Strongly Connected Mixed Graphs and Connected Detachments of Graphs

C.St.J.A. Nash-Williams*

Department of Mathematics
University of Reading
Whiteknights, P.O. Box 220
Reading RG6 6AF, England

ABSTRACT. Let G be a finite strongly connected mixed graph (i.e. a graph with both undirected and directed edges, in which each vertex can be reached from every other vertex if directed edges can only be traversed in their direction of orientation). We establish a necessary and sufficient condition for it to be possible to transform some undirected edges of G into directed edges so that each vertex becomes the head of a prescribed number of newly directed edges and G remains strongly connected. A special case of this result yields a new proof (not requiring matroid techniques) of a necessary and sufficient condition for it to be possible to split each vertex of a finite connected graph into a prescribed number of vertices whilst preserving connectedness.

1 Introduction

A "b-detachment" of a graph G is, informally, a graph D obtained from G by splitting each vertex ξ into a specified number $b(\xi)$ of vertices: D has the same edges as G and an edge joining vertices ξ and η of G becomes an edge of D joining one of the vertices into which ξ splits to one of the vertices into which η splits. In [4] and [5], matroids were used in two different proofs of Theorem 2 below, which asserts a necessary and sufficient condition for G to have a connected b-detachment, when G is finite. Efforts (hitherto unsuccessful) to extend Theorem 2 to infinite graphs have stimulated a

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search for new proofs in the finite case. This in turn led to the discovery of a theorem (Theorem 1) about mixed graphs, i.e. graphs in which some edges are directed and some are undirected. Specifically, suppose that a finite mixed graph G is strongly connected, i.e. such that any vertex can be reached from any other if directed edges must be treated as "one-way streets". Is it possible to orient (i.e. convert to directed edges) some of the undirected edges of G in such a way that each vertex ξ becomes the head of a specified number $b(\xi)$ of newly oriented edges and strong connectedness is preserved? Theorem 1 gives a necessary and sufficient condition for this to be possible. Although a weaker result would suffice for a new proof of Theorem 2, Theorem 1 seems interesting in its own right, and might conceivably be helpful in extending Theorem 2 to infinite graphs: it is therefore proved in Section 3. In Section 4, we use a corollary of Theorem 1 to give a new proof of Theorem 2, not involving matroids, and we sketch two variants of this argument which avoid the need to prove Theorem 1 in full generality. Finally, we show that Theorem 2 implies an apparently more general result, characterizing the smallest possible number of components of a b-detachment of G even when this number is not 1.

When writing this paper, I had forgotten that [1] and [2] contain somewhat similar ideas and results. These do not concern mixed graphs, but Corollary 1a of the present paper is equivalent to Theorem 7 of [1]. The interesting survey paper [3] also mentions many relevant results, including some concerning mixed graphs. Moreover, its author has drawn my attention to a fairly easy argument whereby from Theorem 3.7 of [3] one can deduce a necessary and sufficient condition for the existence of a way of orienting some undirected edges of a strongly connected finite mixed graph so that each vertex ξ becomes the head of exactly $b(\xi)$ newly oriented edges and strong connectedness is preserved. However, this necessary and sufficient condition and the one in Theorem 1 below do not seem to be obviously identical. Therefore the approach described here may retain some independent interest.

Although this work was prompted largely by thinking about infinite graphs, the results in this paper concern finite graphs only. It will therefore henceforward be understood that all graphs, digraphs and mixed graphs considered here are finite.

2 Basic Definitions

The set of non-negative integers and the set of positive integers will be denoted by $\mathbb{Z}_+$ and N respectively. If S is a finite set, $X \subseteq S$ and $f\colon S \to \mathbb{Z}_+$ is a function then $f.X$ denotes $\sum_{x\in X} f(x)$.

A *mixed graph* is a (finite) graph G in which $E(G)$ is the union of two disjoint sets $U(G)$, $\Delta(G)$ whose elements are called *undirected* and *directed* edges respectively. An edge λ joins two (not necessarily distinct) vertices which are regarded as constituting an unordered pair if λ is undirected and an ordered pair $(\lambda t, \lambda h)$ if λ is directed, in which case λt is the *tail* of λ and λh is its *head*. It is intuitively helpful to think of an undirected edge as a two-way street and of a directed edge λ as a one-way street along which we may only travel from λt to λh. We place no restriction (except finiteness) on the number of undirected or directed edges which may join the same pair of vertices. We call G an *undirected graph* if $\Delta(G) = \emptyset$ and a *digraph* if $U(G) = \emptyset$. Both of these are regarded as special types of mixed graphs, and thus the term "mixed" allows but does not require the presence of both undirected and directed edges.

A *subgraph* of a mixed graph G is a mixed graph H such that $V(H) \subseteq V(G)$, $U(H) \subseteq U(G)$, $\Delta(H) \subseteq \Delta(G)$, each edge of H joins the same vertices in H as in G and each directed edge of H has the same tail and the same head in H as in G. Two subgraphs H, K of G are *edge-disjoint* if $E(H) \cap E(K) = \emptyset$.

