A hundred years of whist tournaments

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ABSTRACT. The first serious mathematical study of whist tournament designs was carried out in the 1890s by E.H. Moore. In this survey I shall outline briefly the subsequent work which culminated in the proof of the existence of whist tournaments of all possible orders by Baker, Wilson and Hanani in the 1970s, and then describe some more recent work, mainly by N.J. Finizio, Y.S. Liaw and the author, on the construction of cyclic whist tournaments. In particular, triple whist tournaments will be discussed.

1 A brief history

A whist tournament $Wh(4n)$ for $4n$ players is a schedule of games, each involving two players playing against two others, such that

(i) the games are arranged into $4n-1$ rounds, each of n games,

(ii) each player plays in exactly one game in each round,

(iii) each player partners every other player exactly once,

(iv) each player opposes every other player exactly twice.

Conditions (iii) and (iv) are called the *whist conditions.* We denote each game by a 4-tuple (a, b, c, d) in which the pairs $\{a, c\}$, $\{b, d\}$ designate partnerships, and the pairs $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{d, a\}$ designate opponent pairs. It is convenient to think of a, b, c, d as the order of the four players round a table; partners face each other across the table.

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Example 1.1 A $Wh(8)$, with players $\infty, 0, 1, \ldots, 6$.

Round 1	$(\infty, 4, 0, 5)$	(1,2,3,6)
Round 2	$(\infty, 5, 1, 6)$	(2,3,4,0)
⋮		
Round 7	$(\infty, 3, 6, 4)$	(0,1,2,5)

The mathematical study of whist tournaments apparently started in the 1890s. The journal 'Whist' first appeared in 1891, and at the same time J.T. Mitchell [41] published the first edition of his work "Duplicate Whist". These, along with Foster's 1894 book [32] and the second edition of Mitchell's book, provide examples of $Wh(4n)$ for all $n \leq 10$, most of the examples being due to Mitchell, Safford, Howell and Whitfeld (see Section 5). Most of them possess the *cyclic* structure possessed by Example 1.1; each round is obtained from the previous one by adding 1 (mod $4n-1$) to each non-∞ element. We shall call such a $Wh(4n)$ a *Z-cyclic* whist tournament, emphasizing the fact that the cyclic structure is over $\mathbf{Z}_{4n-1}$, the integers modulo $4n-1$, and not over a Galois field $GF(p^\alpha)$ as is often the case in combinatorial constructions.

In 1896, E.H. Moore, head of the newly founded mathematics department at Chicago, published his remarkable paper *Tactical Memoranda* I-III [42] in which, among many other things, he proved the existence of $Wh(4n)$ whenever $4n = 3p + 1$ (p prime, $p \equiv 1 \pmod 4$) and whenever $4n = 2^\alpha (\alpha \geq 2)$. We shall return to Moore's work later, in Sections 2 and 6. It remained the only major contribution to the subject for over half a century. References to whist tournaments (but under the name of bridge tournaments, contrary to the now-accepted mathematical meaning of that term) appeared in various books on recreational mathematics; for example Dudeney, in problem 265 of *Amusements in Mathematics* [22] asks for (and gives) a (Z-cyclic) $Wh(12)$, and Rouse Ball [15] gives Z-cyclic $Wh(4n)$ for $n = 2, 3, 4$.

Closely tied to the definition of a $Wh(4n)$ is that of a $Wh(4n+1)$. In a $Wh(4n+1)$ there are $4n+1$ players and $4n+1$ rounds, and a different player 'sits out' in each round; conditions (iii) and (iv) remain unaltered. In a Z-cyclic $Wh(4n+1)$ there is no player ∞; the players are the elements of $\mathbf{Z}_{4n+1}$ and, conventionally, player 0 sits out in the first round.

Example 1.2 A Z-cyclic $Wh(13)$.

Round 1	(1,8,12,5)	(2,3,11,10)	(4,6,9,7)
Round 2	(2,9,0,6)	(3,4,12,11)	(5,7,10,8)
⋮			
Round 13	(0,7,11,4)	(1,2,10,9)	(3,5,8, 6)

It is not clear who first introduced the idea of $Wh(4n+1)$. The first general results of which I am aware appear in a short paper by G.L. Watson [46] in 1954, where it is proved that a $Wh(4n+1)$ exists (a) whenever $4n+1$ is a product of primes all of which are $\equiv 1 \pmod 4$, and (b) whenever $g.c.d.(n, 6) = 1$. Watson's tournaments in (a) are all Z-cyclic. He also showed that (c) a $Wh(4n)$ exists wherever $g.c.d.(n, 6) = 1$. These results seem to have remained unknown to subsequent researchers; for example (b) reappears in [12].

