A Direct Proof of a Theorem on Detachments of Finite Graphs

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ABSTRACT. Let G be a finite graph with vertices $\xi_1, \ldots, \xi_n$ and let $S_1, \ldots, S_n$ be disjoint non-empty finite sets. We give a new proof of a theorem characterizing the least possible number of components of a graph D such that $V(D) = S_1 \cup \cdots \cup S_n$, $E(D) = E(G)$ and, when an edge λ joins vertices ξ_i, ξ_j in G, it is required to join some element of S_i to some element of S_j in D (so that, informally, D arises from G by splitting each vertex ξ_i into $|S_i|$ vertices).

1 Introduction

Definitions. The set of all positive integers will be denoted by $\mathbf{N}$. The *first constituent* x of an ordered pair (x, y) will be denoted by $p((x, y))$. The symbol G will always denote a graph. In this paper, graphs may be finite or infinite, and may have loops and multiple edges. If $X \subseteq V(G)$ then E_X or E_X^G denotes the set of those edges of G which are incident with at least one element of X and $G - X$ denotes the subgraph of G such that $V(G-X) = V(G) \backslash X$, $E(G-X) = E(G) \backslash E_X$. The number of components of G is denoted by $c(G)$. A *G-set* is a set Ω of ordered pairs such that $p(\theta) \in V(G)$ for every $\theta \in \Omega$ and each vertex of G is the first constituent of at least one element of Ω. If Ω is a G-set then $\Omega\xi$ denotes $\{\theta \in \Omega : p(\theta) = \xi\}$ for each $\xi \in V(G)$ and ΩX denotes $\{\theta \in \Omega : p(\theta) \in X\}$ for each $X \subseteq V(G)$. If $b \colon V(G) \to \mathbf{N}$ is a function, then $b.X$ denotes $\sum_{\xi \in X} b(\xi)$ for each finite subset X of $V(G)$ and a *(G, b)-set* is a G-set Ω such that $|\Omega\xi| = b(\xi)$ for

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every $\xi \in V(G)$. We define an *Ω-detachment* of G (where Ω is a G-set) to be a graph D such that $V(D) = \Omega$, $E(D) = E(G)$ and, for each edge λ of G, the vertices joined by λ in G are the first constituents of the vertices joined by λ in D. In particular, λ is a loop of G iff it joins vertices of D which have the same first constituent. A graph is a *detachment* of G if it is an Ω-detachment of G for some G-set Ω, and is a *b-detachment* of G if it is an Ω-detachment of G for some (G, b)-set Ω. Thus, informally, a b-detachment of G is obtained from G by splitting each $\xi \in V(G)$ into $b(\xi)$ vertices. (These definitions of "detachment" and "b-detachment" are more restrictive than those in [1], [2] and [3] since they require vertices of a detachment to be ordered pairs, but this makes no difference "up to isomorphism".)

This paper presents a proof of the following known theorem which is more direct than previous proofs involving matroids [1, 2] or orientation of some edges of G to produce a mixed graph [3].

Theorem. *If G is finite and $b\colon V(G) \to \mathbb{N}$ is a function then the minimum of $c(D)$ over all b-detachments D of G is equal to*

