Graphs with Geometric Properties

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ABSTRACT. There are many graphs with the property that every subgraph of a given simple isomorphism type can be completed to a larger subgraph which is embedded in its ambient parent graph in a nice way. Often, such graphs can be classified up to isomorphism. Here we survey theorems on polar space graphs, graphs with the cotriangle property, copolar graphs, Fischer spaces, and generalized Fischer spaces, as well as graphs with the odd coclique property.

1 Introduction

For the purposes of this survey, a "geometric property" of a graph Γ will always mean that all induced subgraphs X of a fixed isomorphism type $\mathcal{T}$, wherever they are encountered, are always contained in some larger subgraph X^* which is embedded in Γ in some canonical way. One should think of the assertion that any subgraph X of type $\mathcal{T}$ completes to X^* as a sort of 'axiom' governing the structure of the graph Γ.

Of course such hypotheses are generally strong, as graphs go, and so we should expect very strong conclusions, in most cases a complete classification up to isomorphism.

A very simple illustration is provided by the so-called friendship theorem, familiar to most graph theorists.

Theorem. **(Friendship)** Suppose that in a finite community of n people any two distinct individuals have exactly one mutual friend. Then n is an odd number, and one of these persons, P_0 is acquainted to all others. Among the remaining persons, each is matched with a unique friendly colleague.

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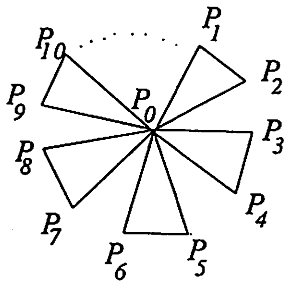
The unique graph of the friendship relation is given below.



This theorem seems to go back to unpublished remarks of Graham Higman in 1968 (see Wilf [27]), and has been generalized to digraphs by Hammersley [21].

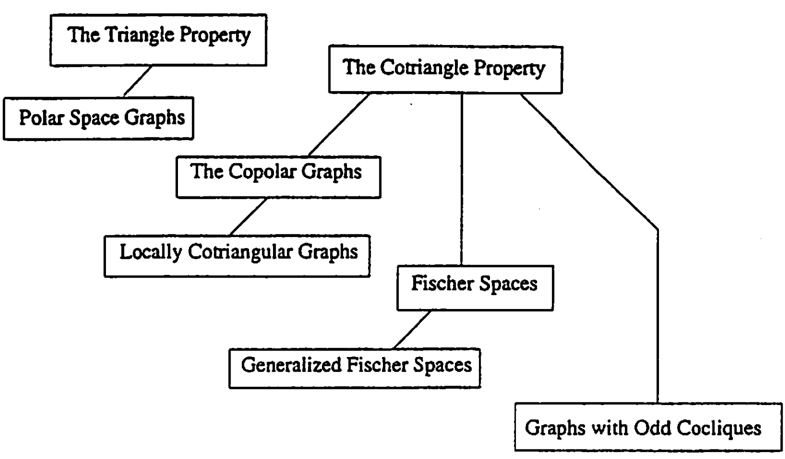
The hypothesis of the Friendship Theorem imposes two distinct geometric properties of the graph in the sense intended above:

Property 1: Every edge lies in a unique triangle.

Property 2: Every pair of non-adjacent vertices lies in a unique 2-claw (the bipartite graph $K_{1,2}$).

In each property, a subgraph is asserted to lie in a slightly larger subgraph, and the uniqueness of the larger graph is an assertion about its embedding.

The subjects visited in this survey are depicted below. A box connected to a box above it indicates that the former is some kind of generalization of the latter.



All graphs which we shall consider are undirected and without multiple edges and loops. By a **clique** we mean an induced subgraph that happens to be complete. It does not mean that it is a maximal such graph, as occurs in some mostly older literature. A **coclique** is an induced subgraph having no edges at all. We will introduce further terminology as it is needed.

2 The Triangle Theorem

The property below is about edges in a graph $\Gamma = (V, E)$. As one can see, it bears a faint resemblance to the basic property of the Friendship Theorem.

(The Triangle Property): Given an edge $e = \{u, v\}$ of the graph Γ, their exists a third vertex w adjacent to both u and v, such that every further vertex of the graph not in the triangle $T = \{u, v, w\}$ is adjacent to 1 or 3 of the vertices of the triangle T.

Example: The graph $\overline{T}_6$ (the dual triangular graph on 6 letters). Its vertices are the fifteen 2-subsets of a 6-set $\Omega = \{1, \ldots, 6\}$. Two such 2-subsets are declared to be adjacent if and only if their intersection is empty. Given an edge consisting of two disjoint 2-subsets U and V, the triangle whose existence is asserted is obtained by adjoining the vertex $W = \Omega - (U + V)$. It is then clear that $\{U, V, W\}$ is a partition of the set Ω into three disjoint 2-sets, and that any further 2-subset X has one of its two letters in two components of this partition, and is disjoint from just one of them. That is, any vertex X outside the triangle $\{U, V, W\}$ is adjacent to exactly one vertex of the triangle.

Of course, this is a special example, for here each edge lies in a unique triangle. We can thus think of the system of 15 vertices and 15 triangles as an incidence system $\Gamma = (\mathcal{P}, \mathcal{L})$, of points and lines. It then satisfies these axioms:

(GQ1) Two distinct points are incident with at most one line.

(GQ2) Given a line L and a point p not incident with L, there exists a unique line on p which intersects L.

Any incidence system $(\mathcal{P}, \mathcal{L})$ which satisfies the axioms (GQ1) and (GQ2) is called a **generalized quadrangle**. If it happens that there are exactly $s+1$ points on each line and exactly $t+1$ lines on each point the generalized quadrangle is said to have **order** (s, t). The generalized quadrangle defined by our example $\overline{T}_6$ has order $(2, 2)$. A generalized quadrangle is said to be **non-degenerate** if no point is collinear with all remaining points; and if it is degenerate it consists of a single point P_0, and a collection of lines pairwise intersecting at this point. The vertices and triangles of the graph in the friendship theorem is a degenerate generalized quadrangle. It is known that in a non-degenerate generalized quadrangle, if some line has at least three points and some point is on at least three lines, then the quadrangle has order (s, t) for some cardinal numbers s and t.

There are just three types of non-degenerate generalized quadrangles of order $(2, t)$. (This has been proved at least five times in the literature, [11], [13], [23], [24], [25].) The cases are:

(1) The 3-by-3 grid having 9 points and 6 lines divided into two parallel classes.

(2) The example $\overline{T}_6$ given above, having 15 points and 15 lines.

(3) A generalized quadrangle of 27 points and 45 lines. (This graph is related to the famous theorem on the 27 lines of an irreducible cubic.)

The respective quadrangles have order $(2, 1)$, $(2, 2)$ and $(2, 4)$, and the graphs of the collinearity relation on vertices all have the triangle property.

These, of course, represent the special case that each edge lies in a unique triangle, such that every vertex outside that triangle is adjacent to exactly one vertex within the triangle. What about the general case? The result is

The Triangle Theorem. *Let Γ be a finite graph having at least one edge and possessing the triangle property. Suppose no vertex in Γ is adjacent to all other vertices of Γ. Then Γ is isomorphic to one of the following graphs:*

1. $Sp(2n, 2)$
2. $S^+(2n, 2)$
3. $S^-(2n, 2)$.

