A Note On $Y - \Delta$ and $\Delta - Y$ Graphs

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ABSTRACT. A 2-connected graph is called $Y - \Delta$ (respectively $\Delta - Y$) *reducible* or simply a $Y - \Delta$ (respectively $\Delta - Y$) graph if it can be reduced to a single edge using a sequence of $Y - \Delta$ (respectively $\Delta - Y$, series and parallel reductions. This paper addresses the problem of decomposing $Y - \Delta$ and $\Delta - Y$ graphs in connection with a new method for decomposing 3-connected graphs proposed recently by Coullard, Gardner, and Wagner.

1 Introduction

A 2-connected graph is called $Y - \Delta$ *reducible* (or simply a $Y - \Delta$ graph) if it can be reduced to a single edge using a sequence of $Y - \Delta$, series and parallel reductions. The class of $Y - \Delta$ graphs is precisely the class of (2-connected) partial 3-trees [1]. A 2-connected graph is called $\Delta - Y$ reducible (or a $\Delta - Y$ graph) if it can be reduced to a single edge using a sequence of $\Delta - Y$, series and parallel reductions. The $Y - \Delta$ and $\Delta - Y$ graphs possess a variety of structural and algorithmic properties and have been studied extensively in the literature [1] [5] [7] [9] [10] [11].

This paper addresses the issue of decomposing $Y - \Delta$ and $\Delta - Y$ graphs in connection with a method for decomposing 3-connected graphs proposed recently by Coullard, Gardner, and Wagner [4]. (A $Y - \Delta$ or $\Delta - Y$ graph

JCMCC 19 (1995), pp. 259-272

is at most 3-connected.) In particular, a 3-connected $Y - \Delta$ graph is decomposable into twirls, wheels and cubes. (A graph is called a *twirl* if it is isomorphic to the graph $K_{3,n}$ for some $n \geq 3$. A graph having at least four vertices is called a *wheel* if it is isomorphic to a connected loopless graph G with a vertex v such that $G \setminus \{v\}$ is a cycle every vertex of which is adjacent to v. A graph isomorphic to $K_2 \times C_4$ is called a *cube*.) The converse is not true. A 3-connected graph having at least four vertices is $\Delta - Y$ if and only if it is decomposable into wheels. Using techniques in this paper, unique decompositions of $Y - \Delta$ and $\Delta - Y$ graphs can be obtained if the graphs are minimally 3-connected. The minimally 3-connected $\Delta - Y$ graphs include the well-known class of Halin graphs.

The results in this paper are motivated in part by a desire to identify classes of graphs that decompose into well-structured classes of graphs. (The twirls, wheels and cubes are examples of such classes.) Graph decomposition techniques and well-structured classes of graphs related to these decompositions have an extensive literature. Whitney and Tutte devised decompositions for 1- and 2-connected graphs respectively [14] [12]. The Coullard-Gardner-Wagner decomposition is the 3-connected equivalent of the Tutte decomposition for 2-connected graphs. The well known *series-parallel* graphs, those 2-connected graphs that can be reduced to a single edge using series and parallel reductions, are precisely the graphs decomposable using the Tutte decomposition into polygons and bonds [13]. The results in this paper can be viewed as a continuation of these efforts. As mentioned, the 3-connected $\Delta - Y$ graphs are precisely those graphs decomposable into wheels. Also, the Whitney and Tutte decompositions lead to unique decompositions. Classes of graphs that decompose into classes of well-structured graphs may admit polynomial-time algorithms for problems that are NP-hard for graphs in general. A number of such decomposition-based algorithms exist for the class of series-parallel graphs [2] [3]. In the present context, Gardner has devised a polynomial-time algorithm for a certain generalization of the minimum-weight cycle problem, called the *minimum-load cycle problem.* The algorithm runs on any class of 3-connected graphs whose members are decomposable into graphs on which this problem has a polynomial-time solution [6]. The minimum-load cycle problem is in general NP-complete, but can be solved in polynomial-time on twirls, wheels and cubes. Consequently, Gardner's algorithm can be applied to the $Y - \Delta$ and $\Delta - Y$ graphs.

The remainder of this paper is organized as follows. In Section 2, the necessary terminology from graph theory is presented as well as a terse description of the Coullard-Gardner-Wagner decomposition. The main results are located in Sections 3. Section 4 presents characterizations of the class of $\Delta - Y$ graphs in terms of the decomposition.

2 Preliminaries

A general familiarity with graphs is assumed. For clarity, however, some definitions and notation will now be established. Graphs are assumed to be undirected with loops and parallel edges allowed. Let $\mathbf{G} := (V, E)$ be a graph where V and E are the sets of vertices and edges of $\mathbf{G}$ respectively. The sets of vertices and edges of $\mathbf{G}$ are also denoted by $V(\mathbf{G})$ and $E(\mathbf{G})$ respectively. If $\mathbf{H}$ is a graph, then $\mathbf{H} \subseteq \mathbf{G}$ denotes that $\mathbf{H}$ is a subgraph of $\mathbf{G}$. If $\phi \neq D \subseteq E$ (respectively $\phi \neq D \subset V$), then $\mathbf{G}[D]$ denotes the subgraph of $\mathbf{G}$ induced by D, and $\mathbf{G} \setminus D$ denotes the subgraph obtained by deleting all the edges (respectively vertices) in D. If $\phi \neq D \subset E$, let $A(\mathbf{G}, D) := V(\mathbf{G}[D]) \cap V(\mathbf{G}[E - D])$. In addition, if $D \subseteq E$, then $\mathbf{G}/D$ is the graph obtained from $\mathbf{G}$ by contracting the edges in D. The graph $\mathbf{G}$ is *contractible* to a graph $\mathbf{H}$ if $\mathbf{G}/D$ is isomorphic to $\mathbf{H}$ for some $D \subseteq E$.