$$\max_{X \subseteq V(G)} (b.X + c(G - X) - |E_X|). \tag{1}$$

2 The new proof

Further Definitions. If $L \subseteq E(G)$ then $G - L$ denotes the subgraph of G such that $V(G - L) = V(G)$, $E(G - L) = E(G) \setminus L$. We write $G - \omega$ in place of $G - \{\omega\}$ if ω is a vertex or edge of G. If $\xi \in V(G)$ then $G[\xi]$ denotes the subgraph of G such that $V(G[\xi]) = \{\xi\}$, $E(G[\xi]) = \emptyset$. The set of components of G is denoted by $\mathcal{C}(G)$. An edge λ of G is a *bridge* if it joins vertices which belong to distinct components of $G - \lambda$. The *valency* of a vertex incident with p loops and q other edges is $2p + q$. A *path* is a non-empty connected graph in which no vertex has valency greater than 2 and every edge is a bridge. A *circuit* is a non-empty finite connected graph in which every vertex has valency 2. A *$\xi\eta$-edge* is an edge joining (not necessarily distinct) vertices ξ, η. A path P is a *$\xi\eta$-path* if ξ, η are (not necessarily distinct) vertices of P and the addition of a $\xi\eta$-edge would convert P into a circuit. A *circuit in G* is a subgraph of G which is a circuit, and "path in G" and "$\xi\eta$-path in G" are similarly defined. A subgraph S of G and subset X of $V(G)$ *meet* each other if $V(S) \cap X \neq \emptyset$. A vertex ξ of G *separates* a subset X of $V(G)$ if two or more components of $G - \xi$ meet X. If λ is a $\xi\eta$-edge of G and $\zeta \in V(G)$ then $G \circ \lambda(\eta \to \zeta)$ will denote the graph G' such that $V(G') = V(G)$, $E(G') = E(G)$, $G' - \lambda = G - \lambda$ and λ joins ξ to ζ in G'. In other words, $G \circ \lambda(\eta \to \zeta)$ is obtained from G by "detaching one end of λ from η and re-attaching it to ζ". (The vertices ξ, η, ζ need not be distinct.)

Let G be a graph, Ω be a G-set and D be an Ω-detachment of G. Then $S(D)$ will denote the set of all vertices ξ of G such that $\Omega\xi$ meets at least two components of D. If $\xi, \eta \in V(G)$, the statement $\xi \overrightarrow{_D} \eta$ will mean that some element of $\Omega\xi$ separates $\Omega\eta$ in D. We define a *D-sequence* to be a non-empty finite sequence $\xi_0, \xi_1, \ldots, \xi_r$ of distinct vertices of G such that $\xi_0 \in S(D)$ and $\xi_i \overrightarrow{_D} \xi_{i+1}$ for every non-negative integer $i < r$. In particular, a sequence with just one term ξ_0 is a D-sequence if $\xi_0 \in S(D)$. We define subsets $R(D)$, $R(D, r)$, $R(D, < r)$ of $V(G)$ by the rule that a vertex of G belongs (i) to $R(D)$ iff it is the last term of some D-sequence (or, equivalently, iff it is a term of some D-sequence), (ii) to $R(D, r)$ iff it is the last term ξ_r of some D-sequence $\xi_0, \xi_1, \ldots, \xi_r$ with $r+1$ terms and is not the last term of any D-sequence with fewer than $r+1$ terms, (iii) to $R(D, < r)$ iff it is the last term of some D-sequence with fewer than $r+1$ terms. We shall say that D is a *tight* Ω-detachment of G if $c(D)$ is finite and no Ω-detachment of G has fewer components than D.

Lemma 1. *If Ω is a G-set and D is a tight Ω-detachment of G then no circuit in D meets $\Omega S(D)$.*

Proof: If a circuit C in D met $\Omega S(D)$ then there would exist vertices $\xi \in S(D)$, $\theta \in V(C) \cap \Omega\xi$ and, since $\xi \in S(D)$, there would exist $\phi \in \Omega\xi$ such that θ, ϕ are in distinct components of D. We could then choose an edge λ of C incident with θ and $D \circ \lambda(\theta \to \phi)$ would be an Ω-detachment of G with fewer components than D, contradicting the tightness of D.

Lemma 2. *Let Ω be a G-set, D be an Ω-detachment of G and $D' = D \circ \lambda(\theta \to \phi)$, where θ, ϕ are vertices in distinct components of D such that $p(\theta) = p(\phi)$ and λ is an edge of D incident with θ. Let ξ, η be vertices of G such that $\eta \notin S(D)$ and $\xi \overrightarrow{_D} \eta$. Then D' is an Ω-detachment of G and $\xi \overrightarrow{_{D'}} \eta$.*