Let G be a mixed graph and X, Y denote subsets of $V(G)$ and L denote a subset of $E(G)$. Then $\overline{X}$ denotes $V(G) \setminus X$, $X \square Y$ denotes the set of those undirected edges of G which join an element of X to an element of Y, $X \triangleright Y$ denotes $\{\lambda \in \Delta(G)\colon \lambda t \in X, \lambda h \in Y\}$ and $X \triangledown Y$ denotes $(X \square Y) \cup (X \triangleright Y) \cup (Y \triangleright X)$. We let $E(X) = X \triangledown V(G)$, $U(X) = X \square V(G)$, $e(X) = |E(X)|$, $u(X) = |U(X)|$. Elements of $(X \triangleright \overline{X}) \cup (X \square \overline{X})$ are *exits* of X and elements of $(\overline{X} \triangleright X) \cup (X \square \overline{X})$ are *entries* of X. *Exits* and *entries* of a subgraph H of G are exits and entries of $V(H)$, respectively. We define $G[X]$, $G - L$ to be the subgraphs of G such that $V(G[X]) = X$, $E(G[X]) = X \triangledown X$, $V(G - L) = V(G)$, $E(G - L) = E(G) \setminus L$; and $G - X$ means $G[\overline{X}]$. Furthermore, $G\langle L \rangle$ denotes the smallest subgraph of G whose set of edges is L, i.e. $E(G\langle L \rangle) = L$ and $V(G\langle L \rangle)$ is the set of those vertices of G which are incident with at least one element of L. If $\xi, \eta \in V(G)$ and $\lambda \in E(G)$, the expressions $G - \{\lambda\}$, $U(\{\xi\})$, $\{\xi\} \square \{\eta\}$, $\{\xi\} \triangleright \{\eta\}$, $\{\xi\} \triangledown \{\eta\}$, $\{\xi\} \square X$, $X \triangleright \{\xi\}$ may be written without braces as $G - \lambda$, $U(\xi)$, $\xi \square \eta$ etc. For every subset X of $V(G)$, $G[X]$ is an *induced* subgraph of G. We say that G is *connected* if $X \triangledown \overline{X}$ is non-empty for every non-empty proper subset X of $V(G)$. We say that G is *strongly connected* or, for brevity, *strong* if every non-empty proper subset of $V(G)$ has at least one exit or, equivalently, if every non-empty proper subset of $V(G)$ has at least one exit and at least one entry. If G is non-empty (i.e. if $V(G) \neq \emptyset$), its maximal connected subgraphs and maximal strong subgraphs will be called *components* and *dicomponents* of G, respectively. Clearly these are all induced subgraphs of G. (We adopt the term "dicomponent" in preference to "strong component" because a dicomponent is not necessarily a component.) The empty graph is considered as having no components or dicomponents. An *initial* dicomponent of G is one which.has no entries.

We let $C(G)$ denote the set of components of G and write $c(G) = |C(G)|$. We note that

$$c(G-L) \leq c(G) + |L| \text{ if } L \subseteq E(G), \tag{1}$$

because removing an edge increases the number of components of a mixed graph by at most 1. Taking $L = E(G)$ in (1) gives

$$|V(G)| \leq |E(G)| + c(G). \tag{2}$$

If $\xi, \eta \in V(G)$, a *$\xi\eta$-dipath* in G is a connected subgraph P of G such that $\xi, \eta \in V(P)$ and each edge λ of P is an exit of a component of $P - \lambda$ which includes ξ and does not include η, i.e. a path P from ξ to η in the usual graph-theoretic sense, such that each edge of P is either undirected or directed and "pointing in the direction from ξ to η along P". A subgraph of G is an *$X\eta$-dipath* if it is a $\xi\eta$-dipath for some $\xi \in X$, and is an *XY-dipath* if it is a $\xi\eta$-dipath for some $\xi \in X$ and some $\eta \in Y$. We say that η is *accessible* from ξ in G if there exists a $\xi\eta$-dipath in G or, equivalently, if every set X such that $\xi \in X \subseteq V(G) \setminus \{\eta\}$ has at least one exit.

When two or more mixed graphs G, H etc. are under consideration, we shall use subscripts, hyphens or additional words when necessary to indicate the mixed graph in which words or symbols are interpreted, e.g. λt_H, $X \square_H Y$ "H-exit", "accessible in H" etc. However, when one of the mixed graphs under consideration is denoted by the symbol G, all words and symbols which require interpretation in some mixed graph will refer to G unless the contrary is explicitly indicated. For example, in these circumstances $\overline{X}$, $X \square Y$, λt and "exit of X" will mean $V(G) \setminus X$, $X \square_G Y$, λt_G and "exit of X in G" respectively, unless otherwise specified.

3 Strong Orientations of Mixed Graphs

We shall say that a mixed graph H is an *orientation* of a mixed graph G if $V(H) = V(G)$, $E(H) = E(G)$, $U(H) \subseteq U(G)$, each edge of G joins the same vertices in H as in G and each directed edge of G has the same tail and the same head in H as in G. In these circumstances, we say that H is obtained from G by *orienting* the edges in $U(G) \cap \Delta(H)$ and that G is obtained from H by *de-orienting* these edges. Thus an orientation of a mixed graph G is another mixed graph obtained from G by turning some (or all or none) of its undirected edges into directed edges. If $b: V(G) \rightarrow \underset{\sim}{Z}_+$ is a function and H is an orientation of G and each vertex ξ of G is the head, in H, of exactly $b(\xi)$ elements of $U(G) \cap \Delta(H)$ then we shall call H a *b-orientation* of G: thus each vertex ξ becomes the head of $b(\xi)$ *newly* directed edges when G is transformed into a b-orientation of G.

Lemma 1. *If G is a mixed graph and b is a function from $V(G)$ into $\underset{\sim}{Z}_+$ and $u(X) \geq b.X$ for every subset X of $V(G)$ then G has a b-orientation.*

Proof: Let $\xi_1, \xi_2, \ldots, \xi_s$ (where $s = b.V(G)$) be a sequence of vertices of G such that each vertex ξ of G appears $b(\xi)$ times in the sequence. If $I \subseteq \{1, 2, \ldots, s\}$ and $X = \{\xi_i : i \in I\}$ then $|\bigcup_{i \in I} U(\xi_i)| = u(X) \geq b.X \geq |I|$. Therefore the sets $U(\xi_1), \ldots, U(\xi_s)$ have distinct representatives $\lambda_1, \ldots, \lambda_s$ respectively and so G has a b-orientation H such that $U(G) \cap \Delta(H) = \{\lambda_1, \ldots, \lambda_s\}$ and $\lambda_i h_H = \xi_i$ for $i = 1, \ldots, s$.

Lemma 2. *Every non-empty mixed graph has at least one initial dicomponent.*

Proof: Let G be a non-empty mixed graph. Choose a vertex α of G which is accessible from as few vertices as possible. Then the set X of all vertices from which α is accessible has no entry. Moreover, if $\xi \in X$, then ξ is accessible from α because otherwise the set of all vertices from which ξ is accessible would be a subset of $X \setminus \{\alpha\}$, contradicting the definition of α. Hence $G[X]$ is an initial dicomponent of G.

Definition. If G is a mixed graph and $X \subseteq V(G)$, then $\mathcal{S}(G, X)$ denotes the set of all induced subgraphs S of $G - X$ such that

$$\overline{X \cup V(S)} \,\square\, V(S) = \overline{V(S)} \triangleright V(S) = \emptyset$$

(i.e. all entries of S belong to $X \,\square\, V(S)$).

