 2.1 we have that $P[W(m,n,\mu) \leq k] = (k+1)/(n+1)$. Hence, for $x \in [0,1]$, we have

$$P[X_n \leq x] = P[W(m,n,\mu) \leq xn] = \frac{\lfloor xn \rfloor + 1}{n+1} \to x, \text{ as } n \to \infty.$$

□

The next proposition deals with the limiting distribution in Case 2.

Proposition 3.2. *If $r > \mu$, then $X_n \to 0$ in distribution, i.e. X_n converges to the degenerate random variable with a unit mass at 0.*

To prove the proposition, we need the following lemma.

Lemma 3.1. *Let*

$$A_j = \frac{(r+1)(r-\mu)}{r} \cdot \frac{1}{j+1} \cdot \binom{(\mu+1)j+\mu-1}{j} \left(\frac{r^\mu}{(r+1)^{\mu+1}}\right)^{j+1}, \tag{3.1}$$

where r, μ are positive real numbers, $r > \mu$, and j is a nonnegative integer. Then

$$\sum_{j=0}^{\infty} A_j = \frac{r-\mu}{r}. \tag{3.2}$$

A proof of the lemma is given in the appendix.

Proof of Proportion 3.2: From Lemmas 2.2 and 2.3 we have

$$\begin{aligned} P[X_n = \frac{k}{n}] &= P[W(m,n,\mu) = k] \\ &= \frac{m+1-\mu n}{m+1} - \binom{m+n}{n}^{-1} \cdot \sum_{j=0}^{k-1} |T^c(j)|, \quad k = 0,1,\ldots,n, \end{aligned} \tag{3.3}$$

where the sum is zero if $k = 0$, and $|T^c(j)|$ is given in (2.6). By considering the binomial expressions in (3.3) and (2.6) as ratios of polynomials in n, we obtain

$$\lim_{n\to\infty} P[X_n = \frac{k}{n}] = \frac{r-\mu}{r} - \sum_{j=0}^{k-1} A_j.$$

where A_j is as defined in (3.1). Therefore, from Lemma 3.1 we have

$$\lim_{n\to\infty} P[X_n = \frac{k}{n}] = \frac{r-\mu}{r} - \sum_{j=k}^{\infty} A_j.$$

Now,

$$\lim_{n\to\infty} P[X_n \leq \frac{k_0}{n}] = \sum_{j=0}^{k_0} \sum_{j=k}^{\infty} A_j$$

$$= \sum_{j=0}^{\infty} (1 + j \wedge k_0) A_j \geq \sum_{j=0}^{k_0} (1 + j) A_j.$$

Differentiating both sides of (3.2) with respect to r, we obtain

$$\sum_{j=0}^{\infty} (1 + j) A_j = 1.$$

Thus, for any given $\varepsilon > 0$ we can find a k_0 such that $\sum_{j=0}^{\infty} (1+j) A_j > 1 - \varepsilon$. Hence,

$$\lim_{n\to\infty} P[X_n \leq \frac{k_0}{n}] > 1 - \varepsilon$$

and the conclusion follows. □

The next proposition gives the limiting distribution in Case 3. The approach is similar to the one used in the proof of Proposition 3.2, hence we omit the proof.

Proposition 3.3. *If $0 \leq m - \mu n < \mu - 1$, then X_n converges in distribution to the uniform random variable on the interval* $[0, 1]$.

It can be similarly shown that X_n converges in distribution to the uniform random variable on [0,1] if $m = \mu n + b$, where b is a constant such that $b \geq \mu - 1$.

Finally, we conjecture that, if $r > \mu$, all the moments of $W(m, n, \mu)$ are bounded as $n \to \infty$. This conjecture arises from computations and heuristic arguments.