Proof: Since D is an Ω-detachment of G and $p(\theta) = p(\phi)$, clearly D' is an Ω-detachment of G. Let D_θ, D_ϕ be the (distinct) components of D which include θ, ϕ respectively and σ be the vertex joined to θ by λ in D. Let Q be the component of $D_\theta - \lambda$ which includes σ and R be the graph obtained from $Q \cup D_\phi$ by adding λ as a $\sigma\phi$-edge. Then clearly R is a component of D', λ is a bridge of R and Q, D_ϕ are the components of $R - \lambda$. Since $\xi \overrightarrow{_D} \eta$, there exist $\rho \in \Omega\xi$ and $\psi, \chi \in \Omega\eta$ such that ψ, χ are in distinct components of $D - \rho$. Therefore $D - \rho$ contains no $\psi\chi$-path, and so any $\psi\chi$-path in $D' - \rho$ must include λ. Therefore the existence of such a path in $D' - \rho$ would require one of ψ, χ to be in Q and the other in D_ϕ, contradicting (since $Q \subseteq D_\theta$) the hypothesis that $\eta \notin S(D)$. Therefore there is no $\psi\chi$-path in $D' - \rho$, and so $\xi \overrightarrow{_{D'}} \eta$.

Lemma 3. *If Ω is a G-set and D is a tight Ω-detachment of G then no circuit in D meets $\Omega R(D)$.*

Proof: Suppose that some circuit J in a tight Ω-detachment D of G meets $\Omega R(D)$. Then we can choose D, J, r so that J meets $\Omega R(D,r)$ and r is as small as possible. By Lemma 1, J cannot meet $\Omega S(D) = \Omega R(D,0)$ and so $r \geq 1$. Since J meets $\Omega R(D,r)$, it meets $\Omega\xi$ for some $\xi \in R(D,r)$. By the definition of $R(D,r)$, ξ is the last term ξ_r of some D-sequence $\xi_0, \xi_1, \ldots, \xi_r$ and if $i \in \{1,\ldots,r\}$ then $\xi \notin R(D, r-i)$ and so $\xi_i, \xi_{i+1}, \ldots, \xi_r$ is not a D-sequence and consequently $\xi_i \notin S(D)$. Since $\xi_0 \overrightarrow{_D} \xi_1$ there are vertices $\theta \in \Omega\xi_0$ and $\psi, \chi \in \Omega\xi_1$ such that ψ, χ are in distinct components D_ψ, D_χ respectively of $D - \theta$. Since $\xi_1 \notin S(D)$ it follows that ψ, χ are in the same component D_0 (say) of D. Therefore $\theta \in V(D_0)$ and D_ψ, D_χ are components of $D_0 - \theta$. Since D_0 is connected, it has an edge λ joining θ to a vertex σ of D_χ. Since $\xi_0 \in S(D)$, some component $D_1 \neq D_0$ of D includes a vertex $\phi \in \Omega\xi_0$. Let $D' = D \circ \lambda(\theta \to \phi)$. Since D is an Ω-detachment of G and $p(\theta) = \xi_0 = p(\phi)$, clearly D' is an Ω-detachment of G. By Lemma 1, no circuit in D can include the vertex $\theta \in \Omega\xi_0 \subseteq \Omega S(D)$ and so λ is a bridge of D. Therefore $D_0 - \lambda$ has two components, namely D_χ and a component D_0' such that $\theta \in V(D_0')$ and $D_\psi \subseteq D_0'$. Clearly $C(D') = (C(D) \setminus \{D_0, D_1\}) \cup \{D_0', H\}$, where H is the graph obtained from $D_\chi \cup D_1$ by adding λ as a $\sigma\phi$-edge. Therefore $c(D') = c(D)$, so that D' is tight, and ψ, χ are in distinct components D_0', H of D', so that $\xi_1 \in S(D')$. Moreover, since $\xi_i \overrightarrow{_D} \xi_{i+1}$ for $0 \leq i < r$ and $\xi_1, \ldots, \xi_r \notin S(D)$ and θ, ϕ are in distinct components D_0, D_1 of D it follows by Lemma 2 that $\xi_i \overrightarrow{_{D'}} \xi_{i+1}$ for $0 \leq i < r$. Therefore $\xi_1, \ldots, \xi_r$ is a D'-sequence and so $\xi \in R(D', < r)$. Since no circuit in D includes θ, it follows that $\lambda \notin E(J)$ and so J is a circuit in D'; and J meets $\Omega R(D', < r)$ since it meets $\Omega\xi$. Thus our choice of D, J, r so as to minimize r is contradicted.