Lemma 3. *Let G be a mixed graph, X be a subset of $V(G)$ and C be a subgraph of G. Then the following statements are equivalent:*

(S1) C is non-empty and strong and $C \in \mathcal{S}(G, X)$.

(S2) C is an initial dicomponent of $G - X$ and $X \triangleright V(C) = \emptyset$.

Proof: If C satisfies (S2) then it is a non-empty strong induced subgraph of $G - X$. Moreover $X \triangleright V(C) = \emptyset$ by (S2) and

$$\overline{X \cup V(C)} \triangleright V(C) = \overline{X \cup V(C)} \,\square\, V(C) = \emptyset$$

since C is an initial dicomponent of $G - X$ and so $C \in \mathcal{S}(G, X)$. Therefore (S1) is true.

Conversely, suppose that C satisfies (S1). Since $C \in \mathcal{S}(G, X)$, it has no entries in $G - X$ and is therefore an initial dicomponent of $G - X$. Moreover $X \triangleright V(C) \subseteq \overline{V(C)} \triangleright V(C) = \emptyset$ since $C \in \mathcal{S}(G, X)$, and so (S2) is true.

Definition. If G is a mixed graph and $X \subseteq V(G)$, then $\mathcal{C}(G, X)$ denotes the set of all subgraphs C of G which satisfy (S1), or equivalently (by Lemma 3) the set of all subgraphs C of G which satisfy (S2). The reader should keep in mind both versions of this definition. We write $c(G, X) = |\mathcal{C}(G, X)|$ and $f(G, b, X) = u(X) - b.X - c(G, X)$ for any function $b \colon V(G) \to \mathbb{Z}_+$.

Lemma 4. *If G is a mixed graph and $X \subseteq V(G)$ then every non-empty member of $\mathcal{S}(G, X)$ contains a member of $\mathcal{C}(G, X)$.*

Proof: This follows from Lemma 2 and the fact that every initial dicomponent of a member of $\mathcal{S}(G, X)$ satisfies (S1) and so belongs to $\mathcal{C}(G, X)$.

Theorem 1. *Let G be a mixed graph and b be a function from $V(G)$ into $\mathbb{Z}_+$. Then G has a strong b-orientation iff G is strong and $u(X) \geq b.X + c(G, X)$ for every non-empty subset X of $V(G)$.*

Proof: Suppose first that G has a strong b-orientation H. Then it is easily seen that G is itself strong. Let X be a non-empty subset of $V(G)$. Let $C \in \mathcal{C}(G, X)$. Since H is strong, $V(C)$ has an entry λ_C in H. Since λ_C must be an entry of C in G and $C \in \mathcal{S}(G, X)$, it follows that $\lambda_C \in U(X)$, but clearly $\lambda_C \notin V(G) \triangleright_H X$. Hence $U(X)$ has at least $c(G, X)$ elements which do not belong to $V(G) \triangleright_H X$. Moreover $b.X$ elements of $U(X)$ belong to $V(G) \triangleright_H X$ since H is a b-orientation of G, and so $u(X) \geq b.X + c(G, X)$.

To prove that, conversely, if this inequality holds for every non-empty $X \subseteq V(G)$ and G is strong then G has a strong b-orientation, we make the following definitions. A *couple* is an ordered pair (G, b) such that G is a mixed graph and b is a function from $V(G)$ into $\mathbb{Z}_+$. A couple (G, b) is *good* if $u(X) \geq b.X + c(G, X)$ for every non-empty subset X of $V(G)$. A couple (G, b) is *perverse* if it is good and G is strong but has no strong b-orientation. A couple (F, a) *precedes* a couple (G, b) if either (i) $|V(F)| + |E(F)| < |V(G)| + |E(G)|$ or (ii) $|V(F)| + |E(F)| = |V(G)| + |E(G)|$ and $a.V(F) > b.V(G)$. A couple (G, b) is *minimally perverse* if it is perverse and no perverse couple precedes (G, b). If there exists a perverse couple then clearly there exists a minimally perverse couple and so it suffices to prove that the latter cannot exist. Therefore, during the remainder of the proof of Theorem 1 (i.e. the remainder of Section 3), we assume that (G, b) is a minimally perverse couple. When this assumption has led to a contradiction, Theorem 1 will be proved.

Lemma 5. *G is non-empty.*

Proof: If G was empty, then G would trivially be a b-orientation of itself, contradicting the hypothesis that (G, b) is perverse.

Lemma 6. *If $\alpha \in V(G)$ and $\lambda \in \alpha \nabla \alpha$ then $\lambda \in U(G)$ and $b(\alpha) = 0$.*

Proof: Suppose that $\lambda \in \Delta(G)$ or $b(\alpha) > 0$. Let $a \colon V(G) \to \mathbb{Z}_+$ be the function such that $a(\xi) = b(\xi)$ for every $\xi \in V(G) \setminus \{\alpha\}$, $a(\alpha) = b(\alpha)$ if λ is directed and $a(\alpha) = b(\alpha) - 1$ if λ is undirected. It is easily seen that $f(G - \lambda, a, X) = f(G, b, X)$ for every $X \subseteq V(G)$, and so $(G - \lambda, a)$ is good since (G, b) is good. Moreover $G - \lambda$ is strong since G is strong, and

$(G - \lambda, a)$ is not perverse since it precedes (G, b). Therefore $G - \lambda$ has a strong a-orientation, which becomes a strong b-orientation of G when λ is replaced as a directed loop; and this contradicts the perversity of (G, b).

We shall say that a subset X of $V(G)$ is *critical* if $f(G, b, X) = 0$.

Lemma 7. *Every vertex of G belongs to a critical subset of $V(G)$.*

Proof: Suppose that a vertex α of G belongs to no critical subset of $V(G)$. Then, since (G, b) is good, $f(G, b, X) \geq 0$ for every non-empty $X \subseteq V(G)$ and $f(G, b, X) \geq 1$ whenever $\alpha \in X \subseteq V(G)$. Therefore $f(G, a, X) \geq 0$ for every non-empty $X \subseteq V(G)$, where $a(\xi) = b(\xi)$ for every $\xi \in V(G) \setminus \{\alpha\}$ and $a(\alpha) = b(\alpha) + 1$. Therefore (G, a) is good. However, (G, a) cannot be perverse since it precedes (G, b). Therefore G has a strong a-orientation H. In H, α is the head of $a(\alpha) = b(\alpha) + 1$ elements of $U(G) \cap \Delta(H)$, and de-orienting one of these edges converts H into a strong b-orientation of G, contradicting the perversity of (G, b).