Explanation of the Graphs

1. Let V be a finite dimensional vector space over the field $\mathbf{Z}/(2)$, of integers modulo 2. A **symplectic form** is a bilinear mapping $B\colon V \times V \to \mathbf{Z}/(2)$, which satisfies $B(u, u) = 0$ for every vector u of V. The radical of B is the set of vectors r such $B(r, u) = 0$ for every vector u of V. B is said to be **non-degenerate** if its radical is the 0-subspace.

Whenever B is a non-degenerate symplectic form on V, then V has a so-called **symplectic basis** $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$, where$B(x_i, y_i) = 1$, $i = 1, \ldots, n$, and all other symplectic inner products among the basis elements are equal to zero. In particular V is a **perpendicular direct sum**

$$V = \langle x_1, y_1\rangle \perp \langle x_2, y_2\rangle \perp \ldots \perp \langle x_n, y_n\rangle$$

and has even dimension $2n$.

The vertices of the graph $Sp(2n,2)$ are the non-zero vectors of the $2n$-dimensional space V admitting a non-degenerate symplectic form as above. Two vectors u and v of the graph are adjacent if and only if they are **perpendicular with respect to** B - i.e., $B(u,v)=0$.

2. Orthogonal geometries over $\mathbf{Z}/(2)$ can be thought of in the following way: Let V be a $2n$-dimensional space over $\mathbf{Z}/(2)$ admitting a non-degenerate symplectic form B as above. An **orthogonal geometry over** $\mathbf{Z}/(2)$ is a partition of the non-zero vectors of V into two sets S and $\mathcal{N}$ subject to these rules:

Let u and v be any two distinct nonzero vectors in V . Then

1. If (u,v) is in $S\times S$ or in $\mathcal{N}\times\mathcal{N}$ then $u+v$ is in S if and only if $B(u,v)=0$.
2. If (u,v) is in $S\times\mathcal{N}$, then $u+v$ is in S if and only if $B(u,v)=1$.

When V has dimension only 2, there are just two possibilities: (Q) all three non-zero vectors of V belong to $\mathcal{N}$, or (D) two of the non-zero vectors of V belong to S, the other to $\mathcal{N}$.

In general, then, V is a perpendicular direct sum of non-degenerate 2-dimensional subspaces of types "Q" or "D", and it can be shown that the 4-spaces of types $D\perp D$ and $Q\perp Q$ are isomorphic. Thus in general there are just two cases:

$$O^+: \text{(Hyperbolic) } D\perp D\perp\ldots\perp D \text{ (all } D's)$$

and

$$O^-: \text{(Elliptic) } Q\perp D\perp\ldots\perp D \text{ (exactly one } Q).$$

Thus graph $S^+(2n,2)$ and $S^-(2n,2)$ are the graphs $(S,\perp)$ of the perpendicular relation on the vertex set S in the hyperbolic and elliptic cases respectively, when $dim(V)=2n$.

The sets S and $\mathcal{N}$ are called respectively the sets of **singular** and **non-singular** vectors of the orthogonal geometry. This is an opportunity to define two further graphs which we shall need later on, namely the graphs $\mathcal{N}^+(2n,2)$ and $\mathcal{N}^-(2n,2)$ which record the Perp-relation on the set of non-singular vectors $\mathcal{N}$ in the two respective cases, hyperbolic and elliptic.

In low dimensions these graphs are often familiar in other contexts.
$S^+(4,2)$ is the graph of the 3-by-3 grid (9 vertices).
$\mathcal{N}^-(4,2)$ is the famous Petersen graph (10 vertices).
$Sp(4,2)$ is the graph $\overline{T}_6$ of our example (15 vertices).
$\mathcal{N}^+(6,2)$ is the graph $\overline{T}_8$ (28 vertices).
$\mathcal{N}^-(6,2)$ is Burnside's graph of the double sixes (36 vertices).
$Sp(6,2)$ is the Perp-graph on the lines generated by the root system of the

Lie algebra of type E_7 (63 vertices).
$\mathcal{N}^+(8,2)$ is the Perp-graph on the lines generated by the root system of type E_8 (120 vertices).

The triangle theorem was proved in [23] as an offshoot of a characterization of the symplectic groups over $GF(2)$ by the graph of the commuting relation on its class of transvections. The proof was a simple induction. Since one could assume that one was dealing with more than a generalized quadrangle, the induced graph on the set $S(a,b)$ of all vertices adjacent to two non-adjacent vertices a and b, could be seen to be a connected subgraph which also had the triangle property. This was enough structure to determine the whole graph.

3 Polar Spaces

How should one generalize the triangle theorem? One possibility is to enlarge the triangle to some special clique with a similar relation to the graph as a whole: thus one might consider:

(The Polar Space Property): *Given any edge $e = \{u, v\}$ of the graph Γ, there exists a clique $C(e)$ containing e such that every vertex x of Γ outside $C(e)$ is adjacent either to exactly one or to all of the vertices of $C(e)$.*

One should note that the point-collinearity graphs of the class of generalized quadrangles fulfills this condition. For these graphs, each external vertex x is adjacent to just one vertex of $C(e)$, that is, the option for adjacency to all vertices of the clique $C(e)$ never arises for any edge e. Unfortunately, despite many excellent characterization theorems (see Chapter 10 of the forthcoming Handbook of Incidence Geometry) there is presently no exhaustive classification theory for generalized quadrangles, and the prospects look more and more dim as new infinite families of generalized quadrangles keep emerging.

Notation: If Γ is any graph, let $Rad(\Gamma)$ be the set of vertices of Γ which are adjacent to all remaining vertices. If x is a vertex of Γ, we let $x^\perp$ denote the vertex x together with the set $\Gamma(x)$ of all vertices adjacent to x. Thus a vertex x belongs to $Rad(\Gamma)$ if and only if $x^\perp = V(\Gamma)$, the full vertex set of Γ.

If Γ is a graph with the polar space property, we say that Γ is **non-degenerate** if and only if $Rad(\Gamma)$ is empty. There is a graph homomorphism from the induced subgraph of non-radical vertices $V(\Gamma) - Rad(\Gamma)$ onto a graph Γ^*, obtained by identifying vertices x and y for which $x^\perp = y^\perp$. It is easily seen that the graph Γ^* is now a non-degenerate graph with the polar space property. Moreover, the isomorphism type of Γ can be completely recovered from the isomorphism type of the graph Γ^* and the cardinality of $Rad(\Gamma)$: one simply blows up each vertex u^* of Γ^* to a clique U of the cardinality of $Rad(\Gamma)$, inserts all possible edges or none between cliques

U and V according as their images u^* and v^* are adjacent in Γ^* or not, and then adjoins a copy of $Rad(\Gamma)$ whose vertices are made adjacent to all the vertices of the newly-created cliques U. The result is a reconstruction of Γ.

Thus to classify the graphs with the polar space property, there is no loss of generality in assuming that the graph has empty radical. The resulting theorem has a simple hypothesis, but a fairly complicated conclusion: indeed the theorem yields a simple characterization of all of the classical groups.

The Polar Space Theorem Stated As A Theorem In Graph Theory
Let Γ *be graph with these properties:*

(i) Each edge e lies in a clique C_e, of three or more vertices, such that each vertex outside this clique is adjacent to exactly one, or to all of its vertices.

(ii) No vertex is adjacent to all remaining vertices.

(iii) Γ possesses at least one edge.