4 Concluding remarks

Extension to more general models

The results of Section 2 easily extend to the binomial scenario. Suppose that instead of knowing m and n, we only know the total N, and that

$$\begin{aligned}P[\text{a customer is of type I}] &= p \\ &= 1 - P[\text{a customer is of type II}], \text{ for some } p \in (0,1).\end{aligned}$$

Then m is now a binomial random variable, and by the law of total probability,

$$\begin{aligned}P[W(N,p,\mu) = k] &= \sum_{i=0}^{N} P[W(i, N-i, \mu) = k] \cdot P(m = i) \\ &= \sum_{i=0}^{N} P[W(i, N-i, \mu) = k] \cdot \binom{N}{i} p^i (1-p)^{N-i}.\end{aligned}$$

The probability $P[W(i, N-i, \mu) = k]$ is obtained by referring to the appropriate case in Section 2. We can further assume that N is also a random variable with a distribution $h(x)$, $x = 1, 2, \ldots$ (for example, a truncated Poisson). In such a case

$$P[W = k] = \sum_{j=1}^{\infty} \sum_{i=1}^{j} P[W(i, j-i, \mu) = k] \cdot \binom{j}{i} p^i (1-p)^{j-i} \cdot h(j).$$

Applications

The distribution of W has applications similar to those of the ballot problem, especially in the area of nonparametric statistics. For example, W can be used as a test statistic for testing the randomness of sequences of binary observations (to be discussed in a future work).

Applications also exist in other fields such as inventory models. For a simple example, suppose that items in a manufacturing facility are produced one at a time as a Poisson process. Orders are received in batches of size μ, also as a Poisson process, independently of the production process. The above results can be used to find the distribution of the proportion of the times an order cannot be filled.

Open problems for future research

The queuing model presented in this paper gives rise to several other problems. We mention two of them, which are currently studied.

1. *The distribution of W under different queue disciplines* – It can be assumed for example, that Type II customers who cannot get change are sent to the end of the line and are served last (provided that eventually there is change for them). This assumption (which is applicable to certain models) would yield a different distribution of W. Other queue disciplines can be considered as well.

2. *An $M_p\backslash M\backslash 1$ queue* – Our model can be studied in the $M\backslash M\backslash 1$ queue formulation with two types of customers and $P[\text{a customer is of type I}] = p$, $p \in (0,1)$. Such a model can be appropriately labeled as $M_p\backslash M\backslash 1$ queue. The usual random variables such as the number of customers in the system, the waiting time and the busy period, will be distributed differently than in the standard $M\backslash M\backslash 1$ case.

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Appendix

Proof of Lemma 3.1: First we introduce the variable $s = 1/(r+1)$, so that

$$\frac{r^{\mu}}{(r+1)^{\mu+1}} = s(1-s)^{\mu}.$$

Then (3.2) can be written as

$$\sum_{j=0}^{\infty} \frac{1}{j+1} \binom{(\mu+1)j+\mu-1}{j} s^j (1-s)^{\mu(j+1)} = 1. \tag{A.1}$$

By Stirling's formula we have,

$$\binom{(\mu+1)j+\mu-1}{j} \sim \frac{1}{\sqrt{j}} \left(\frac{\mu+1^{\mu+1}}{\mu^{\mu}}\right)^j, \text{ as } j \to \infty,$$

which guarantees uniform (and absolute) convergence of the series in (A.1). Now, (A.1) is equivalent to

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{j+1} \binom{(\mu+1)j+\mu-1}{j} \binom{(j+1)\mu}{k} s^{j+k} = 1,$$

or

$$\sum_{M=0}^{\infty}\sum_{j=0}^{M} \frac{(-1)^{M-j}}{j+1} \binom{(\mu+1)j+\mu-1}{j} \binom{(j+1)\mu}{M-j} s^{M} = 1. \tag{A.2}$$

The equality in (A.2) is true if and only if

$$\sum_{j=0}^{M} \frac{(-1)^j}{j+1} \binom{(\mu+1)j+\mu-1}{j} \binom{(j+1)\mu}{M-j} = 0, \ M = 1, 2, 3, \ldots$$

or equivalently, if and only if

$$\sum_{j=0}^{M} (-1)^j \binom{M}{j} \binom{(j+1)\mu+j-1}{M-1} = 0, \ M = 1, 2, 3, \ldots \qquad \text{(A.3)}$$

Finally, the left hand side of (A.3) is the Mth difference of

$$p(j) =: \binom{(j+1)\mu+j-1}{M-1},$$

which is a polynomial in j of degree $M-1$. Therefore, (A.3) is true since the Mth difference of a polynomial of degree less than M is always 0. Since (A.3) is equivalent to (3.2), this completes the proof. □

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