Lemma 8. *If $\omega \in V(G)$ and $V(G) \triangleright \omega = \emptyset$ and $b(\omega) = 0$ then $\{\omega\}$ is a maximal critical subset of $V(G)$.*

Proof: By Lemma 7, ω belongs to some maximal critical subset X of $V(G)$. Let

$$\mathcal{C}' = \{C \in \mathcal{C}(G, X) : \omega \,\Box\, V(C) \neq \emptyset\},$$
$$P = \bigcup\{V(C) : C \in \mathcal{C}'\}, Y = X \setminus \{\omega\}.$$

Clearly $\mathcal{C}(G, X) \setminus \mathcal{C}' \subseteq \mathcal{C}(G, Y)$ and $|\mathcal{C}'| \leq |\omega \,\Box\, P|$. Therefore

$$c(G, X) \leq c(G, Y) + |\omega \,\Box\, P| - |\mathcal{C}(G, Y) \setminus \mathcal{C}(G, X)|. \tag{3}$$

Since $\overline{X \cup V(C)} \,\Box\, V(C) = \overline{V(C)} \triangleright V(C) = \emptyset$ for every $C \in \mathcal{C}'$ and $V(G) \triangleright \omega = \emptyset$, it follows that $G[\{\omega\} \cup P] \in \mathcal{S}(G, Y)$ if $\omega \,\Box\, \overline{X \cup P} = \emptyset$. Moreover $G[\{\omega\} \cup P]$ is strong since each member of $\mathcal{C}'$ is strong and $\omega \,\Box\, V(C) \neq \emptyset$ for each $C \in \mathcal{C}'$. Therefore $G[\{\omega\} \cup P] \in \mathcal{C}(G, Y) \setminus \mathcal{C}(G, X)$ if $\omega \,\Box\, \overline{X \cup P} = \emptyset$. Moreover $|\omega \,\Box\, P| \leq |\omega \,\Box\, \overline{X}|$ and if $\omega \,\Box\, \overline{X \cup P} \neq \emptyset$ then $|\omega \,\Box\, P| < |\omega \,\Box\, \overline{X}|$. From these remarks and (3) it follows that $c(G, X) < c(G, Y) + |\omega \,\Box\, \overline{X}|$. Moreover $b.X = b.Y$ since $b(\omega) = 0$; and $u(X) \geq u(Y) + |\omega \nabla \overline{X}|$. Therefore $f(G, b, X) > f(G, b, Y)$, which implies that $Y = \emptyset$ because (G, b) is good and X is critical. Therefore $\{\omega\} = X$, which is a maximal critical subset of $V(G)$.

Lemma 9. *If X is a non-empty subset of $V(G)$ and $C \in \mathcal{C}(G, X)$ then $X \,\Box\, V(C) \neq \emptyset$.*

Proof: Since G is strong, $V(C)$ has at least one entry in G. Since $C \in \mathcal{S}(G, X)$, any such entry must belong to $X \,\Box\, V(C)$.

Definitions. If X is a non-empty subset of $V(G)$, then G_X will denote a mixed graph obtained from G by contracting its subgraph $G[X]$ to a vertex $\hat{X}$. More precisely, let $\hat{X}$ be an object which is not a vertex or edge of G and let ϕ: $V(G) \to \overline{X} \cup \{\hat{X}\}$ be the function such that $\phi(\xi) = \xi$ for every $\xi \in \overline{X}$ and $\phi(\xi) = \hat{X}$ for every $\xi \in X$. Then $V(G_X) = \overline{X} \cup \{\hat{X}\}$, $E(G_X) = E(\overline{X})$ and for all $\xi \in \overline{X}$, $\eta \in V(G)$ we have

$$\xi \Box \eta \subseteq \xi \Box_{G_X} \phi(\eta), \qquad \xi \triangleright \eta \subseteq \xi \triangleright_{G_X} \phi(\eta),$$
$$\eta \triangleright \xi \subseteq \phi(\eta) \triangleright_{G_X} \xi.$$

We also define a function $b_X: V(G_X) \to Z_+$ by letting $b_X(\xi) = b(\xi)$ for every $\xi \in \overline{X}$ and $b_X(\hat{X}) = 0$.

Lemma 10. *If X is a non-empty subset of $V(G)$ then G_X is strong and (G_X, b_X) is good.*

Proof: Let F denote G_X and ϕ: $V(G) \to V(F)$ be the function defined above. If Z is a non-empty proper subset of $V(F)$ then, since G is strong, $\phi^{-1}(Z)$ has at least one exit in G and therefore Z has at least one exit in F. Hence F is strong.

Now let $\hat{X} \in Y \subseteq V(F)$, $Y' = Y \setminus \{\hat{X}\}$ and let $\mathcal{C}'(F, Y')$ be the set of those members of $\mathcal{C}(F, Y')$ which do not include $\hat{X}$. If $C \in \mathcal{C}(F, Y)$ then either $C \in \mathcal{C}'(F, Y')$ or $\hat{X} \Box_F V(C) \neq \emptyset$: therefore

$$c(F, Y) \leq |\mathcal{C}'(F, Y')| + |\hat{X} \Box_F (V(F) \setminus Y)|. \tag{4}$$

Moreover, it is easily seen that $\mathcal{C}'(F, Y') \subseteq \mathcal{C}(G, Y')$ and that if $\hat{X}$ is a vertex of some $C \in \mathcal{C}(F, Y')$ then $G[\phi^{-1}(V(C))]$ belongs to $\mathcal{S}(G, Y')$ and so contains by Lemma 4 a member of $\mathcal{C}(G, Y')$. Therefore $c(G, Y') \geq c(F, Y')$ and consequently

$$f(F, b_X, Y') \geq f(G, b, Y'). \tag{5}$$

To prove that (F, b_X) is good it suffices to show that

(i) $f(F, b_X, \{\hat{X}\}) \geq 0$,

(ii) if $\hat{X} \in Y \subseteq V(F)$ and $Y \neq \{\hat{X}\}$ then $f(F, b_X, Y) \geq f(F, b_X, Y') \geq 0$.