Two nonloop edges of $\mathbf{G}$ are called *parallel* if they have the same ends. A graph $\mathbf{G}'$ is said to be obtained from $\mathbf{G}$ by a parallel reduction if $\mathbf{G}' := \mathbf{G} \setminus \{f\}$ where e and f are parallel edges of $\mathbf{G}$. Two edges e and f of $\mathbf{G}$ are said to be in *series* if $\mathbf{G}[\{e, f\}]$ is a path whose internal vertex has degree two in $\mathbf{G}$. A graph $\mathbf{G}'$ is said to be obtained from $\mathbf{G}$ by a *series reduction* if $\mathbf{G}' := \mathbf{G}/\{f\}$. If $\mathbf{H} \subseteq \mathbf{G}$ is a cycle with three vertices, then $\mathbf{H}$ is called a *triangle* (also "delta" or "Δ") of $\mathbf{G}$, and if $\mathbf{H}$ is induced by the edges incident to a degree-three vertex of $\mathbf{G}$, then $\mathbf{H}$ is called a *triad* (also "wye" or "Y") of $\mathbf{G}$. Suppose $\{v_1, v_2, v_3\} \subseteq V$, and choose $t, e_1, e_2, e_3 \notin V \cup E$. Add t to $\mathbf{G}$ as a vertex and e_i as an edge so that e_i has ends t and v_i. The resulting graph is said to be obtained from $\mathbf{G}$ by a *triad addition* at $\{v_1, v_2, v_3\}$. If $\mathbf{H}$ is a triangle of $\mathbf{G}$ and $\mathbf{G}'$ is obtained from $\mathbf{G}$ by a triad addition at $V(\mathbf{H})$, then the graph $\mathbf{G}' \setminus E(\mathbf{H})$ is said to be obtained from $\mathbf{G}$ by a *$\Delta - Y$ reduction*. Conversely, $\mathbf{G}$ is said to be obtained from $\mathbf{G}' \setminus E(\mathbf{H})$ by a $Y - \Delta$ reduction. In Figure 2.1 below, the graph $\mathbf{G}'$ is obtained from $\mathbf{G}$ by a $\Delta - Y$ reduction, and $\mathbf{G}$ is obtained from $\mathbf{G}'$ by a $Y - \Delta$ reduction.

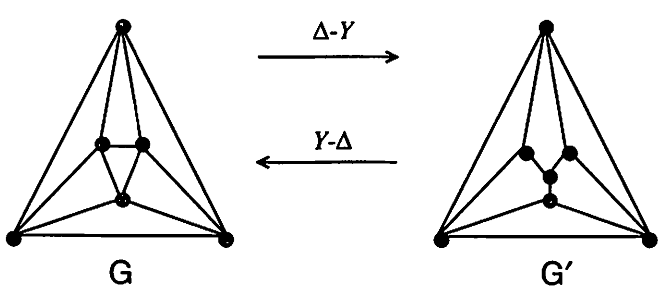

Figure 2.1

If $k \geq 1$, a partition $\{D_1, D_2\}$ of E is called a *k-separation* of $\mathbf{G}$ if

$|D_i| \geq k$, $i = 1, 2$, and $|A(\mathrm{G}, D_1)| \leq k$. For $n \geq 2$, G is called *n-connected* if it has no k-separation for $k < n$. If G is an n-connected graph with at least $n + 1$ vertices and $u, v \in V$, then G has at least n internally disjoint (u, v)-paths [8]. The graph G is called *minimally n*-connected if for each edge e of G, $\mathrm{G} \setminus \{e\}$ has a $(n-1)$-separation.