It was in the 1970s that the existence of a $Wh(4n)$ and a $Wh(4n+1)$ was established for all positive integers n. This work was carried out by R.M. Wilson, R.D. Baker and H. Hanani. The 'prime source' [14] was in fact never published, but an account of the $4n$ case can be found in [13], while both the $4n$ and $4n+1$ cases are treated in the theses of Baker [12] and Hartman [33] and also in the author's book [1].

The main ingredients of the constructions are the following.

(1) Direct construction of $Wh(q)$ where $q = p^\alpha$ is a prime power, $q \equiv 1 \pmod 4$, by using a primitive element θ of the Galois field $GF(q)$. If $q = 4t+1$, such a θ has multiplicative period $4t$, and the games

$$(1, \theta^t, \theta^{2t}, \theta^{3t}), (\theta, \theta^{t+1}, \theta^{2t+1}, \theta^{3t+1}), \ldots, (\theta^{t-1}, \theta^{2t-1}, \theta^{3t-1}, \theta^{4t-1})$$

i.e.

$$(1, \theta^t, -1, -\theta^t) \times 1, \theta, \theta^2, \ldots, \theta^{t-1} \tag{1.1}$$

form the initial round of a cyclic $Wh(p^\alpha)$ over $GF(p^\alpha)$.

Comment on (1). Watson's construction [46] agrees with this in the special case when $q = p$, but generalizes that special case in a different way. Construction (1.1) appears in Baker [12] and Baker and Wilson [14]. Prior to it, Bose and Cameron [18], in a paper dealing with the simpler problem when resolvability is not required (i.e. there is no requirement to group the games into rounds), gave an alternative approach to constructing a $Wh(q)$. By Mann's lemma [39], if θ is a primitive element of $GF(q)$, there exist *odd* integers α, β s.t. $\theta^\alpha + 1 = \theta^\beta(\theta^\alpha - 1)$. It then follows that the games

$$(1, \theta^\alpha, -1, -\theta^\alpha) \times 1, \theta^2, \theta^4, \ldots, \theta^{2t-2} \tag{1.2}$$

constitute the initial round of a cyclic $Wh(q)$.

Note that both (1.1) and (1.2) have, in the initial round, partnerships of the form $\{x, -x\}$. These pairs are said to form a *patterned starter* in the additive group of $GF(q)$. Whist tournaments with such partnerships are studied further by Finizio [25]; on the other hand, many similar constructions of $Wh(p)$ with non-patterned partnerships are provided by Liaw [37].

(2) Product Theorems. If a $Wh(v)$ and a $Wh(w)$ exist, then a $Wh(vw)$ exists.

Comment on (2). This is really three results, depending on whether v and w are $\equiv 0$ or $1 \pmod 4$.

Moore [42] also had a product theorem, but it was not so general. With v and w both $\equiv 0 \pmod 4$, he proved that a $Wh(vw)$ exists whenever a $Wh(v)$ exists and a $TWh(w)$ exists, where a $TWh(w)$ is a particular type of $Wh(v)$ known as a *triplewhist* tournament. Triplewhist tournaments will be discussed in Sections 2, 3 and 6.

(3) Using pairwise balanced designs. These are used as scaffolding on which to combine smaller whist tournaments into larger ones. A $PBD(K, v)$ is a collection of subsets (blocks) of a v-set, each block size being in the set K, such that each pair of elements occurs as a subset of precisely one block. For example, we can take a finite protective plane of order 4, i.e. a $PBD(\{5\}, 21)$, and from it construct a $Wh(21)$ by constructing a $Wh(5)$ on each block and arranging the games suitably in rounds. Similarly, a $Wh(v)$ can be constructed from a $Wh(29)$ and a $Wh(37)$ for all sufficiently large $v \equiv 1 \pmod 4$ since there exists [47] a $PBD(\{29, 37\}, v)$ for all sufficiently large $v \equiv 1 \pmod 4$.

Comment on (3). It was Moore [42] who gave the first example of such a construction. If we take a resolvable $(v, 4, 1)$ design where, necessarily, $v \equiv 4 \pmod{12}$, we can replace the blocks in each resolution class by three rounds of a $Wh(4)$ on each block, thereby obtaining three rounds of a $Wh(v)$.