Taking $Y = \{\hat{X}\}$ in (4) establishes (i) if we observe that $b_X(\hat{X}) = 0$, $u_F(\{\hat{X}\}) = |\hat{X} \Box_F (V(F) \setminus \{\hat{X}\})|$ and that $\mathcal{C}(F, \emptyset) = \{F\}$ since F is strong and consequently $\mathcal{C}'(F, \emptyset) = \emptyset$. To prove (ii), suppose that $\hat{X} \in Y \subseteq V(F)$ and $Y \neq \{\hat{X}\}$. Then $b_X.Y = b_X.Y'$ (since $b_X(\hat{X}) = 0$) and $|\mathcal{C}'(F, Y')| \leq c(F, Y')$ and $u_F(Y) = u_F(Y') + |\hat{X} \Box_F (V(F) \setminus Y)|$. From these observations and (4), it follows that $f(F, b_X, Y) \geq f(F, b_X, Y')$, which is non-negative by (5) and the goodness of (G, b).

Lemma 11. *If X is a maximal critical subset of $V(G)$ and $C \in \mathcal{C}(G,X)$ then $V(C) = \{\omega\}$ for some vertex ω such that $b(\omega) = 0$ and $V(G) \rhd \omega = \emptyset$.*

Proof: Since $f(G,b,\emptyset) < 0$ by Lemmas 2 and 5, X is non-empty. Let $W = V(C)$ and let a denote the restriction of b to W. Suppose that $\emptyset \neq Z \subseteq W$. Then

$$b.(X \cup Z) = b.X + a.Z. \tag{6}$$

Since $C \in \mathcal{S}(G,X)$ it follows that

$$u(X \cup Z) = u(X) + u_C(Z). \tag{7}$$

From the version of the definition of $\mathcal{C}(G,X)$ involving (S1), we see that $(\mathcal{C}(G,X) \setminus \{C\}) \cup \mathcal{C}(C,Z) \subseteq \mathcal{C}(G,X \cup Z)$ and therefore

$$c(G,X \cup Z) \geq c(G,X) + c(C,Z) - 1. \tag{8}$$

By (6), (7) and (8),

$$f(G,b,X \cup Z) \leq f(G,b,X) + f(C,a,Z) + 1.$$

But $f(G,b,X \cup Z) \geq 1$ and $f(G,b,X) = 0$ because (G,b) is good, X is critical and (by the maximality of X) $X \cup Z$ is not critical. Therefore $f(C,a,Z) \geq 0$. This proves that (C,a) is good. Moreover (C,a) precedes (G,b) since $X \neq \emptyset$. Therefore (C,a) is not perverse and so C has a strong a-orientation C'.

Suppose that $|W| \geq 2$. Then (G_W, b_W) also precedes (G,b) and so is not perverse. Therefore, by Lemma 10, G_W has a strong b_w-orientation J. Clearly G has a b-orientation H such that C' is a subgraph of H and $H_W = J$. To prove that H is strong, we must show that an arbitrary non-empty proper subset Y of $V(G)$ has at least one exit in H. If Y and $\overline{Y}$ both meet $V(C)$ then $Y \cap V(C)$ is a non-empty proper subset of $V(C)$ and so has an exit in C' since C' is strong, and therefore Y has an exit in H. If Y, $\overline{Y}$ do not both meet $V(C)$ then one of them is a non-empty proper subset of $V(G_W)$ and therefore has both an entry and an exit in J since J is strong, and therefore once again Y has an exit in H. Hence H is a strong b-orientation of G, contradicting the perversity of (G,b). We infer that $|W| < 2$ and so $V(C) = W = \{\omega\}$ for some vertex ω.

Clearly $\mathcal{C}(G,X)\setminus\{C\} \subseteq \mathcal{C}(G,X\cup\{\omega\})$ and so $c(G,X\cup\{\omega\}) \geq c(G,X)-1$. Since $V(C) = \{\omega\}$ and $C \in \mathcal{S}(G,X)$, Lemma 6 gives $u(X \cup \{\omega\}) = u(X)$ if $b(\omega) > 0$. Moreover $b.(X\cup\{\omega\}) = b.X + b(\omega)$. Therefore $f(G,b,X\cup\{\omega\}) \leq f(G,b,X) + 1 - b(\omega)$ if $b(\omega) > 0$. Hence $b(\omega) = 0$ since (G,b) is good and X is critical and (by the maximality of X) $X \cup \{\omega\}$ is not critical. Finally, $V(G) \rhd \omega = \emptyset$ since $V(C) = \{\omega\}$ and $C \in \mathcal{S}(G,X)$ and $\omega \rhd \omega = \emptyset$ by Lemma 6. Lemma 11 is thus proved.

Since (G,b) is good, $u(X) - b.X \geq f(G,b,X) \geq 0$ when $\emptyset \neq X \subseteq V(G)$, and obviously $u(\emptyset) = b.\emptyset = 0$. Therefore

$$u(X) \geq b.X \text{ for every } X \subseteq V(G). \tag{9}$$

We shall call a subset X of $V(G)$ *crucial* if $u(X) = b.X$.

Lemma 12. *Every maximal critical subset of $V(G)$ is crucial.*

Proof: Suppose that a maximal critical subset X of $V(G)$ is not crucial. Since X is critical but not crucial, $b.X + c(G,X) = u(X) \neq b.X$ and so we can choose some $C \in \mathcal{C}(G,X)$. By Lemma 11, $V(C) = \{\omega\}$ for some vertex ω such that $b(\omega) = 0$ and $V(G) \triangleright \omega = \emptyset$. By Lemma 8, $\{\omega\}$ is a maximal critical subset of $V(G)$. Since X is not crucial and therefore non-empty, there exists by Lemma 9 an edge $\lambda \in X \square V(C) = \omega \square X$. Moreover $\omega \square V(D) \neq \emptyset$ for every $D \in \mathcal{C}(G,\{\omega\})$ by Lemma 9, and $|\omega \square V(G)| = c(G,\{\omega\})$ since $\{\omega\}$ is critical and $b(\omega) = 0$. Therefore every element of $\omega \square V(G)$ joins ω to a vertex of a member of $\mathcal{C}(G,\{\omega\})$ and so λ joins ω to a vertex ω_0 of some $D_0 \in \mathcal{C}(G,\{\omega\})$. We observe that $\omega_0 \in X$ since $\lambda \in \omega \square X$. Since $\{\omega\}$ is a maximal critical subset of $V(G)$, it follows by Lemma 11 that $V(D_0) = \{\omega_0\}$ and $b(\omega_0) = 0$ and $V(G) \triangleright \omega_0 = \emptyset$. Therefore $\{\omega_0\}$ is a maximal critical subset of $V(G)$ by Lemma 8, and so $X = \{\omega_0\}$ because X is critical and $\omega_0 \in X$. Since $C \in \mathcal{S}(G,X)$ and $D_0 \in \mathcal{S}(G,\{\omega\})$ it follows that