Then one of the following is true:

(1) Γ is a generalized quadrangle.

(2) There are two maximal parabolic subgroups H and K of a compact form of an algebraic group G of type E_7. The vertices are the cosets of H in G; two of them being adjacent if and only if they have a nonempty intersection with a common coset of K.

(3) Γ is isomorphic to one of the following three graphs defined by a (possibly infinite-dimensional) right vector space V over a division ring D:

(i) There exists a non-degenerate (σ, ϵ)-Hermitian form f on V and the vertices of Γ are the totally isotropic 1-spaces of V, two of them being adjacent if and only if they are perpendicular with respect to the form f.

(ii) D has characteristic 2, and V admits a non-degenerate pseudoquadratic form Ω . The vertices are the totally Q-singular 1-spaces, and the cliques C_e are the totally Q-singular 2-subspaces of V (each viewed as a collection of 1-spaces).

(iii) V is a 4-dimensional vector space over the non-commutative division ring D, the vertices Γ are the 2-dimensional subspaces of V, and two of them are adjacent if and only if they intersect at a 1-subspace.

The graphs of the theorem are allowed to be infinite (whether or not the cliques C_e, which must all have the same cardinality, are finite or not). Its proof is not at all graph-theoretic in any natural way, but is basically a geometric proof since it intrinsically utilizes the theory of **Buildings** as developed by Prof. J. Tits ([26]). Moreover, the proof of this theorem has gone through many stages over the years, beginning with the seminal work of F.D. Veldkamp ([27]). It was then generalized and adapted for flag complexes of buildings by Tits ([26]), had its hypotheses simplified further by Buekenhout and Shult ([4]), and later relaxed by Peter Johnson to allow infinite rank ([22]). Finally, Cuypers, Johnson and Pasini introduce further simplifications of Tits' proof for singular rank at least four ([6], [7]). Nonetheless, having said all that, it is a theorem that characterizes graphs, and does so by a geometric property in the sense of the introduction .

4 Graphs with the Cotriangle Property

We turn now to a theorem which does have a natural graph-theoretic proof. We say that a subset X of the vertices of a graph Γ, is an **odd set** if and only if $x^{\perp}$ intersects X at a set of odd cardinality for each vertex x of Γ (whether in X or not). Thus in Section 2, the triangles of the triangle theorem are odd 3-cliques.

We now consider a variation on the theme of Section 2. Consider:

(The Cotriangle Property) *For each 2-coclique (non-adjacent pair of vertices) $\{u,v\}$ in the graph Γ, there exists a third vertex w such that $T = \{u,v,w\}$ is an odd cotriangle - that is T possesses no edges, and every vertex outside of T is adjacent to an odd number of vertices of T.*

Example. Petersen's Graph. The odd cotriangles are the maximal cocliques of size three (there are also 4-cocliques here, but they are not odd).

Reductions for the Classification

1. Γ can be assumed to be **co-connected** - that is, the complementary graph Γ is connected.

Suppose Γ partitions into disjoint subgraphs

$$\Gamma = \Gamma_1 + \Gamma_2 + \dots \Gamma_k$$

where each vertex in Γ_i is adjacent to every vertex in Γ_j, for $1 \leq i < j \leq k$. Then it is easy to see that Γ has the cotriangle property if and only if each co-component Γ_i has it.

2. One can assume Γ is **reduced** - that is, if $x^{\perp} = y^{\perp}$, then $x = y$.

The relation $x^{\perp} = y^{\perp}$ is an equivalence relation on the vertices of Γ. The equivalence class containing the vertex x is denoted by the symbol x^*. It is clearly a clique, and if x^* and y^* are two such classes, then either every possible edge exists between x^* and y^* or there are no edges at all between x^* and y^*, this, according as x is adjacent to y or not.

We can thus make a new graph Γ whose vertices are the equivalence classes x^* of the previous paragraph, two being adjacent if and only all possible edges exist between them. The graph Γ is clearly reduced; that is, $(x^*)^{\perp} = (y^*)^{\perp}$ implies $x^* = y^*$. Moreover we have a graph homomorphism $\Gamma \to \Gamma^*$ which takes each vertex x to its equivalence class x^*.

The point of all this is that Γ possesses the cotriangle property if and only if Γ^* does.

Conversely if Γ is a known graph, then the isomorphism type of the reconstructed Γ is completely determined by merely assigning cardinalities to the classes x^* at each fibre of the homomorphism .

Thus, in order to classify graphs with the cotriangle property, it suffices to classify only such graphs which are (1) coconnected and (2) reduced.

We can now state:

The Cotriangle Theorem *Let Γ be a finite reduced coconnected graph possessing the cotriangle property. Then Γ is one of the following graphs:*

1. The dual triangular graph, $\overline{T}_n$.
2. $Sp(2n, 2)$
3. $\mathcal{N}^+(2n, 2)$
4. $\mathcal{N}^-(2n, 2)$.

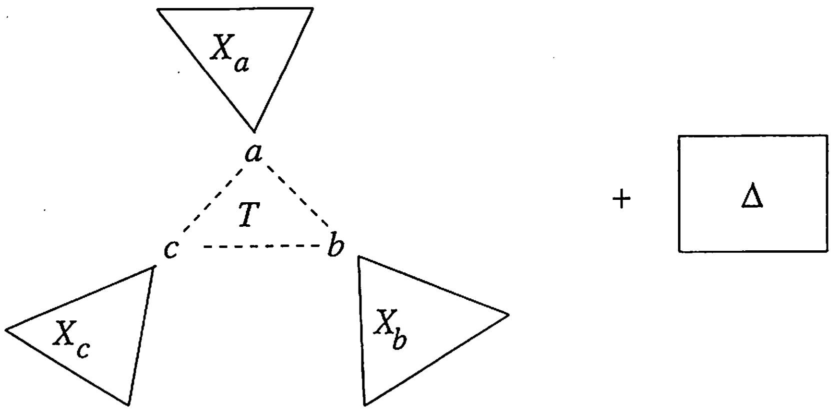
The proof proceeds as follows:

First, if a 2-coclique $\{u, v\}$ lived in *two* odd cotriangles $T_i = \{u, v, w_i\}$, $i = 1, 2$, then $(w_1)^{\perp}$ and $(w_2)^{\perp}$ would coincide. Thus since Γ is a reduced graph, $w_1 = w_2$; in other words, each 2-coclique $\{u, v\}$ lies in a unique odd cotriangle whose third vertex can unambiguously be denoted $t(u, v)$.

Now consider a fixed odd cotriangle $T = \{a, b, c\}$. The rest of the vertices of Γ can be partitioned into four sets:

$$\Gamma - T = X_a + X_b + X_c + \Delta$$

where X_u are the vertices of $\Gamma - T$ adjacent only to vertex u in T, and $\Delta = T^{\perp}$.