More recently a lot of work has been done on the construction of *cyclic* whist tournaments, in particular **Z**-*cyclic* ones. In a series of papers, Anderson and Finizio [2-10], Finizio [23-31] and Liaw [37] have constructed many new families of **Z**-cyclic $Wh(4n)$ and $Wh(4n+1)$ designs; these constructions will be described in Sections 4-6.

2 Some special types of whist tournament

(2a) Directed whist tournaments. For such tournaments we replace the whist condition (iv) by:

(iv)(D): each player has every other player once as an opponent on his left, and once as an opponent on his right.

We use $DWh(v)$ to denote a directed whist tournament for v players, and observe that Baker's tournaments $Wh(p^\alpha)$ given by (1.1) are $DWh(p^\alpha)$. Directed whist tournaments have been widely studied under the alternative name of resolvable perfect Mendelsohn designs with block size 4. Indeed, Baker's construction (1.1) reappears in the context of Mendelsohn designs in Keedwell's 1984 paper [35].

In Baker's thesis [12] it is proved that a $DWh(4n+1)$ exists for all sufficiently large n, and that infinitely many $DWh(4n)$ exist. Following the work of Bennett [16] it is now known that a $DWh(4n+1)$ exists for all $n \geq 1$. Further, it has recently been shown by Zhang Xuebin [48] that a $DWh(4n)$ exists for all $n \geq 85$. Recently, Finizio [27] has commenced a study of Z-cyclic $DWh(4n+1)$, extending Baker's construction to $4n+1 = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, where each $p_i \equiv 1 \pmod 4$. It is stated in [14] that no Z-cyclic $DWh(4n)$ exists.

We also remark that Baker [12] found a Z-cyclic $DWh(33)$.

Example 2.1 Initial round of a Z-cyclic $DWh(33)$.

$$(25, 27, 14, 32,), (26, 6, 10, 29), (23, 21, 4, 13), (16, 12, 7, 15),$$
$$(3, 2, 24, 31), (5, 17, 20, 11), (30, 22, 28, 9), (18, 8, 19, 1).$$

(2b) Triplewhist tournaments. These go right back to Moore [42]. In the game (a, b, c, d) we shall say that a, b (and c, d) are *opponents of the first kind*, and a, d (and b, c) are *opponents of the second kind*. We then call $Wh(v)$ a *triplewhist tournament* $TWh(v)$ if condition (iv) is replaced by

(iv)(T): each player has every other player once as an opponent of the first kind, and once as an opponent of the second kind.

The name 'triplewhist' tournament arises because from it *three* whist tournaments can be derived; for the partner pairs, or the opponent pairs of the first kind, or the opponent pairs of the second kind, can be taken as partner pairs of a $Wh(v)$. Example 1.1 is in fact a $TWh(8)$, and Moore's $Wh(3p+1)$ and $Wh(4n)$ were all triplewhist tournaments. Moore proved that a $TWh(vw)$ exists whenever both a $TWh(v)$ and a $TWh(w)$ exist. Baker and Wilson [14] established that a $TWh(4n+1)$ exists for all sufficiently large n, and a $TWh(4n)$ exists for all sufficiently large $n \not\equiv 2 \pmod 4$. Since then it has been established that a $TWh(4n)$ exists for all n, except possibly for a tiny set of small values of n; I believe that the present list of uncertainties is 14,54,62,70. See section 7. In many of the cases the existence of a $TWh(4n)$ is deduced from the existence of certain self orthogonal latin squares; this connection is explained in Section 3 (Lemma 3.2).

Recently, Anderson and Finizio have studied Z-cyclic $TWh(vw)$. They have extended Moore's construction of a $TWh(3p+1)$ to obtain Z-cyclic $TWh(v)$ whenever $v = 3p^n + 1$ (p prime, $p \equiv 1 \pmod 4$) [4] and whenever $v = 3p_1^{\alpha_1} \dots p_r^{\alpha_r} + 1$ [6,7] whenever the p_i are *compatible* primes $\equiv 1 \pmod 4$, i.e. whenever each $p_i - 1$ is divisible by the *same* power of 2. They have also constructed many other families of Z-cyclic $TWh(4n)$ and $TWh(4n+1)$; these will be studied in Section 6.

(2c) Three-person whist tournaments. A whist tournament is said to have the three-person property if no two games have more than two players in common, i.e. if each set of three players plays together in at most one game. This property is first discussed in the context of block designs in a paper by Mendelsohn [40], and Finizio has used the ideas there to construct several infinite families of three person whist tournaments [24]. All Z-cyclic three person whist tournaments with $v \leq 21$ are listed by Finizio in [23]. Unfortunately, there does not seem to be a simple method of checking whether or not a given Z-cyclic $Wh(v)$ has the three-person property other than by systematic inspection of all blocks. Perhaps the first examples of three-person whist tournaments in the literature are the $Wh(20)$ and $Wh(32)$ given by Hartman in [34].