Suppose G is 3-connected and $\{D_1, D_2\}$ is a 3-separation of G such that neither $\mathrm{G}[D_1]$ nor $\mathrm{G}[D_2]$ is a triad. Such a 3-separation is called *cyclic.* Let $A(\mathrm{G}, D_1) := \{v_1, v_2, v_3\}$. Choose $t, e_1, e_2, e_3 \notin V \cup E$. For $i = 1, 2$, let H_i be the graph obtained from $\mathrm{G}[D_i]$ by a triad addition at $\{v_1, v_2, v_3\}$ such that e_j is an edge with ends t and v_j, $1 \leq j \leq 3$. Let $S_i \subseteq \{e_1, e_2, e_3\}$ be the set of edges of H_i incident to a degree-two vertex, and let W_i be the set of these degree-two vertices. If $S_i \neq \phi$, let $\mathrm{H}'_i := \mathrm{H}_i / S_i$ where $V(\mathrm{H}'_i) = V(\mathrm{H}_i) - W_i$. Finally, a certain edge renaming procedure is executed. Specifically, if e_j is contracted in H_i, then e_j is renamed f in $\mathrm{H}'_{i'}$, $i' \neq i$, where f is the edge of H_i incident to e_j at a degree-two vertex. (By 3-connectivity, an edge e_j cannot be contracted in both H_1 and H_2.) If G_i is the graph obtained from H'_i by this renaming procedure, then $\{G_1, G_2\}$ is called a *simple decomposition* of G. The edges in the set $E(\mathrm{G}_1) \cap \{e_1, e_2, e_3\} = E(\mathrm{G}_2) \cap \{e_1, e_2, e_3\}$ are called *marker edges*, and t is called the *marker vertex.* Let E_3 be the set of edges incident to a degree-one vertex in $\mathrm{G}[D_1]$ or $\mathrm{G}[D_2]$. Let $E_i := D_i - E_3$, $i = 1, 2$. If $\{F_1, F_2\}$ is a partition of E_3, note that $\{E_1 \cup F_1, E_2 \cup F_2\}$ is a 3-separation of G and that any two such 3-separations give rise to the same simple decomposition $\{\mathrm{G}_1, \mathrm{G}_2\}$. The triple $\{E_1, E_2; E_3\}$ is called the *split associated with* the 3-separation $\{D_1, D_2\}$, and the decomposition $\{\mathrm{G}_1, \mathrm{G}_2\}$ will be referred to as the simple decomposition *associated with* either the 3-separation $\{D_1, D_2\}$ or the split $\{E_1, E_2; E_3\}$.

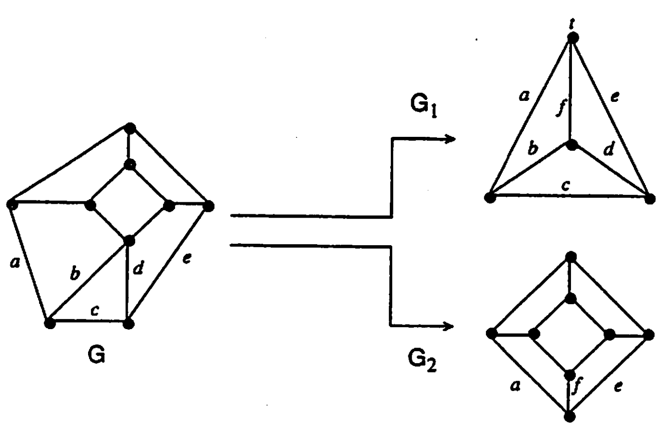

Figure 2.2

In Figure 2.2, $\{G_1, G_2\}$ is the simple decomposition of G associated with the split $\{E_1, E_2; E_3\}$ where $E_1 = \{b, c, d\}$ and $E_3 = \{a, e\}$. The edge f is the (unique) marker edge.

The following result is a fundamental property of simple decompositions.

Proposition 2.1 ([4, Lemmas 2.2 and 2.5]). *If $\{G_1, G_2\}$ is a simple decomposition of a 3-connected (respectively minimally 3-connected) graph G, then both G_1 and G_2 are 3-connected (respectively minimally 3-connected).* □

If G is a 3-connected graph, then a *decomposition* Γ of G is either $\{G\}$ or the set $(\Gamma' - \{H\}) \cup \{H_1, H_2\}$ where Γ' is a decomposition of G and $\{H_1, H_2\}$ is the simple decomposition of some $H \in \Gamma'$. In the latter case, $(\Gamma' - \{H\}) \cup \{H_1, H_2\}$ is called a *simple refinement* of Γ'. If $\Gamma_1, \ldots, \Gamma_k$ is a sequence of decompositions of G where Γ_{i+1} is a simple refinement of Γ_i assuming $i + 1 \leq k$, then Γ_k is called a *refinement* of Γ_1. If $k > 1$, then Γ_k is a proper refinement of Γ.

3 Decompositions of $Y - \Delta$ and $\Delta - Y$ Graphs

The purpose of this section is to show that a 3-connected $Y-\Delta$ (respectively $\Delta - Y$) graph G having at least four vertices has a decomposition that is minimal with respect to the property that every member is a twirl, a wheel or a cube (respectively wheel). If G is minimally 3-connected, this decomposition is unique.

A graph G is a *3-tree* if G is a triangle or G is obtained from a 3-tree by a triad addition at the set of vertices of a triangle of G. A *partial* 3-tree is a subgraph of a 3-tree. As noted in Section 1, the class of 2-connected $Y - \Delta$ graphs is precisely the class of 2-connected partial 3-trees. It will be convenient to view $Y - \Delta$ graphs as partial 3-trees throughout the sequel.

The following result has been proved independently by a number of authors.

Lemma 3.1. *Suppose G is a partial 3-tree and H is a $\Delta - Y$ graph. If G (respectively H) is contractible to a graph K, then K is also a partial 3-tree (respectively $\Delta - Y$ graph).* □

The first step in establishing that a 3-connected partial 3-tree G has the type of decomposition mentioned above is to verify that the graphs in a simple decomposition of G are also partial 3-trees.

