A Few More Z-Cyclic Whist Tournaments

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ABSTRACT. Z-cyclic whist tournaments for $q+1$ players, $Wh(q+1)$, where q is a prime, $q \equiv 3 \pmod 4$, are quite rare. Solutions for $q = 3, 7, 11, 19, 23$, and 31 were known in the early to mid 1890's. Since that time no new such $Wh(q+1)$ have appeared. Here we present Z-cyclic $Wh(q+1)$ for $q = 43, 47, 59$. Also presented for the first time is a Z-cyclic $Wh(45)$ and a Z-cyclic $Wh(40)$ that has the three person property. All of these results were obtained via the computer.

1 Introduction

A whist tournament on v players, denoted by $Wh(v)$, is a resolvable $(v, 4, 3)$ balanced incomplete block design (BIBD) wherein each block (a, b, c, d) designates that the unordered pairs $\{a, c\}$ and $\{b, d\}$ denote partnerships and that all other (unordered) pairs denote opponents. The design is subject to the conditions that each player partners every other player exactly once and opposes every other player exactly twice.

Example 1.1. A $Wh(4)$ is given by the following three (3) blocks (alt. tables or games). Note that each block constitutes a parallel class (alt. round) of the BIBD (a, b, c, d), (a, c, b, d), (a, b, d, c).

It has been known since the late 1970's that $Wh(v)$ exist if and only if $v \equiv 0, 1 \pmod 4$ [1, 3, 5]. For a recent survey of the Whist Tournament Problem see [2]. A problem of current interest relates to the existence of Z-cyclic $Wh(v)$, where by Z-cyclic is meant that (i) if $v \equiv 0 \pmod 4$ the players are elements in $Z_{v-1} \cup \{\infty\}$ and if $v \equiv 1 \pmod 4$ the players are elements in Z_v, and (ii) round $j + 1$ is obtained by adding $+1 \pmod N$ to each non-∞ element in round j. Here $N = v - 1$ if $v \equiv 0 \pmod 4$ and $N = v$ if $v \equiv 1 \pmod 4$. We note that for Z-cyclic $Wh(v)$ it is enough to indicate the games for an initial round, for the remaining rounds can

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be obtained by developing this initial round (mod N), with N as above. Furthermore for a cyclic resolvable BIBD any parallel class can be taken as initial round. For Z-cyclic $Wh(v)$ it is conventional to take as initial round that which omits 0 when $v \equiv 1 \pmod 4$ and that for which ∞ and 0 are partners when $v \equiv 0 \pmod 4$. The method of differences [1] can be employed to verify that a set of games serves as an initial round of a Z-cyclic $Wh(v)$.

Example 1.2. The following are the initial rounds of a Z-cyclic $Wh(4)$, $Wh(5)$, and $Wh(8)$, respectively.

$$(\infty, 1, 0, 2); (1, 3, 4, 2); (\infty, 4, 0, 5), (1, 2, 3, 6).$$

Z-cyclic $Wh(q + 1)$, q a prime, $q \equiv 3 \pmod 4$ are quite rare. Examples for $q = 3$, 7, 11, 19, 23, and 31 were known by 1897 [6, 7] but no new such $Wh(v)$ have appeared since. In Section 2 we provide Z-cyclic $Wh(v)$, $v = 44$, 48, and 60 corresponding to $q = 43$, 47, and 59 respectively. We also present for the first time an example of a Z-cyclic $Wh(45)$ and a $3P$ Z-cyclic $Wh(40)$. The latter example extends the existence of $3P$ $Wh(v)$ [4]. All of these results were generated on a computer.

2 New Z-Cyclic $Wh(v)$

In each of the following examples we give the initial round of the corresponding $Wh(v)$.

Example 2.1. $Wh(44)$.

$$(\infty, 20, 0, 35), (13, 30, 29, 19), (12, 17, 31, 38), (25, 1, 32, 2),$$
$$(3, 14, 5, 24), (33, 28, 16, 8), (21, 18, 39, 23), (27, 42, 36, 34),$$
$$(7, 41, 37, 4), (10, 6, 22, 9), (11, 40, 15, 26).$$

Example 2.2. $Wh(48)$.

$$(\infty, 45, 0, 17), (36, 9, 20, 15), (18, 1, 33, 5), (6, 19, 7, 29),$$
$$(43, 46, 21, 35), (32, 39, 27, 22), (38, 40, 12, 28), (37, 16, 24, 23),$$
$$(25, 41, 34, 2), (42, 31, 13, 4), (44, 3, 30, 26), (8, 11, 10, 14).$$

Example 2.3. $Wh(60)$.

$$(\infty, 33, 0, 34), (21, 3, 23, 15), (4, 31, 17, 58), (25, 42, 53, 56),$$
$$(49, 36, 5, 45), (1, 14, 11, 20), (40, 24, 47, 32), (7, 9, 18, 28),$$
$$(37, 8, 57, 12), (43, 55, 48, 19), (26, 2, 50, 27), (13, 6, 39, 35),$$
$$(41, 29, 44, 46), (51, 30, 10, 52), (22, 54, 38, 16).$$

Example 2.4. $Wh(45)$.

$$(1,12,2,40),(32,3,25,5),(27,10,36,22),(44,34,31,19),$$
$$(35,20,29,43),(14,42,24,13),(37,33,26,7),(6,11,9,15),$$
$$(18,21,23,39),(16,17,30,38),(41,28,4,8).$$

A whist tournament is said to have the Three Person Property (3P) if the intersection of any two tables in the tournament is at most two (2). Each of the $Wh(44)$, $Wh(48)$, $Wh(60)$ listed above have this property. The following is an example of a $Wh(40)$ that has the Three Person Property. Thus Examples 2.1, 2.2, 2.3, and 2.5 extend the existence of $3P$ $Wh(v)$ [4].

Example 2.5. $3P$ $Wh(40)$.

$$(\infty,24,0,30),(13,21,34,14),(32,37,19,26),(1,3,2,5),$$
$$(6,38,9,28),(17,22,25,8),(7,18,29,23),(27,11,36,15),$$
$$(4,31,16,12),(35,33,20,10).$$

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