$$V(C) \square \overline{X \cup V(C)} = V(D_0) \square \overline{\{\omega\} \cup V(D_0)} = \emptyset,$$

i.e. $\{\omega,\omega_0\} \square (V(G) \setminus \{\omega,\omega_0\}) = \emptyset$. Since in addition $V(G) \triangleright \omega = V(G) \triangleright \omega_0 = \emptyset$, it follows that $\{\omega,\omega_0\}$ has no entries in G and so $\{\omega,\omega_0\} = V(G)$ because G is strong. Since $b(\omega) = b(\omega_0) = 0$ it follows that G is a strong b-orientation of itself, which contradicts the perversity of (G,b) and thus proves Lemma 12.

Let Y, Z be crucial subsets of $V(G)$. Then $u(Y) = b.Y$, $u(Z) = b.Z$ and clearly $U(Y \cap Z) \subseteq U(Y) \cap U(Z)$ and $U(Y \cup Z) = U(Y) \cup U(Z)$: these facts and (9) give

$$\begin{aligned} b.Y + b.Z &= u(Y) + u(Z) \geq u(Y \cap Z) + u(Y \cup Z) \\ &\geq b.(Y \cap Z) + b.(Y \cup Z) = b.Y + b.Z. \end{aligned}$$

From this and (9) it follows that $u(Y \cap Z) = b.(Y \cap Z)$ and $u(Y \cup Z) = b.(Y \cup Z)$. Therefore the union (and intersection) of any two crucial sets is crucial and hence by induction the union of any finite number of crucial sets is crucial. Since $V(G)$ is by Lemmas 7 and 12 the union of finitely many crucial sets, it is crucial, i.e. $|U(G)| = b.V(G)$. By (9) and Lemma 1, G has a b-orientation H, which is a digraph since $|U(G)| = b.V(G)$. To prove that H is strong, consider a non-empty proper subset X of $V(G)$.

By Lemma 2, $G - X$ has an initial dicomponent C. If $C \notin \mathcal{C}(G, X)$ then $\emptyset \neq X \triangleright V(C) \subseteq X \triangleright_H V(C)$ and so X has an H-exit. If $C \in \mathcal{C}(G, X)$ then $u(X) > b.X$ since (G, b) is good and so, since H is a digraph and the H-heads of exactly $b.X$ elements of $U(X)$ belong to X, X once again has an H-exit. Hence every non-empty proper subset of $V(G)$ has an H-exit, and so H is a strong b-orientation of G. This contradicts the perversity of (G, b) and thus completes the proof of Theorem 1.

4 Connected Detachments of Graphs

Definitions. Let G be an undirected graph and $b\colon V(G) \to \mathbf{N}$ be a function. If D is an undirected graph and $E(D) = E(G)$, a *b-coalescence* of D onto G is a function p from $V(D)$ onto $V(G)$ such that $|p^{-1}(\{\xi\})| = b(\xi)$ for every $\xi \in V(G)$ and $p^{-1}(\{\xi\}) \nabla_D p^{-1}(\{\eta\}) = \xi \nabla \eta$ for all $\xi, \eta \in V(G)$. An undirected graph D is a *b-detachment* of G if $E(D) = E(G)$ and there exists a b-coalescence of D onto G. Informally, this means that D is obtained from G by splitting each vertex ξ into $b(\xi)$ vertices, namely the elements of $p^{-1}(\{\xi\})$, where p is a b-coalescence of D onto G.

In [4] and [5], matroids were used to establish a necessary and sufficient condition for G to have a connected b-detachment. We shall now deduce this result, without using matroids, from the following corollary (equivalent to Theorem 7 of [1]) of Theorem 1.

Corollary 1a. *Let G be an undirected graph, $\alpha \in V(G)$ and b be a function from $V(G)$ into $\mathbb{Z}_+$. Then G has a b-orientation in which every vertex is accessible from α iff*

$$e(X) \geq b.X + c(G - X) - |\{\alpha\} \cap \overline{X}| \tag{10}$$

for every subset X of $V(G)$.

Proof: Let M be a mixed graph obtained from G by adding, for each $\xi \in V(G) \setminus \{\alpha\}$, a directed edge with tail ξ and head α. Then M is strong iff G is connected, i.e. iff (10) holds when $X = \emptyset$. Moreover, if $\emptyset \neq X \subseteq V(G)$ then $u_M(X) = e(X)$ and $c(M, X) = c(G - X) - |\{\alpha\} \cap \overline{X}|$ because $\mathcal{C}(M, X)$ is the set of those components of $G - X$ which do not include α. Therefore $u_M(X) \geq b.X + c(M, X)$ for every non-empty subset X of $V(M) = V(G)$ iff (10) holds for every such X. Hence, by Theorem 1, M has a strong b-orientation iff (10) holds for every $X \subseteq V(G)$, and clearly G has a b-orientation in which every vertex is accessible from α iff M has a strong b-orientation.