Proposition 3.2. *If $\{G_1, G_2\}$ is the simple decomposition associated with the split $\{E_1, E_2; E_3\}$ of a 3-connected partial 3-tree G, then G_1 and G_2 are also 3-connected partial 3-trees.*

Proof: The graphs G_1 and G_2 are 3-connected by Proposition 2.1. It will be shown that G_1 is a partial 3-tree.

Let $A(\mathbf{G}, E_1) := \{u, v, w\}$. Suppose first that $V(\mathbf{G}[E_2]) - A(\mathbf{G}, E_1) \neq \phi$. Choose a vertex $x \in V(\mathbf{G}[E_2]) - A(\mathbf{G}, E_1)$. By the 3-connectivity of $\mathbf{G}$, there exists internally disjoint (x,u)-, (x,v)- and (x,w)-paths. Choose three such paths and denote them $\mathbf{P}_u$, $\mathbf{P}_v$ and $\mathbf{P}_w$ respectively. Being a subgraph of a partial 3-tree, the graph $\mathbf{G}[E_1 \cup E(\mathbf{P}_u) \cup E(\mathbf{P}_v) \cup E(\mathbf{P}_w)]$ is a partial 3-tree. Finally, by contracting all but one of the edges of each of the paths $\mathbf{P}_u$, $\mathbf{P}_v$ and $\mathbf{P}_w$, it is seen that $\mathbf{G}[E_1 \cup E(\mathbf{P}_u) \cup E(\mathbf{P}_v) \cup E(\mathbf{P}_w)]$ is contractible to $\mathbf{G}_1$, and so by Lemma 3.1, $\mathbf{G}_1$ is a partial 3-tree.

Now suppose that $V(\mathbf{G}[E_2]) - A(\mathbf{G}, E_1) = \phi$. In this case, $\mathbf{G}[E_2]$ is a triangle such that $V(\mathbf{G}[E_2]) = A(\mathbf{G}, E_1)$. By definition, there exists a 3-tree $\mathbf{H}$ such that $\mathbf{G} \subseteq \mathbf{H}$. Let $\mathbf{H}'$ be the graph obtained from $\mathbf{H}$ by a triad addition at $A(\mathbf{G}, E_1)$. By definition, $\mathbf{H}'$ is a 3-tree. If D is the set of edges of the added triad, then $\mathbf{H}'[E_1 \cup D]$ is a partial 3-tree. The proof is now complete since $\mathbf{H}'[E_1 \cup D]$ is isomorphic to $\mathbf{G}_1$. □

The following result is a version of a result of Arnborg and Proskurowski [1, Theorem 3,4].

Theorem 3.3. *If* $\mathbf{G}$ *is a 3-connected partial 3-tree having at least five vertices, then* $\mathbf{G}$ *has a subgraph* $\mathbf{J}$ *that is isomorphic to one of the graphs* $\mathbf{J}_1$ *through* $\mathbf{J}_3$ *depicted in Figure 3.1 where* $A(\mathbf{G}, E(\mathbf{J})) = \{u, v, w\}$ *(say).*

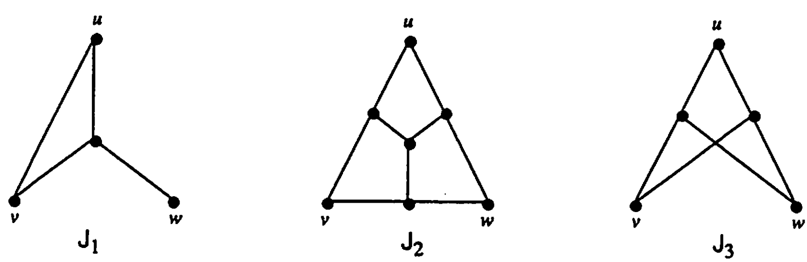

Figure 3.1

The following proposition is one of the main results of this section.

Proposition 3.4. *If* $\mathbf{G}$ *is a 3-connected partial 3-tree having at least four vertices, then* $\mathbf{G}$ *has a decomposition every member of which is isomorphic to* $\mathbf{K}_{3,3}$, $\mathbf{K}_4$ *or* $\mathbf{K}_2 \times \mathbf{C}_4$.

Proof: Since $\mathbf{G}$ has at least four vertices, $\mathbf{G}$ has at least six edges. The proof will proceed by induction on $|E(\mathbf{G})|$. If $\mathbf{G}$ has exactly six edges, then $\mathbf{G}$ is isomorphic to $\mathbf{K}_4$. Suppose then that the result holds for all 3-connected partial 3-trees having less than k edges, $k \geq 7$. Let $\mathbf{G}$ be a 3-connected partial 3-tree having k edges. In this case, $\mathbf{G}$ has at least five vertices. By Theorem 3.3, G has a subgraph J isomorphic to one of the graphs depicted in Figure 3.1 where $A(\mathbf{G}, E(\mathbf{J})) = \{u, v, w\}$ (say).