Second, $\{x, y\}$ is an edge in X_a if and only if $\{x, t(y, c)\}$ is not an edge if and only if $\{t(x, c), t(y, c)\}$ is an edge. Thus the mapping $X_a \rightarrow X_b$ defined by sending u to $t(u, c)$, is an isomorphism of induced subgraphs of Γ. It **follows that the isomorphism type of X_a completely determines the isomorphism type of the induced subgraph of**

$$T + X_a + X_b + X_c.$$

Third, there is a surjective mapping f from the set $[X_a, X_a]^*$ of 2-cocliques of X_a to Δ, whose fibres are determined by a certain equivalence relation R on $[X_a, X_a]^*$. Moreover, two R-classes in $[X_a, X_a]^*$ represent nonadjacent vertices of Δ if and only if a pair of representative 2-cocliques from each class together comprise either a 3-coclique or a 4-vertex subgraph with only one or three edges. Moreover, a vertex x of X_a is adjacent to a vertex $f(u, v)$ of Δ if and only if x is adjacent to 0 or 2 members of the coclique $\{u, v\}$ of $[X_a, X_a]^*$. It follows that **the isomorphism type of Γ is completely determined by the isomorphism type of the subgraph X_a.**

Fourth, no vertex in X_a is adjacent to all remaining vertices of X_a. (Any such vertex would have the same "perp" as a, against Γ being reduced.)

Fifth, if $e = \{u, v\}$ is an edge within X_a, then $t(u, b)$ is not adjacent to $t(v, c)$, and $w = t(t(u, b), t(v, c))$ is a vertex of X_a such that $T = \{u, v, w\}$ is an odd triangle. Thus X_a **satisfies the hypothesis of the Triangle Theorem.** (This was an observation of J.J. Seidel.)

The third, fourth and fifth steps show that Γ is uniquely determined. Thus when X_a possesses no edges at all (totally disconnected), Γ is T_n. When X_a is $Sp(2n, 2)$, Γ is $Sp(2n + 2, 2)$, and when X_a is $S^\epsilon(2n, 2)$,then Γ is $\mathcal{N}^\epsilon(2n, 2)$, $\epsilon = \pm 1$.

The Infinite Cotriangle Theorem

It should be remarked that recently J.I. Hall has given an essentially different proof of the Cotriangle Theorem - one that does not require finiteness of the graph ([16]). In his proof the vertices of a reduced co-connected graph with the cotriangle property are embedded into a vector space V over the field $GF(2)$ so that

(i) the embedded vertices span V and

(ii) the vectors of an embedded cotriangle always sum to 0.

The new graphs of the conclusion are just infinite versions of the same finite graphs:
$\overline{T}(Q)$: the disjointness relation on all 2-subsets of some (infinite) set Ω.
$Sp(V)$: The perpendicular relation on all 1-spaces of a $GF(2)$-space V with respect to a non-degenerate symplectic form.
$\mathcal{N}(V,q)$: Vertices are all non-singular non-radical vectors of an infinite $GF(2)$-space V with respect to a non-degenerate quadratic form q, two vertices x and y being adjacent if and only if $q(x+y) = q(x) + q(y)$.

Locally Cotriangular Graphs

These are graphs Γ in which the induced subgraph $\Gamma(x)$ on the set $x^{\perp} - \{x\}$ of vertices adjacent to x, is a graph with the cotriangle property. This is still an instance of a geometric property of a graph in the spirit of the introduction of this survey. It says that we are considering graphs in which every *2-claw lies in a 3-claw such that every exterior vertex which happens to be adjacent to its center is also adjacent to 1 or 3 of its three terminal vertices.* It does not make any assertion about exterior vertices which are not adjacent to its center - which is all the better!

The first real theorem in this area was Jonathan Hall's theorem classifying all graphs which are locally the Petersen graph [(14]). This was a hard theorem. As one gets more experienced in trying to characterize graphs locally, a certain principle emerges: It is easier if $\Gamma(x)$ is a "richly structured" graph in the sense that it is reduced and all circuits reduce to sums of triangles. The trouble here is that Petersen's graph has girth five. Just by looking at the conclusion set one can see that there must be difficulties in the theorem. There are three graphs: all are graphs of the commuting relation on a class of involutions in some group, and the groups are (i) Sym(7) (the $\overline{T}_7$ graph on the 21 transpositions; (ii) 3.Sym(7) (the 63 preimages of the transpositions) and (iii) $P\Gamma L(2,25)$ (the 65 involutions conjugate to the action induced by a field automorphism).

In the general case that $\Gamma(x)$ is a graph with the cotriangular property, one has to contend with the case that $\Gamma(x)$ is not coconnected, and that

$\Gamma(x)$ is not reduced. Hall and I could not handle the former, and for reasons that can be easily described, nothing useful can be said about the cases that $\Gamma(x)^*$ is K_1 or K_3. So suppose $\mathcal{D}$ is the class of finite coconnected graphs possessing the cotriangle property which is not the disjoint union of 1 or 3 cliques. We say that a graph Γ is **locally** $\mathcal{D}$, if for every vertex x, $\Gamma(x)$ belongs to the class $\mathcal{D}$. Note that this is far from asserting that $\Gamma(x)$ is of a fixed isomorphism type. This is the theorem:

Theorem On Locally Cotriangular Graphs ([20]). *Let Γ be a finite graph which is locally $\mathcal{D}$. Then the connected components of Γ are in the following list:*

1. $\overline{T}_n$, $n \geq 7$;
2. $Sp(2n, 2)$ minus a (possibly empty) polar subspace.
3. All non singular vectors of an orthogonal geometry with the points of a coclique of singular points adjoined.
4. The subgraph of $Sp(2n, 2)$ obtained by inducing an orthogonal geometry on the set of lines on a point p, and removing all points except p from those lines on p corresponding to singular "points" of the residue of p.
5. The locally Petersen graphs on 63 and 65 vertices, and the 117-vertex graph $\mathcal{N}^+(6,3)$ of square-norm non-singular 1-spaces of the quadric of type $\Omega^+(6,3)$, under the perpendicular relation.

But what if a graph is not locally coconnected? Only very recently have we learned how to handle this. Jon Hall's paper "Local indecomposability of certain geometric graphs" ([18]) is a brilliant adaptation of ideas of Aschbacher on component problems in groups to graphs. (This is no mean accomplishment when one realizes that without the vertices being group elements, the vertices are "dead" - that means that every argument where a group-action was previously available now has to be fought out on its own ground.) This paper is at the core of what we call **groups and geometries**: it is a paper every graph-theorist should read.

But all of this section on locally cotriangular graphs is a side-issue.

5 Copolar Graphs

Just as polar space graphs display a generalization of the triangle property in which the triangle has been replaced by a larger clique with the "one or all" property, similarly the cotriangle property admits a generalization in which the cotriangle has been replaced by a larger coclique which has the same "one or all" property.

A **copolar graph** is a graph Γ such that for every pair of distinct non-adjacent vertices $\{a, b\}$, there exists a coclique $C = C(a, b)$ in Γ such that

(i) C contains $\{a, b\}$

(ii) every exterior vertex x in $\Gamma - C$ is adjacent to exactly one or all of the vertices of C.

The study of copolar graphs is amenable to many of the same reductions that were available to graphs with the cotriangle property. Thus:

1) If Γ decomposes into coconnected components - that is

$$\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_k$$

where each vertex in Γ_i is adjacent to every vertex in Γ_j, for $1 \leq i < j \leq k$, then Γ is a copolar graph if and only if each component subgraph Γ_i is.

2) If $\Gamma \to \Gamma^*$ is the graph homomorphism that takes each vertex x to its equivalence class x^*, then Γ is a copolar graph if and only if Γ^* is.

Theorem On Copolar Graphs. (J.I. Hall [15]) *A finite graph Γ is copolar if and only if every coconnected component A of Γ^* is isomorphic to one of the following:*

(1) a coclique of size at least three;

(2) a Moore graph;

(3) a dual triangular graph $\overline{T}_n$;

(4) $\mathcal{N}^{\epsilon}(2n, 2)$;

(5) $Sp(2n, q)$;

(6) the complementary graph of any graph without triangles.