Proof of the Theorem: Let M denote the maximum in (1). Let Ω be a (G,b)-set and D be a tight Ω-detachment of G. Then clearly $c(D)$ is minimized over all b-detachments D of G and so it suffices to prove that $c(D) = M$.

If S is a subgraph of G, let ΩS be the subgraph of D such that $V(\Omega S) = \Omega V(S)$, $E(\Omega S) = E(S)$. If F is a subgraph of D, let $p(F)$ be the subgraph of G such that $V(p(F)) = \{p(\theta) \colon \theta \in V(F)\}$, $E(p(F)) = E(F)$.

Let X be a subset of $V(G)$. In (1), E_X means E_X^G, which is equal to $E_{\Omega X}^D$. Therefore $D - E_X$ is the union of $b.X + c(G-X)$ disjoint non-empty subgraphs $D[\theta]$ $(\theta \in \Omega X)$, ΩC $(C \in C(G-X))$ and so $c(D - E_X) \geq b.X + c(G - X)$. Since removing an edge from a graph increases the number of components by at most 1, it follows that $c(D) \geq c(D - E_X) - |E_X| \geq b.X + c(G-X) - |E_X|$. Since X was arbitrary, we infer that $c(D) \geq M$.

Let $R(D) = R$. Suppose that $\xi \in V(G) \setminus R$ and $\Omega\xi$ includes vertices ψ, χ which are in distinct components of $D - \Omega R$. Since $S(D) \subseteq R(D)$, it follows that $\xi \notin S(D)$ and so ψ, χ are in the same component of D. Therefore D contains a $\psi\chi$-path, which must include a vertex $\theta \in \Omega R$ since ψ, χ are in distinct components of $D - \Omega R$. By Lemma 3, no circuit in D includes θ, and so ψ, χ are in distinct components of $D - \theta$. Therefore $p(\theta) \overrightarrow{_D} \xi$, which, since $p(\theta) \in R$, implies that $\xi \in R$ by the definition of $R(D) = R$. Since this contradicts the assumption that $\xi \in V(G) \setminus R$, we conclude that $\Omega\xi$ cannot meet two distinct components of $D - \Omega R$ if $\xi \in V(G) \setminus R$. Therefore the graphs $p(F)$ $(F \in C(D - \Omega R))$ are disjoint subgraphs of $G - R$. Since clearly these graphs are non-empty and connected and have union $G - R$, they are the components of $G - R$ and so

$$c(G - R) = c(D - \Omega R) = c(D - E^D_{\Omega R}) - |\Omega R|.$$

Moreover $c(D) = c(D - E^D_{\Omega R}) - |E^D_{\Omega R}|$ because all elements of $E^D_{\Omega R}$ are bridges of D by Lemma 3; and clearly $E^D_{\Omega R} = E^G_R$ and $|\Omega R| = b.R$. Therefore $c(D) = b.R + c(G - R) - |E^G_R| \le M$.

3 Detachments of infinite graphs

What can one prove about infinite graphs on the lines of the above Theorem? The methods of proof in [1], [2] and [3] seem difficult to extend to infinite graphs, but the method used here seems more promising in this respect. As an indication of initial progress, I state below two results hitherto obtained by this method, deferring proofs to possible future papers.

Definitions. An infinite path P is *one-way infinite* if exactly one vertex of P has valency 1 in P. If Ω is a G-set, C is component of an Ω-detachment of G and $\xi, \eta \in V(G)$ then the statement $\xi \overrightarrow{_C} \eta$ will mean that some element of $\Omega\xi \cap V(C)$ separates $\Omega\eta \cap V(C)$ in C.

Proposition 1. *If G is an infinite graph, Ω is a G-set and P is a one-way infinite path in a tight Ω-detachment D of G then $V(P) \cap \Omega R(D)$ is finite.*

Proposition 2. *If G is an infinite graph, Ω is a G-set and C is a component of a tight Ω-detachment D of G then there is no infinite sequence $\xi_0, \xi_1, \xi_2, \ldots$ of distinct vertices of G such that $\xi_0 \in S(D)$ and $\xi_{i-1} \overrightarrow{_C} \xi_i$ for every $i \in \mathbb{N}$.*

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