Suppose first that J is isomorphic to $\mathbf{J}_1$. Let D be the set of edges of the triangle of J. Since $|V(\mathbf{G})| \geq 5$, $\{D, E(\mathbf{G})-D\}$ is a cyclic 3-separation of G. Let $\{E_1, E_2; E_3\}$ be the split associated with this 3-separation. In this case, $|E_3| \geq 1$. Therefore, if $\{G_1, G_2\}$ is the associated simple decomposition of G, $\mathbf{G}_1$ is isomorphic to $\mathbf{K}_4$ and $|E(\mathbf{G}_2)| \leq |E(\mathbf{G})| - 1$. By Proposition 3.2, $\mathbf{G}_2$ is a 3-connected partial 3-tree, and so by induction, $\mathbf{G}_2$ has a decomposition Γ every member of which is isomorphic to $\mathbf{K}_{3,3}$, $\mathbf{K}_4$ or $\mathbf{K}_2 \times \mathbf{C}_4$. Therefore, $\Gamma \cup \{\mathbf{G}_1\}$ is such a decomposition.

Now suppose that J is isomorphic to either $\mathbf{J}_2$ or $\mathbf{J}_3$. In this case, let D be the set of edges of J. Suppose $\{D, E(\mathbf{G})-D\}$ is not a 3-separation of G. Assume first that J is isomorphic to J_2. Let s be the unique vertex of J that is not a degree-two vertex of J or adjacent to one. It follows that G contains a subgraph K that is isomorphic to $\mathbf{J}_1$ where $A(\mathbf{G}, E(\mathbf{K})) = \{u, w, s\}$ or $A(\mathbf{G}, E(\mathbf{K})) = \{u, v, s\}$ since $\mathbf{G} \setminus \mathbf{J}_2$ consists of two edges joining u and w, u and v, or v and w. If J is isomorphic to $\mathbf{J}_3$, it again follows that G contains a subgraph K isomorphic to $\mathbf{J}_1$ where $A(\mathbf{G}, E(\mathbf{K})) = \{u, v, w\}$. The result follows. Assume therefore that $\{D, E(\mathbf{G}) - D\}$ is a 3-separation. Suppose this 3-separation is not cyclic. If J is isomorphic to $\mathbf{J}_2$, then G is a cube. If, however, J is isomorphic to $\mathbf{J}_3$, then G is isomorphic to $\mathbf{K}_{3,3}$. Finally, if the 3-separation $\{D, E(\mathbf{G}) - D\}$ is cyclic and $\{\mathbf{G}_1, \mathbf{G}_2\}$ is the associated simple decomposition of G, then $\mathbf{G}_1$ is isomorphic to either $\mathbf{K}_{3,3}$ or $\mathbf{K}_2 \times \mathbf{C}_4$, and $|E(\mathbf{G}_2)| < |E(\mathbf{G})|$. The result now follows by induction. □

The next result concerns the decomposition of a 3-connected $\Delta - Y$ graph.

Proposition 3.5. *If* G *is a 3-connected* $\Delta - Y$ *graph having at least four vertices, then* G *has a decomposition every member of which is isomorphic to* $\mathbf{K}_4$.

Proof: For convenience, if H is a graph, let $d(\mathbf{H}) := \Sigma\{d_{\mathbf{H}}(v) | v \in V(\mathbf{H})$ and $d_{\mathbf{H}}(v) \geq 4\}$ where $d_{\mathbf{H}}(v)$ denotes the degree of v in H. Suppose the proposition is false, and let $\mathbf{G}^*$ be a counterexample with $d(\mathbf{G}^*) + |V(\mathbf{G}^*)|$ a minimum.

Since $\mathbf{G}^*$ is a $\Delta - Y$ graph, there exists a sequence of $\Delta - Y$, series and parallel reductions that can be used to reduce $\mathbf{G}^*$ to a single edge. Since $\mathbf{G}^*$ is 3-connected and has at least four vertices, the first reduction in this sequence must be a $\Delta - Y$ reduction. Let D denote the set of edges of the triangle involved in this reduction. If the 3-separation $\{D, E(\mathbf{G}^*) - D\}$ is not cyclic, then $\mathbf{G}^*$ is isomorphic to $\mathbf{K}_4$, a contradiction. Otherwise, let $\{E_1, E_2; E_3\}$ be the split associated with $\{D, E(\mathbf{G}^*)-D\}$, and let $\{\mathbf{G}_1^*, \mathbf{G}_2^*\}$ be the simple decomposition associated with this split. If $\hat{\mathbf{G}}^*$ is the graph obtained from $\mathbf{G}^*$ by the $\Delta - Y$ reduction involving the edges in D, then $\mathbf{G}_2^*$ is isomorphic to a graph that can be obtained from $\hat{\mathbf{G}}^*$ by contracting at most three edges of $\hat{G}^*$. By Lemma 3.1, G_2^* is a $\Delta - Y$ graph.