The Graphs of the Conclusion of Hall's Copolar Theorem

A **Moore graph** is a graph of valence k and diameter 2 having no triangles and no 4-circuits. Such a graph has exactly $1 + k^2$ vertices. These graphs were introduced by Hoffman and Singleton who proved that k can assume only the values $k = 2, 3, 7$, or 57 and that the graphs were unique in the first three cases, namely the pentagon, Petersen's graph, and a unique 50-vertex graph (now called the Hoffman-Singleton graph). To my knowledge it is still not known whether a 3,250-vertex Moore graph of valence 57 can exist.

The one-or-all cocliques here are the sets of neighbors of a vertex.

$Sp(2n, q)$ is the graph whose vertices are the 1-subspaces of a vector space over $GF(q)$ admitting a non-degenerate symplectic form B. Two 1-spaces are adjacent if and only if they are distinct and perpendicular with respect to B.

The one-or-all cocliques here are the 1-spaces belonging to a non-degenerate 2-space (hyperbolic line).

Cases (3), (4) and (5) with $q = 2$ are graphs with the cotriangle property.

Cases (1) and (6) are degenerate in some sense. In case (1) no vertex is adjacent to all vertices of the one-or-all cocliques while in case (6) the one-or-all cocliques have size just 2. Still, many complex configurations can exist, any transversal design yields an example of the former, while anyone would admit that graphs without triangles are numerous to say the least. But it is the best that can be said in these cases, for all of them do yield copolar graphs.

A word about Hall's proof. In the reduced co-connected case all of the one-or-all cocliques have the same cardinality $q+1$, which can be assumed to be at least 4. The set $\Gamma(x)$ of neighbor's of a vertex x also is a copolar graph. An earlier version of some of the argument's now in Hall's paper on local indecomposability mentioned above, allowed one to assume that $\Gamma(x)$ is coconnected. Induction then shows $(\Gamma(x))^*$ must be one of the graphs (1)-(6).

We end this subsection with two questions:

Can infinite copolar graphs be classified?

Can some of the locally copolar graphs be classified?

6 Fischer Spaces

There is another generalization of the cotriangle property, which, for lack of a better name, I will call

The Fischer Property. For each 2-coclique $\{v, v\}$ in the graph Γ, there is a third vertex w, such that

(i) $T = \{u, v, w\}$ is a cotriangle (3-coclique), and

(ii) for each vertex x of $\Gamma - T$ is adjacent to 0, 1 or all three members of T.

Again, there are two reductions to the classification problem that take place just as they did for copolar spaces.

1. Γ can be assumed coconnected.

2. Γ can be assumed to be reduced - that is, $x^{\perp} = y^{\perp}$ implies $x = y$.

But there is also a third kind of reduction. We define an equivalence relation θ on the vertex set of Γ by writing $x\theta y$ if and only if the sets $\Gamma(x)$ and $\Gamma(y)$ of vertices adjacent to x and y, respectively, coincide. Then θ-equivalence classes are cocliques, and for any two distinct θ-classes, either there are no edges between them or else all possible edges between them exist, so that any two of them together form either a larger coclique or a complete bipartite graph. In the latter case we say that the classes are "adjacent". There is an obvious graph homomorphism $\theta\colon \Gamma \to \theta\Gamma$, taking each vertex to its θ-class. One has

Γ has the Fischer property if and only if $\theta\Gamma$ does.

This means that the isomorphism type of $\theta\Gamma$ and the θ-class cardinalities determine the isomorphism type of Γ. Thus it suffices to classify graphs with the Fischer property only in the case that they are coconnected, reduced, and θ-reduced (i.e., all θ-classes are single-vertex sets) - a combination of conditions which we will call "**irreducible**".

At the moment, it seems hopeless to classify irreducible graphs having the Fischer property. But there is hope, if some assertion is made about cotriangle-closed subgraphs which contain three points.

As an earlier step in this direction, Jon Hall noticed in [16] that there is a very close relation between the following three sorts of objects:

I. Graphs which are locally a $3 \times d_x$ grid (d_x depending on the local vertex x).

II. Partial linear spaces with three points on each line, and every pair of intersecting lines generating a dual affine plane of order 2.

III. 3-transposition groups (G, D) with no non-trivial normal solvable subgroups in which no three elements of D generates a subgroup of order 18 or 54.

This three fold correspondence arose in connection with his proof of the infinite version of the Cotriangle Theorem.

The linear spaces of II result when we regard the system of cotriangles of a reduced indecomposable graph as the lines of the partial linear space. The fact that two intersecting lines generate a dual affine plane follows from the cotriangle property and the fact the graph is reduced.

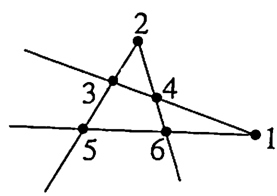
If one hopes for such a correspondence between groups and geometries in the case that the cotriangle property is replaced by the weaker Fischer property, the structure of the "planes" generated by two intersecting lines will have to be hypothesized since they are no longer deducible from the Fischer property alone.

A point-line geometry $\Pi = (\mathcal{P}, \mathcal{L})$ is a rank 2 incidence system of so-called "points" $\mathcal{P}$, and "lines", $\mathcal{L}$, such that each line is incident with at least two distinct points. Π is called a **partial linear space** if and only if distinct lines are incident with at most one common point. A **subspace** is a subset S of $\mathcal{P}$ such that any line having at least two of its incident points in S, in fact have all of their incident points in the set S. Each subspace S can be regarded as a point line geometry of its own, with its lines being those lines of $\mathcal{L}$ with all incident points in S. Clearly the intersection of two subspaces is a subspace. If X is a subset of P, the **subspace generated by** X, and denoted $\langle X \rangle$, is the intersection of all subspaces of Π containing X.

There are two point-line geometries of special interest to us here:

The affine plane of order 3 - denoted $AG(2,3)$ - has as its points the 9 vectors of a 2-dimensional vector space over $GF(3)$, and as its lines the 12 additive cosets (or translates) of its 1-dimensional subspaces.

The dual affine plane of order 2 is the system of 6 points and 4 lines drawn below:



A **Fischer space** is a point-line geometry $\Pi = (\mathcal{P}, \mathcal{L})$ such that

(Fi) Π is a partial linear space with three points per line; and

(Fii) The subspace generated by any two intersecting lines is either an affine plane of order 3 or a dual affine plane of order 2.

This notion was introduced by Francis Buekenhout in [3].

A **3-transposition group** (G, D) is a group G generated by a conjugacy class of involutions (elements of order two) such that for any two distinct involutions d and e of D, the product de is an element of order 2 or 3 .

The connection between Fischer spaces and 3-transposition groups with no noncentral solvable normal subgroups

First suppose Π is a Fischer space. For each point u in Π let t_u be that permutation of the points of Π which fixes u and takes each point v collinear with but distinct from u, to the unique third point of the line on u and v, and which fixes every point of Π which is not collinear with u. Then t_u is well defined since Π is a partial linear space. The precise importance of the two types of planes (affine over $GF(3)$ or dual affine over $GF(2)$) is that because of them, t_u preserves the full set of lines $\mathcal{L}$ of the Fischer space Π. Thus t_u is an automorphism of Π, of order 2. Now the following can easily be shown:

If u and v are not collinear, then t_u and t_v commute; otherwise $t_u t_v$ is an automorphism of the graph Γ of order 3

Thus letting G be the subgroup of $Aut(\Gamma)$ generated by all t_v as v ranges over the vertex set of Γ, G will be a 3-transposition group, provided D is just one conjugate class in G. Being coconnected and θ-reduced will insure this.