Sketch of Alternative Proof (not requiring Theorem 1). If G has a b-orientation H in which every vertex is accessible from α and $X \subseteq V(G)$ then, for every component C of $G - X$ which does not include α, $E(X) \setminus (V(G) \triangleright_H X)$ must include an H-entry of C, and so (10) holds. Now assume (10) for every $X \subseteq V(G)$. Then, by Lemma 1, G has a b-orientation H, which we may choose so as to maximize the cardinality of the set A of vertices accessible from α in H. Suppose that $A \neq V(G)$. Let $L = A \Box_H A$. Let ϕ_1^λ, ϕ_2^λ be the vertices joined by an edge $\lambda \in L$. If $H - \lambda$ contains an $\alpha\phi_i^\lambda$-dipath P and an $\overline{A}\phi_{3-i}^\lambda$-dipath Q with no common edge then G has a b-orientation H', agreeing with H on all edges outside $\{\lambda\} \cup E(Q)$, such that all vertices in $V(P) \cup V(Q)$ are accessible from α in $H'\langle E(P) \cup E(Q) \cup \{\lambda\}\rangle$ and consequently all vertices in $A \cup V(Q)$ are accessible from α in H'. This contradicts the choice of H. Therefore P, Q cannot exist and so there do not exist two edge-disjoint $\rho\tau$-dipaths in the mixed graph J obtained from $H - \lambda$ by adding new vertices ρ, σ, τ and new directed edges $\nu(\rho, \alpha)$, $\nu(\rho, \sigma)$, $\nu(\phi_1^\lambda, \tau)$, $\nu(\phi_2^\lambda, \tau)$ and $\nu(\sigma, \xi)$ for every $\xi \in \overline{A}$, where each $\nu(\eta, \zeta)$ has tail η and head ζ. Therefore, by the appropriate variant of Menger's Theorem, $V(J)$ has a subset Z'_λ such that $\rho \in Z'_\lambda$, $\tau \notin Z'_\lambda$ and Z'_λ has at most one exit in J. If $Z_\lambda = Z'_\lambda \cap V(G)$ then, bearing in mind that ϕ_1^λ, ϕ_2^λ are accessible from α in H, it follows that $\overline{A} \subseteq Z_\lambda$, at least one of ϕ_1^λ, ϕ_2^λ belongs to $\overline{Z}_\lambda$ and either (i) $\alpha \in \overline{Z}_\lambda$ and Z_λ has no exit in H or (ii) $\alpha \in Z_\lambda$ and Z_λ has exactly one exit π_λ in H. Then the set of exits in H of $Z = \bigcap_{\lambda \in L} Z_\lambda$ is $\{\pi_\lambda : \lambda \in L'\}$ for some $L' \subseteq L$ such that $\pi_\lambda \neq \pi_\mu$, when $\lambda, \mu \in L'$ and $\lambda \neq \mu$. [If $L = \emptyset$, take $Z = V(G)$, $L' = \emptyset$.] For $\lambda \in L'$ let π_λ join $\gamma_\lambda \in Z$ to $\delta_\lambda \in \overline{Z}$ and let A_λ be the set of vertices accessible in $H - Z$ from δ_λ. Let A_α be the set of vertices accessible in $H - Z$ from α (so that $A_\alpha = \emptyset$ if $\alpha \in Z$). Then $\overline{Z} = A_\alpha \cup \bigcup_{\lambda \in L'} A_\lambda$ since $\overline{Z} \subseteq A$. If $\psi \in \overline{Z}$ and H contains an $(\{\alpha\} \cup Z)\psi$-dipath R and a $Z\psi$-dipath S then $\psi \in \overline{Z}_\lambda$ and S is a $Z_\lambda \overline{Z}_\lambda$-dipath for some $\lambda \in L$, so that (i) is false for this λ and (ii) gives $E(R) \cap E(S) \neq \emptyset$. This implies that the sets A_θ $(\theta \in L' \cup \{\alpha\})$ must be disjoint, and their disjointness together with their definitions implies that no edge can join vertices in distinct sets A_θ $(\theta \in L' \cup \{\alpha\})$. If $\lambda \in L'$ then (i) is false for λ and so, since $\delta_\lambda \in \overline{Z} \subseteq A$, (ii) implies that $\gamma_\lambda \in A$. Hence no H-exit of Z is in $\overline{A} \nabla \overline{Z}$, and consequently $\overline{A} \nabla \overline{Z} = \emptyset$ because $\overline{Z} \subseteq A$ and A by its definition can have no H-exit. Therefore $\overline{A \cap Z}$ is the union of disjoint non-empty sets $\overline{A}$, A_λ $(\lambda \in L')$ and (if $\alpha \in \overline{Z}$) A_α with no edge joining vertices in distinct sets in this list, and so

$$c(G - (A \cap Z)) \geq |L'| + |\{\alpha\} \cap \overline{A \cap Z}| + 1. \tag{11}$$

Since A has no H-exits, an edge $\lambda \in E(A \cap Z)$ must be a directed edge of the b-orientation H of G whose head is in $A \cap Z$ or an H-exit of Z or an element of $A \Box_H A = L$ and in this last case λ is again an H-exit of Z since ϕ_1^λ or ϕ_2^λ belongs to $\overline{Z}_\lambda$. Since Z has $|L'|$ H-exits, it follows that $e(A \cap Z) \leq b.(A \cap Z) + |L'|$, contradicting (10) and (11).

Definitions. A connected mixed graph T is an *α-arborescence* (where $\alpha \in V(T)$) if every edge λ of T is an exit of the component of $T - \lambda$ which includes α. Informally, this means that T is a tree and every edge of T is either undirected or directed and "pointing in the direction away from α in T". An *α-subarborescence* of a mixed graph G (where $\alpha \in V(G)$) is a subgraph of G which is an α-arborescence.

Theorem 2. *Let G be an undirected graph and b be a function from $V(G)$ into N. Then G has a connected b-detachment iff*

$$e(X) \geq b.X + c(G - X) - 1 \tag{12}$$

for every subset X of $V(G)$.

Proof: Suppose that G has a connected b-detachment D. Then there exists a b-coalescence p of D onto G. If $X \subseteq V(G)$, then $|p^{-1}(X)| = b.X$ since $|p^{-1}(\{\xi\})| = b(\xi)$ for each $\xi \in X$. Therefore $V(D)$ can be partitioned into $b.X + c(G-X)$ non-empty sets $\{\theta\}$ $(\theta \in p^{-1}(X))$, $p^{-1}(V(C))$ $(C \in \mathcal{C}(G - X))$ such that all edges of D joining two vertices belonging to distinct sets in this partition belong to $E(X)$. Therefore $c(D - E(X)) \geq b.X + c(G - X)$ and so, by (1), $b.X + c(G - X) - e(X) \leq c(D) \leq 1$.