Note first that G_2^* has at least four vertices. In addition, observe that $d(G_2^*) \leq d(G^*)$. If $d(G_2^*) = d(G^*)$, then $|E_3| = 3$, and so $|V(G_2^*)| = |V(G^*)| - 2$. Assume now that $d(G_2^*) \leq d(G^*)$. Note that $|V(G_2^*)| \leq |V(G^*)| + 1$. If $|V(G_2^*)| = |V(G^*)| + 1$, then $E_3 = \phi$, and so $d(G_2^*) \leq d(G^*) - 3$. In each case, it follows that $d(G_2^*) + |V(G_2^*)| < d(G^*) + |V(G^*)|$. By induction, G_2^* has a decomposition Γ every member of which is isomorphic to K_4. Consequently, every member of $\Gamma \cup \{G_1^*\}$, a decomposition of G^*, is isomorphic to K_4, a contradiction. □

Two decompositions Γ and Γ' of a 3-connected graph G are called *equivalent* if Γ' can be obtained from Γ by replacing some of the marker edges and vertices of members of Γ by marker edges and vertices of members of Γ'. A decomposition Γ is *unique* with respect to a property π if Γ satisfies π and any other decomposition that satisfies π is equivalent to Γ. Finally, Γ is minimal with respect to π if Γ satisfies π but no decomposition having Γ as a proper refinement satisfies π.

The remainder of this section is devoted to showing that every 3-connected partial 3-tree (respectively $\Delta - Y$) graph with at least four vertices has a decomposition minimal with respect to the property that every member is a twirl, a wheel or a cube (respectively wheel). If the graph is minimally 3-connected, this decomposition is unique. The proof involves a certain notion of graph composition defined below.

Two members of a decomposition Γ of a 3-connected graph are called *adjacent* if they share a marker vertex. (Note that exactly two graphs in Γ share the same marker vertex.) The following two observations are easily made for adjacent $G_1, G_2 \in \Gamma$ that share a marker vertex t:

(i.) the set S of edges of G_1 incident to t is equal to the set of edges of G_2 that are incident to t, and

(ii.) if e is an edge of G_1 with ends v and t where $v \in (V(G_1) \cap V(G_2)) - \{t\}$, then e is an edge of G_2 with ends v and t.

(Any edge satisfying (ii.) is a marker edge. Note however that a marker edge may have ends u and t in G_1 and ends v and t in G_2, $u \neq v$.) Given G_1 and G_2 as above, define a graph G, called the *composition* of G_1 and G_2, as follows. Let $E(G) := (E(G_1) \cup E(G_2)) - \{e \in E(G_1) \cap E(G_2) | e$ has the same ends in both G_1 and $G_2\}$, and let $V(G) := (V(G_1) \cup V(G_2)) - \{t\}$. If $e \in (E(G_1) \cup E(G_2)) - S$, then e has the same ends in G as it does in G_1 or G_2. Finally, if e has ends u and t in G_1 and ends v and t in G_2, $u \neq v$, then e has ends u and v in G. Note that the graph G is the unique graph having $\{G_1, G_2\}$ as a simple decomposition. The next result follows from the above observations and constructions although the proof is somewhat technical and is therefore omitted. It is sufficient to say that enough information is associated with each member of a decomposition Γ of G to uniquely reconstruct G from Γ.

Proposition 3.6 **([6, Lemma 5.2.1]).** *If Γ is a decomposition of a 3-connected graph* H, G_1 *and* G_2 *are adjacent members of* Γ, *and* G *is the composition of* G_1 *and* G_2, *then* $(\Gamma - \{G_1, G_2\}) \cup \{G\}$ *is also a decomposition of* H.

The proof of the following lemma is straightforward.

Lemma 3.7. *If* G *is a twirl (respectively wheel) and* $\{G_1, G_2\}$ *is a simple decomposition of* G, *then* G_1 *and* G_2 *are both twirls (respectively wheels).* □

The following theorem is the main result of this section.

Theorem 3.8. *If* G *is a 3-connected partial 3-tree (respectively $\Delta - Y$ graph) with at least four vertices, then* G *has a decomposition minimal with respect to the property that every member is a twirl, a wheel or a cube (respectively wheel).* □

Proof: Suppose G is a partial 3-tree. (The proof is similar if G is a $\Delta - Y$ graph.) By Proposition 3.4, G has a decomposition Γ every member of which is a twirl, a wheel or a cube. Suppose there exists adjacent graphs K_1 and K_2 in Γ such that K_1 and K_2 are either both twirls or both wheels. If K is the composition of K_1 and K_2 and K is a twirl or a wheel, replace Γ with $(\Gamma - \{K_1, K_2\}) \cup \{K\}$. By Proposition 3.6, $(\Gamma - \{K_1, K_2\}) \cup \{K\}$ is a decomposition of G. Continue this process until there is no pair of adjacent twirls or adjacent wheels whose composition is a twirl or a wheel respectively. Let Γ' be the resulting decomposition of G.