Conversely, suppose (G, D) is a 3-transposition group with no normal non-central solvable subgroups. Let $\mathcal{L}$ be the collection of triplets $T = \{d, e, f\}$ in D, belonging to a Sym(3) generated by any two non-commuting involutions d and e in D. Then $\Pi = (D, \mathcal{L})$ is a partial linear space.

Here is where the group theory comes in! When one examines what is generated by three elements of D one comes up with the 6 transpositions of Sym(4) in two different ways, or the group $3^{1+2}:2$ (called $SU(3,2)'$ by J.I. Hall) which is an extension of an extraspecial 3-group T' of order 3^3 by an involution which inverts the Frattini-factor group of T' while centralizing its center. The two commuting graphs with their attendant cotriangles are the dual affine plane over $GF(2)$ and the affine plane of order 3. Thus (D, all $\{0,1,3\}$-cotriangles) is a Fischer space.

And finally, we come back to the analog of cotriangular graphs. The graph theorist studying the complement of the collinearity graph of a Fischer space is in fact studying a special sort of irreducible graph with the Fischer property - ones for which intersecting cotriangles generate the cotriangle closed 6- and 9-vertex subgraphs corresponding to the two classes of planes. We call such a graph, an **irreducible Fischer graph.** We call a graph Γ a **Fischer graph** if and only if it is a graph with the Fischer property which after θ-reduction and $*$-reduction becomes an irreducible Fischer graph in the sense of the previous sentence.

So in one sense Fischer spaces, reduced Fischer graphs, and certain 3-transposition groups are all the same. But in the sense of classifying objects in their natural contexts they are not the same at all.

1. For the group G involved in a 3-transposition group (G, D), G may well have a center $Z(G)$ which is simply invisible in terms of the action on the irreducible Fischer graph or its Fischer geometry. But as a group-theorist one is obligated to classify G and hence describe its center.

2. If the Fischer graph is not $*$-irreducible, the group theorist is still obligated to describe the non-central normal 2-subgroup structure of G,

while the geometers studying the Fischer space $\Pi(G, D)$ may ignore the center of G, but is still obligated to detail the line structure of any normal 2-subgroup of $G/Z(G)$. (A beautiful series of paper by J.I. Hall, in fact fulfilled the group-theoretical obligation here ([14, [15]).) The graph theorist studying the Fischer graph Γ, by contrast, may ignore all of the solvable normal subgroup structure in classifying the graphs.

3. If the Fischer graph is not θ-reducible, its isomorphism type can be determined from Γ and some cardinality commitments. But for every two vertices in a θ-class of a $*$-reduced graph Γ the unique third member of the cotriangle must live in the same θ-class. Thus each θ-class of Γ is itself a Fischer space of Moufang type (i.e., a linear space in which all planes are affine of order three). For the geometers or group-theorists, this structure must be described, while for the graph-theorist it is merely a totally disconnected graph of some unknown cardinality.

The Classification of Fischer Spaces. In the context of graphs with geometric properties we are interested here only with irreducible Fischer graphs. But it is in fact impossible to describe either the conclusions or the proof without mentioning groups.

In fact, historically everything started the other way around. 3-transposition groups were studied long before there were any Fischer spaces, Fischer graphs or even a cotriangle theorem. In fact, the finite groups involved above were completely classified by Bernd Fischer nearly 24 years ago. Without much ado, the theorem is as follows:

Fischer's 3-Transposition Theorem. *Let (G, D) be a finite 3-transposition group with no non-central solvable subgroups. Then we may identify G with one of the following containments of a class D and a subgroup G:*

(1) The transposition class of a symmetric group;

(2) The transvection class of $Sp(V, B)$ where B is a nondegenerate symplectic form B on V over $GF(2)$;

(3) The class of transvections of a non-degenerate orthogonal geometry over $GF(2)$;

(4) A conjugacy class of reflections of a nondegenerate orthogonal group over $GF(3)$;

(5) The transvection class of a nondegenerate unitary space over $GF(4)$;

(6) A unique class of involutions in one of the five finite groups $\Omega^+(8, 2)$: Sym(3), $\Omega^+(8, 3)$: Sym(3), Fi_{22}, Fi_{23}, or Fi_{24}.

Obviously, from the correspondences given above, this is already enough to classify all finite irreducible Fischer graphs. However, it is my intention here to present the master theorem and a rough plan of its proof as presented by Jon Hall and Hans Cuypers in [10]. (Of course, by this late date the proof has incorporated quite a few efforts of many authors to streamline, and extend the original proof.)

The General 3-Transposition Theorem (Cuypers-Hall)

(Statement) Same as Fischer's theorem except that in the hypothesis the word "finite" is removed and in conclusions (1) through (5) the natural modules for the polar spaces are allowed to be infinite dimensional.

The proof only uses group-theory in a heavy way in getting the conclusions at step (6). There is no way any reformer's proof is going to get around Fischer's innumerable steps - either the basic ones, or the specific ones - which characterize the three sporadic Fischer Groups.

Obviously any proof of the infinite (irreducible) 3-transposition theorem will yield as a dividend the classification of the irreducible Fischer graphs.

Classification Theorem For Irreducible Fischer Graphs. *Let Γ be an irreducible Fischer graph. Then Γ is one of the following:*

(1) Γ is **symplectic type**, that is it is a conclusion of Hall's infinite version of the cotriangle theorem, $Sp(V)$, $N(V)$, $T(\Omega)$, or one of the finite versions in the finite case.

(2) Γ is the graph of the "perp" relation on non-singular 1-spaces of square norm or else all of those of non-square norm in some (possibly infinite dimensional) nondegenerate orthogonal space V over $GF(3)$.

(3) Γ is the "commuting graph on unitary transvections" - that is, the vertices are the singular 1-spaces of a (possibly infinite dimensional) non-degenerate $GF(4)$ - space with respect to a Hermitian form h. The lines are the 3 singular points of any 2-space with a trivial radical.

(4) One of the finite graphs appearing in conclusion (5) of Fischer's 3-transposition theorem.

Historical Note: By now it should be obvious that the cotriangle theorem mentioned above is but a small case of the Fischer graph theorem we have just displayed. Moreover, Fischer's theorem on 3-transposition groups already existed a good 3 years before the cotriangle theorem. So what was the point of even proving the cotriangle theorem? There were several:

(1) At that time, Fischer's theorem (because of its general scope) was a long hard way to prove something which enjoyed a simpler life in one of its special subcases.

(2) If there were ever to be a generalization of the cotriangle theorem in a direction away from the already-existing Fischer groups, it should be on the grounds of geometric properties alone. (God does not normally give the geometers a group to work with. So it made sense to ask why these things existed.)