Now assume (12) for every $X \subseteq V(G)$. Unless $V(G) = \emptyset$ (when the required b-detachment exists trivially), we can choose a vertex $\alpha \in V(G)$. By (12) and Corollary 1a, G has a b_α-orientation J in which every vertex is accessible from α, where $b_\alpha(\xi) = b(\xi)$ for every $\xi \in V(G) \setminus \{\alpha\}$ and $b_\alpha(\alpha) = b(\alpha) - 1$. Let T be a maximal α-rooted subarborescence of J. If $V(T)$ was not $V(G)$ then, since every vertex is accessible from α in J, $V(T)$ would have a J-exit λ and $T \cup J\langle\{\lambda\}\rangle$ would be a larger α-rooted subarborescence of J than T. Therefore $V(T) = V(G)$. Since each $\xi \in V(G)$ is the head of $b_\alpha(\xi)$ directed edges of J, of which at most one is in T if $\xi \neq \alpha$ and none is in T if $\xi = \alpha$, $\Delta(J) \setminus E(T)$ has a subset S such that each $\xi \in V(G)$ is the J-head of exactly $b(\xi) - 1$ edges in S. Let ϕ be a bijection of S onto a set S' disjoint from $V(G) \cup E(G)$. Then G clearly has a b-detachment D such that $V(D) = V(G) \cup S'$, $G - S$ is a subgraph of D and, in D, each edge $\lambda \in S$ joins λt_J to $\phi(\lambda)$. Moreover D is connected since the spanning tree of G corresponding to T is a subgraph of D and each element of S' is D-adjacent to a vertex in $V(G) = V(T)$.

Sketch of Alternative Proof (not requiring Theorem 1 or Corollary 1a). If $X \subseteq V(G)$ let $g(X) = e(X) - c(G - X)$. Assume (12) for every $X \subseteq V(G)$. Let $m\colon V(G) \to N$ be a function such that $m(\xi) \geq b(\xi)$ for every $\xi \in V(G)$ and $g(X) \geq m.X - 1$ for every $X \subseteq V(G)$ and, subject to these requirements, $m.V(G)$ is as large as possible. Then, by Lemma 1, G has an m_α-orientation J, where α is an arbitrarily chosen vertex of G and $m_\alpha(\xi) = m(\xi)$ for every $\xi \in V(G) \setminus \{\alpha\}$ and $m_\alpha(\alpha) = m(\alpha) - 1$. Call a set X *critical* if $g(X) = m.X - 1$. Since $m.V(G)$ has been maximized, every vertex of G must belong to a critical subset of $V(G)$.

Let Y, Z be subsets of $V(G)$ and let $H = G - Y$, $K = G - Z$, $M =$

$(Y \setminus Z) \nabla (Z \setminus Y)$. Then

$$e(Y \cup Z) + e(Y \cap Z) + |M| = e(Y) + e(Z). \tag{13}$$

Form a bipartite graph Γ with $c(H) + c(K)$ vertices ρ_C ($C \in \mathcal{C}(H)$), σ_D ($D \in \mathcal{C}(K)$) and $c(H \cap K)$ edges λ_F ($F \in \mathcal{C}(H \cap K)$), where λ_F joins the vertices ρ_C, σ_D such that $F \subseteq C \cap D$. Then $c(H \cup K) = c(\Gamma)$ and so, by (2) applied to Γ,

$$c(H \cap K) + c(H \cup K) \geq c(H) + c(K). \tag{14}$$

Furthermore $c(H \cup K) = c((G - (Y \cap Z)) - M) \leq c(G - (Y \cap Z)) + |M|$ by (1), and $H \cap K = G - (Y \cup Z)$. From these observations and (13) and (14), it follows that $g(Y \cap Z) + g(Y \cup Z) \leq g(Y) + g(Z)$. Using this fact and an argument like the one following the end of the proof of Lemma 12, we see that the union of any two critical subsets of $V(G)$ is critical and hence (recalling that every vertex belongs to a critical set) that $V(G)$ is critical, i.e. $|E(G)| = m_\alpha . V(G)$. Therefore J is a digraph. The set A of all vertices accessible in J from α has no J-exits and so $\lambda h_J \in A$ for every $\lambda \in E(A)$. Therefore $e(A) = m_\alpha . A = m.A - 1 \leq g(A)$ and so $A = V(G)$, i.e. all vertices of G are J-accessible from α. Since each $\xi \in V(G)$ is the head of $m_\alpha(\xi) \geq b_\alpha(\xi)$ (directed) edges of J, the argument can now be completed as before.

In [4] and [5], a somewhat more general result than Theorem 2 was proved by matroid methods. However, this more general result can be deduced from Theorem 2 (without using matroids), as we now show.

Corollary 2a. *Let G be an undirected graph and b be a function from $V(G)$ into N. Then the minimum of $c(D)$ over all b-detachments D of G is equal to the maximum of $b.X + c(G - X) - e(X)$ over all subsets X of $V(G)$.*

Proof: Let M be the above maximum. If D is a b-detachment of G, the first paragraph of the proof of Theorem 2 shows that $b.X + c(G - X) - e(X) \leq c(D)$ for every $X \subseteq V(G)$, and hence $c(D) \geq M$. Let $n = |V(G)|$, $q = b.V(G) + n + 1$ and H be an undirected graph obtained from G by adding a new vertex ω and for each $\xi \in V(G)$ adding q edges joining ω to ξ. Define b': $V(H) \to N$ by letting $b'(\xi) = b(\xi)$ for $\xi \in V(G)$ and $b'(\omega) = qn + 1 - M$. If $X \subseteq V(G)$ then

$$e_H(X) = e(X) + q|X| \geq b.X = b'.X = b'.X + c(H - X) - 1$$

since $H - X$ is connected, and

$$\begin{aligned} e_H(X \cup \{\omega\}) &= e(X) + qn \geq b.X + c(G - X) - M + qn \\ &= b'.(X \cup \{\omega\}) + c(H - (X \cup \{\omega\})) - 1. \end{aligned}$$

Therefore, by Theorem 2, H has a connected b'-detachment D'. If p is a b'-coalescence of D' onto H then $D_0 = D' - p^{-1}(\{\omega\})$ is a b-detachment of G and, by (1) applied to D',

$$\begin{aligned} c(D_0) &= c(D' - (\omega \nabla_H V(G))) - b'(\omega) \\ &\le c(D') + |\omega \nabla_H V(G)| - b'(\omega) = M. \end{aligned}$$

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