Suppose that Γ' is not minimal with respect to the property that every member is a twirl, a wheel or a cube. Therefore, there exists a decomposition Γ'' of G such that Γ' is a proper refinement of Γ'' and every member of Γ'' is a twirl, a wheel or a cube. Since a cube has no splits, by Lemma 3.7, it may be assumed that Γ' is a simple refinement of Γ''. In this case, $\Gamma' = (\Gamma'' - \{K\}) \cup \{K_1, K_2\}$ where $K \in \Gamma''$ and $\{K_1, K_2\}$ is a simple decomposition of K. Now K is either a twirl or a wheel, and so K_1 and K_2 are either adjacent twirls or adjacent wheels respectively, a contradiction, since K is the composition of K_1 and K_2 and Γ' has, by construction, no adjacent twirls or adjacent wheels whose composition is a twirl or wheel respectively. □

If the graph G in the statement of Theorem 3.8 is minimally 3-connected, then the decomposition Γ' is unique. That is, if Γ is a decomposition minimal with respect to the property that every member of Γ is a twirl, a wheel or a cube, then Γ and Γ' are equivalent. The uniqueness is implied by the following result of Coullard, Gardner, and Wagner [4, Theorem 1.1]. A 3-connected graph is called *cyclically 4-connected* if it has no splits.

Theorem 3.9. *A minimally 3-connected graph* G *having at least four vertices has a unique decomposition* Γ *minimal with respect to the property that every member of* Γ *is a twirl, a wheel or a cyclically 4-connected graph.* □

For a minimally 3-connected partial 3-tree or $\Delta - Y$ graph G, let Γ′ be the decomposition of G guaranteed to exist by Theorem 3.8. If Γ is any decomposition of G that is minimal with respect to the property that every member of Γ is a twirl, a wheel or a cube, then it is minimal with respect to the property that every member is a twirl, a wheel or a cyclically 4-connected graph. Therefore, Γ′ and Γ are equivalent. Without the assumption that G is minimally 3-connected, the decomposition Γ′ need not be unique. For example, the graph in Figure 3.2 below is a partial 3-tree, but it has two nonequivalent minimal decompositions the members of which are wheels.

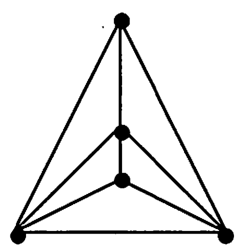

Figure 3.2

4 More on $\Delta - Y$ Graphs

It was shown in the last section that every 3-connected $\Delta - Y$ graph with at least four vertices is decomposable into wheels. Conversely, suppose a 3-connected graph G has a decomposition Γ every member of which is a wheel. Since a wheel is clearly decomposable into graphs isomorphic to K_4, without loss of generality, assume every member of Γ is isomorphic to K_4. If $|\Gamma| = 1$, G is a $\Delta - Y$ graph. Otherwise, choose $K \in \Gamma$, and choose $K' \in \Gamma$ adjacent to K. If K'' is the composition of K and K', then as noted in the previous section, $\Gamma' = (\Gamma - \{K, K'\}) \cup \{K''\}$ is a decomposition of G, and $\{K, K'\}$ is a simple decomposition of K''. Clearly, K can be obtained from K'' by a $\Delta - Y$ reduction followed by a (possibly null) sequence of series reductions. Now (if necessary) repeat the above process with Γ′ playing the role of Γ and K'' playing the role of K. Continuing in this manner, the graph K can be obtained from G by a sequence of $\Delta - Y$ and series reductions. Therefore, G is a $\Delta - Y$ graph. The following result has been proved.

Theorem 4.1. *A 3-connected graph having at least four vertices is $\Delta - Y$ if and only if it is decomposable into wheels.* □

The above discussion suggests that the decomposition might lead to a constructive characterization of the class of $\Delta - Y$ graphs. This characterization is an alternative to the one described by Politof [10]. The details follow.

Let G_1 and G_2 be 3-connected graphs having triads T_1 and T_2 respectively. For $i = 1, 2$, suppose $\{e_i, f_i, g_i\}$ are the edges in T_i, and let v_i, x_i, y_i, and z_i be vertices of G_i such that e_i has ends x_i and v_i, f_i has ends y_i and v_i, and g_i has ends z_i and v_i. Choose $e, f, g \notin V(G_1) \cup V(G_2) \cup E(G_1) \cup E(G_2)$. Define a graph G, called a *graft* of G_1 and G_2, as follows. Let $E(G) := (E(G_1) - \{e_1, f_1, g_1\}) \cup (E(G_2) - \{e_2, f_2, g_2\}) \cup \{e, f, g\}$ and $V(G) := (V(G_1) - \{v_1\}) \cup (V(G_2) - \{v_2\})$. If $h \in (E(G_1) - \{e_1, f_1, g_1\}) \cup (E(G_2) - \{e_2, f_2, g_2\})$, then h has the same ends in G as it does in G_1 or G_2. Finally, let e, f, and g have ends x_1 and x_2, y_1 and y_2, and z_1 and z_2 respectively. The edges e, f, and g are called *graft* edges. Let $D \subseteq \{e, f, g\}$, and suppose the graph G/D is 3-connected. Any graph obtained from G/D (after renaming edges in $\{e, f, g\} - D$ if necessary) is said to be obtained from G_1 and G_2 by a *graft-contract* operation. In Figure 4.1, G is a graft of G_1 and G_2. The final graph is obtained from G by contracting edges e and f.