The scheme behind the proof. As already mentioned, it suffices only to examine the Cuypers-Hall 3-transposition theorem. One must first sort out what the small cases are. At a low-dimensional level much of this can be calculated on any modern Computational Algebra program. We have already described the subgroups generated by three "independent" generators from D. They were the involution commuting graphs of (the transpositions in Sym(4) and involutions of $SU(3,2)' = 3^{1+2}: 2$. But similar computations show that the exceptional (not Moufang or irreducible) conclusions of Fischer's theorem which could result from a subgroup of G generated by at most 5 elements of D comprise a finite list of finite groups. But those which are critical in forming the right case divisions are in the following DIAGNOSTIC LIST:

(1) $2^{1+6}: SU(3,2)'$

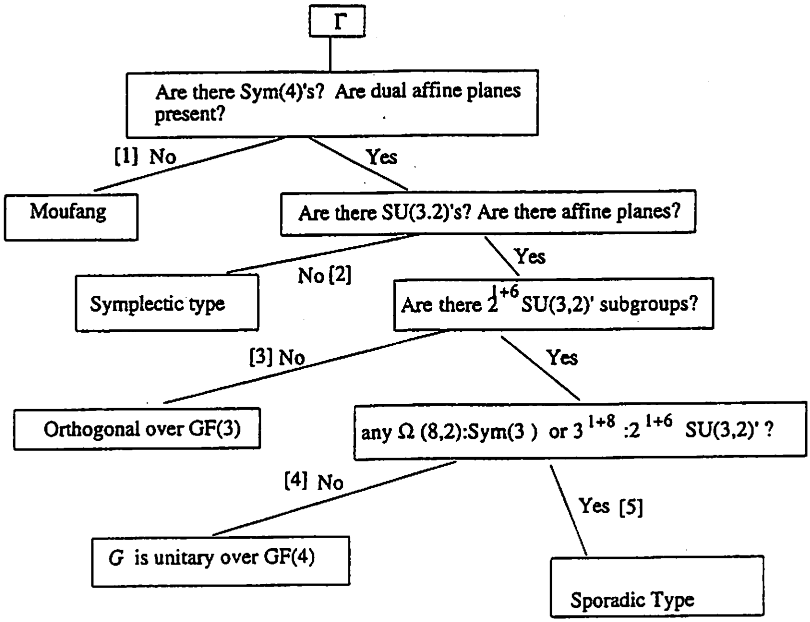
(2) $\Omega^+(8,2)$: Sym(3)

(3) $(3^8+3^8): 2^{1+6}: SU(3,2)'$ or its quotient

(4) $3^8: 2^{1+6}: SU(3,2)'$.

Now the proof proceeds according to the tree-flow-chart as given below. In this figure, an option in advancing downwards is in almost all cases given by a dichotomy that can be described in both group-theoretic and geometric terms. A number in brackets indicates the presence of a theorem classifying all graphs/groups in this case division, so that no further descent from this leaf is necessary. I will try to describe these theorems - many of which only use the group theory lightly.

Here is the tree:



Each stem which reaches a leaf (end-node) in this tree requires a complete classification of everything which falls into such a *cul de sac.* We have numbered these.

[1] This does not concern us graph theorists. The graph simply has no edges. However the group-theory problem here is quite complex. One has a 3-transposition group which is 2-nilpotent - in effect an involution acts on a 3-group so that its conjugacy class in the semidirect product is a 3-transposition class. In fact, this is how the whole 3-transposition problem started: Moufang loops of exponent 3 studied by Marshall Hall.

[2] This is exactly the hypothesis of the infinite version of the cotriangle theorem quoted above ([16]). So it really has a geometric proof.

[3] The way this works is that there is a provable "extended parallel relation" on lines, and the parallel classes themselves have the structure of a polar space over $GF(3)$. From the polar space theorem (stated in a previous section) there is no restriction on the dimension of this space. The actual orthogonal $GF(3)$ space can be reconstructed up to a choice of a class of non-singular 1-subspaces of this polar space.

[4] This step depends on one of those streamlines of a brilliant first theorem (Fischer's) ten years earlier. Weiss gave a not too involved proof that if for some involution d in D, the "sub-3-transposition group" $\langle D \cap C_G(d) \rangle = H$ possesses a non-central normal 2-subgroup then Γ is symplectic or unitary type.

It should be remarked that since the publication of [10], one doesn't need the subgroup $3^{1+8}: 2^{1+6}SU(3,2)'$ in the dichotomy of steps [4] and [5]. When this group is present, so is the former group $\Omega(8,2)$: Sym(3).

[5] For what is left it can be shown that things don't go on forever with one known 3-transposition group perpetually being a local version of a "higher dimensional" parent. So - because things cannot extend beyond the case that a perp of a triangle is centrally of type $U(6,2)$ - one is completely led back to the finite cases classified by Fischer.

I have to admit that this is a strange proof. Normally one expects a tree to *eliminate* the *existence* of pathological subspaces so that at the end of the proof, one has a very tight situation in which the non-existence of these subspaces can be considered as axioms forbidding subconfigurations.

But in the proof diagramed above it is altogether different. Every one of the *eliminated* cases are of the form that **certain subspaces don't exist.** That is, at each stage we know that in the surviving geometry or graph at least one subspace of a certain type exists. This means that after a series of "No" answers, one is left with a fairly complicated space: it *must* have at least a certain complexity in its subspaces. Far from making the final cases hard to classify, they are in fact made easy to classify because somehow the subspaces whose attendance is required don't *mesh.* We shall meet this same phenomenon in the next section on Generalized Fischer spaces. There ought to be a name for this *too-many-incompatible-subspaces* phenomenon.

7 Generalized Fischer Spaces

An obvious generalization of a Fischer graph would be this graph-theoretic property:

Every 2-coclique $\{u, v\}$ of the graph Γ lies in a unique coclique $C = C(u, v)$ of at least three vertices, such that every vertex x not in C is adjacent to either 0, exactly one, or all the vertices of C.

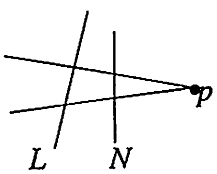
Just as the irreducible Fischer graphs could be converted to Fischer spaces by considering the complementary graph, so also the above property has a corresponding "space" that goes with it:

A partial linear space $\Pi = (\mathcal{P}, \mathcal{L})$ is called a **delta space** if for every line L and point p not on L, p is collinear with no, all but one, or all points of L.

It can also be shown that if the delta space has a connected collinearity graph, that all lines have the same cardinality q.

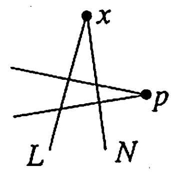
The problem here is that we don't know what all the planes should look like. In the case of Fischer spaces we knew that there had to be just two types of planes because we could classify the subgroups generated by just three 3-transpositions. But when q is larger than 3, there is no group around to tell us what the planes are.

In fact, Hans Cuypers showed that if the lines of a delta space satisfy the so-called Pasch axiom



The configuration

implies



L and N intersect

then all the planes (that is, subspaces generated by two intersecting lines) are either projective planes or dual affine plans.

In this case, if only projective planes occur, then Π is simply a protective space by the celebrated theorem of Veblen and Young.

If only dual affine planes occur, then Π is a copolar space. An infinite version of Hall's copolar space theorem was then proved by Cuypers to show that Π is the geometry of points and hyperbolic lines of some nondegenerate symplectic geometry, or q is 3 and the complement of the collinearity graph is a cotriangular graph and hence known.

But if both types of planes appear Cuypers has shown that Π is a projective space with all the points of a subspace S of codimension at least 3 removed and all lines which non-trivially intersect this subspace removed as well.