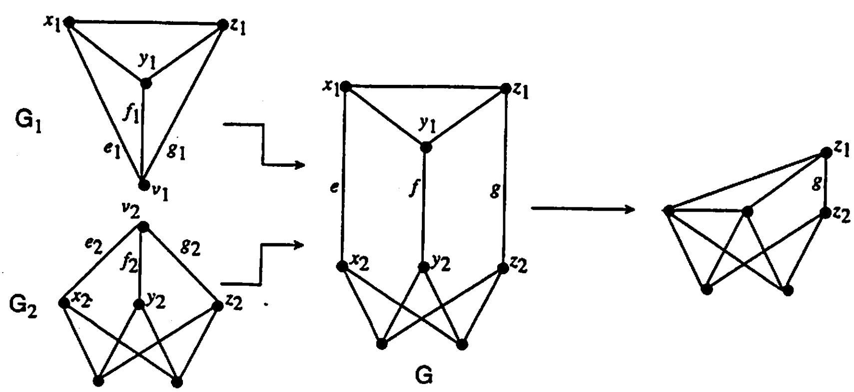

Figure 4.1

Proposition 4.2 ([6, Propositions 2.3.1 and 2.3.2]. *If G and H are 3-connected graphs, then*

(i.) a graft of G and H is 3-connected, and

(ii.) if $e \in E_3$ for some split $\{E_1, E_2; E_3\}$ of **G** *and e is not an edge of a triangle of* **G**, *then* $\mathbf{G}/\{e\}$ *is 3-connected.*

□

The point of (ii.) is to identify when a graft edge can be contracted to maintain 3-connectivity.

Suppose **G** is a $\Delta - Y$ graph, and let **H** be a wheel. Let **K** be obtained from **G** and **H** by a graft-contract operation. If **T** is the triad of **H** involved in the construction of **K** and $\{\mathbf{G}_1, \mathbf{G}_2\}$ is the simple decomposition associated with the 3-separation $\{E(\mathbf{H}) - E(\mathbf{T}), E(\mathbf{G}) - (E(\mathbf{H}) - E(\mathbf{T}))\}$, then $\mathbf{G}_1$ is isomorphic to **H** and $\mathbf{G}_2$ is isomorphic to **G**. Consequently, **K** is a $\Delta - Y$ graph.

Paraphrasing the discussion prior to Theorem 4.1, let Γ be a decomposition of **G** every member of which is a wheel. If $|\Gamma| > 1$, choose $\mathbf{K} \in \Gamma$, and choose $\mathbf{K}' \in \Gamma$ adjacent to **K**. Since any composition $\mathbf{K}''$ can be obtained from **K** and $\mathbf{K}'$ by a graft-contract operation, **G** can be obtained from **K** (say) by a sequence of such operations. The next result follows.

Theorem 4.3. *A 3-connected graph* **G** *is $\Delta - Y$ if and only if either (i.)* **G** *is a wheel or (ii.) there exists sequences $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_k$ and $\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_{k-1}$, $k \geq 2$, such that $\mathbf{G} = \{\mathbf{G}_k\}$, each $\mathbf{H}_i$ is a wheel, and for $2 \leq i \leq k$, $\mathbf{G}_i$ is obtained from $\mathbf{G}_{i-1}$ and $\mathbf{H}_{i-1}$ by a graft-contract operation.* □

5 Concluding Remarks

As shown in Proposition 3.4, every 3-connected partial 3-tree having at least four vertices is decomposable into twirls, wheels and cubes. The converse in not true. For example, the graphs $\mathbf{K}_{2,2,2}$ and $\mathbf{K}_5$ depicted in Figure 5.1 are not partial 3-trees [11] but are decomposable into wheels and twirls. Consequently, the class of graphs decomposable into twirls, wheels and cubes contains the 3-connected partial 3-trees as a proper subclass. Obtaining a good characterization of this class of graphs is a subject for future research.

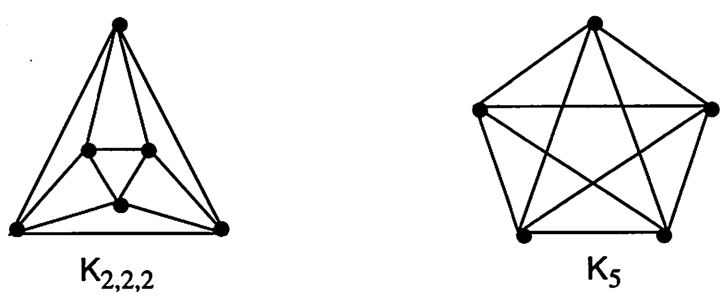

Figure 5.1

Finally, Coullard, Gardner and Wagner showed that any 3-connected graph has a unique minimal decomposition with respect to the property that each of its members satisfies a certain technical condition on its splits [4]. A characterization of the graphs satisfying this condition has not yet been obtained. In this paper, this decomposition has been determined for minimally 3-connected $Y - \Delta$ and $\Delta - Y$ graphs. Describing the unique decomposition mentioned above for $Y-\Delta$ and $\Delta-Y$ graphs is an interesting topic for future research. Such a description for the class of 3-connected partial 3-trees would have to include a class of graphs having the graph in Figure 3.2 as a member.

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