But these are not the planes which make their appearance in Fischer spaces. Accordingly, Hans Cuypers has defined a generalized Fischer space to be a connected point-line geometry $\Pi = (\mathcal{P}, \mathcal{L})$ such that any two intersecting lines generate either an affine plane or a dual affine plane.

Classification Theorem For Generalized Fischer Spaces. *A generalized Fischer space must be one of the following:*

(i) A Fischer space (B. Fischer, [12], H. Cuypers and J. Hall, [10]).

(ii) An affine space (F. Beukenhout, [2]).

(iii) The geometry of hyperbolic lines of a non-degenerate symplectic space ([15], [8]).

(iv) $NU(n,2)$, $q=4$ (H. Cuypers, [5] and Cuypers and Shult, [9]).

Again the proof is a kind of sieve:

If $q = 3$, Π is known by the classification of Fischer spaces.

If only affine planes appear, since $q > 3$, (ii) holds.

If only dual affine planes appear, case (iii) results. (Note that the reference to Cuypers ([5]) yields an infinite version of J. Hall's copolar theorem with specified planes.)

So we must have both kinds of planes.

We say Π has property (P) if and only if for any point p not in one of the affine planes A, p cannot be collinear with all but one point of A. In this case Cuypers showed that Π has as its points the non-singular 1-spaces of an n-dimensional vector space over $GF(4)$ with respect to a non-degenerate Hermitian form. The lines are the quartets of non-singular points belonging to any 2-space with a 1-dimensional radical - i.e., any projective line tangent to the Hermitian variety.

Cuypers and Shult ([9]) then showed that if $q > 3$, no generalized Fischer space can fail to have property (P). This completes the classification.

8 Graphs with Odd Cocliques

Let Γ be a graph. A subset A of the vertex set is called **odd** if $|x^{\perp} \cap A|$ is odd for all vertices x of Γ. Another variation of the cotriangle property is the following

Odd d-Coclique Property: Fix an integer $d > 1$.

$(Cc)_d$ Each $(d-1)$-coclique is contained in some odd d coclique.

$(C1)_d$ There exists an odd d-coclique C and vertex x not in C, such that x is collinear with exactly one vertex of C.

As easily seen, a graph Γ has the odd d-coclique property if and only if each of its coconnected components Γ_i has it, and if its canonical reduced image Γ^* has it. Thus in classifying graphs with the odd d-coclique property it may be assumed without loss that Γ is an irreducible graph, that is, a reduced coconnected graph.

When $d = 3$, such a graph Γ is a connected irreducible graph with the cotriangle property, and so has been classified up to isomorphism by earlier results in this survey.

Theorem (A. Brouwer and E. Shult [1]). *Let Γ be a finite irreducible graph. Then Γ possesses the odd d-coclique property for $d \geq 4$ if and only if Γ is isomorphic to one of the following graphs.*

1. $VO^{\epsilon}(m,2)$ and either $d = 0 \pmod 8$, $m \geq d+1-\epsilon$, or $d = 4 \pmod 8$, $m \geq d+1+\epsilon$.

2. $O^{\epsilon}(m,2)$ and either $d = 1 \pmod 8$, $m \geq d+2-\epsilon$, or $d = 5 \pmod 8$, $m \geq d+2+\epsilon$.

3. $T(O^{\epsilon}(m,2))$ and either $d = 2 \pmod 8$, $m \geq d+1-\epsilon$, or $d = 6 \pmod 8$, $m \geq d+1+\epsilon$.

4. $N^{\epsilon}(2n,2)$ and either $d = 3 \pmod 8$, $2n \geq d+2+\epsilon$, or $d = 7 \pmod 8$, $2n \geq d+1-\epsilon$.

Explanation of the graphs

Let (V, Q) be an m-dimensional vector space over $GF(2)$ with a non-degenerate quadratic form. This means that either m is an even dimension, and the associated symplectic form B is non-degenerate, or else m is an odd dimension and the radical of the symplectic form B is 1-dimensional and non-singular. In the latter case the associated polar space is isomorphic to the polar space of the symplectic geometry $Sp(m-1,2)$.

In case 2, the graphs $O^{\epsilon}(m,2)$ have the Q-singular 1-spaces of V as vertex set, two vertices being adjacent if and only if they are perpendicular with respect to the bilinear form B. Thus when m is odd, we get the graph $Sp(m-1,2)$ defined earlier. In fact, the graphs in this case are precisely the graphs of the **triangle theorem**.

In case 4, the graphs are the perp-graphs on the non-radical non-singular 1-spaces of V. When m is an odd dimension, the graph is again $Sp(m-1,2)$. These are precisely the graphs of the **cotriangle theorem**, with the exception of the dual triangular graphs.

In case 1, the vertices are the actual vectors of the space V, two of them being adjacent if and only if they differ by a nonzero singular vector.

Now let Δ be any connected graph. The **Taylor graph** $T(\Delta)$ is formed as follows:

1. One forms an isomorphic copy Δ' of Δ with the isomorphic image of vertex x being denoted x'. The vertices of $T(\Delta)$ are the disjoint unions of the vertices of Δ and Δ' and two new vertices α and α'.

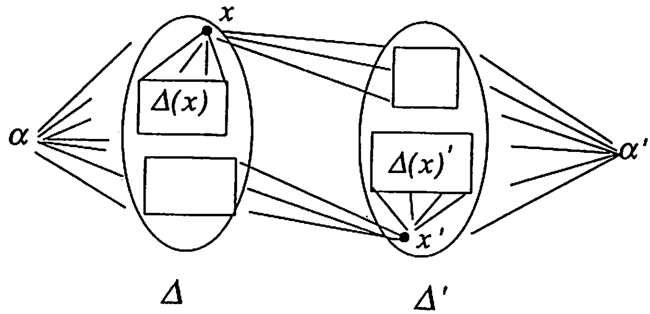
2. The edges of $T(\Delta)$ are

   (i) the edges of Δ and Δ' so that these are induced subgraphs of $T(\Delta)$.

   (ii) α is adjacent to every vertex of Δ; α' is adjacent to every vertex of Δ'.

   (iii) A vertex x in Δ is adjacent to a vertex y' if and only if x and y are distinct non-adjacent vertices of Δ.

Such a graph is a double cover of the complete graph. Clearly if Δ is the complete graph, then $T(\Delta)$ is just two copies of the complete graph $\{\alpha\} + \Delta$. Otherwise, each vertex z has a unique antipodal vertex z' and the graph looks like this:



Now in case 3, we are just forming the Taylor graph construction when Δ is one of the three graphs of the triangle theorem.

Remark: The proof consists in showing that Γ is uniquely determined by the subgraph $C \cup C_a$ where C is an odd d-coclique and for a vertex a in C, C_a is all exterior vertices adjacent only to a in C. Because there is now an infinite version of the cotriangle theorem due to Jon Hall ([16]), induction on d and the proof of uniqueness doesn't use finiteness, we see that *the condition on finiteness can be dropped in the above theorem.*

This only means that in the conclusion, m is now a possibly infinite cardinal. (Of course, when m is infinite, the distinction between $\epsilon = +1$ and $\epsilon = -1$, in the orthogonal geometries disappears.)

We close this survey with a query. Can anyone give an explanation (from the "first principle" of the odd d-coclique property itself) of the unusual and totally unexpected periodicity modulo 4 in the types of graphs which appear in the conclusion of the odd coclique theorem?

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