3 Connections with other combinatorial structures

In this section we describe connections between triple whist tournaments, self orthogonal latin squares with symmetric orthogonal mate (SOLSSOM) and spouse-avoiding mixed doubles round robin tournaments (SAMDRR).

A SAMDRR(n) for n couples is a tournament involving n husband and wife pairs; each game involves two players of opposite sex playing against two other players of opposite sex, and the games are arranged so that every two players of the same sex play against each other exactly once, and each player plays with each member of the opposite sex (excluding spouse) exactly once as partner and exactly once as opponent. A SAMDRR(n) is resolvable if the games can be arranged into $n-1$ rounds with each player playing in exactly one game per round (if n is even), and with every player except the ith husband and wife pair playing in the ith round (if n is odd).

Example 3.1 A resolvable SAMDRR(5).

Round 1	H_5W_3 v H_2W_4	H_4W_5 v H_3W_2
Round 2	H_1W_4 v H_3W_5	H_5W_1 v H_4W_3
Round 3	H_2W_5 v H_4W_1	H_1W_2 v H_5W_4
Round 4	H_3W_1 v H_5W_2	H_2W_3 v H_1W_5
Round 5	H_4W_2 v H_1W_3	H_3W_4 v H_2W_1

Lemma 3.1. *If a $TWh(v)$ exists then a resolvable SAMDRR(v) exists.*

Proof: Replace each game (a, b, c, d) by two games H_aW_c v H_bW_d, H_cW_a v H_dW_b. Note that opponent pairs of the first kind give same sex opponent pairs, while opponent pairs of the second kind give opposite sex opponent pairs. □

Note that the SAMDRR(v) of Lemma 3.1 has very special properties: e.g. if H_a, H_b oppose each other in a particular round, so do their wives

W_a, W_b. So one would not expect the converse of the lemma to be true. However, the following has proved to be particularly useful.

Lemma 3.2. *If a resolvable SAMDRR(n) exists then a $TWh(4n)$ exists.*

Proof: The proof depends on whether n is even or odd. Suppose first that n is odd. Suppose round t omits the pair (H_t, W_t), and is made up of games H_iW_l v H_jW_k. Then we obtain from round t the following 3 rounds of the required $TWh(4n)$:

(i) all games (i_1, l_3, j_1, k_3), (i_2, l_4, j_2, k_4), and (t_1, t_2, t_3, t_4);

(ii) all games (i_4, j_4, l_1, k_1), (i_3, j_3, l_2, k_2), and (t_1, t_3, t_4, t_2);

(iii) all games (i_1, k_2, l_2, j_1), (i_3, k_4, l_4, j_3), and (t_1, t_4, t_2, t_3).

This gives $3n$ rounds. We obtain $n-1$ further rounds by taking a resolvable transversal design $TD(4, n)$ with the jth group, $1 \le j \le 4$, consisting of all elements with suffix j, and with one resolution class consisting of all the blocks $\{i_1, i_2, i_3, i_4\}$, $i \le n$. From each of the remaining $n-1$ classes, obtain a round of the required $TWh(4n)$ by replacing each block $\{i_1, j_2, k_3, l_4\}$ by the game (i_1, l_4, k_3, j_2). □

The proof for n even is similar. These results appear in Baker's work ([12, 13]) although the actual presentation of the proof is a little different, being given in terms of self orthogonal latin squares (see the next lemma).

It is essentially due to the work of Baker and Wilson [14], Wallis [44] and Wang [45] on resolvable SAMDRR(v) that the existence of a $TWh(4n)$ with $n \equiv 2 \pmod 4$ has been confirmed in most cases. This, and related work, is often presented in terms of Latin squares, because of the following result, which extends ideas in [19].

Lemma 3.3 [12]. *A resolvable SAMDRR(n) exists if and only if there exist a self orthogonal Latin square SOLS(n) A of order n and a symmetric Latin square B orthogonal to both A and A^T, having constant diagonal if n is even and having the same diagonal as A if n is odd. (B is called a symmetric orthogonal mate of A.)*

Proof: First suppose a resolvable SAMDRR(n) exists with spouse pairs (H_i, W_i). Define $A = (a_{ij})_{n\times n}$ by

$$a_{ij} = \begin{cases} i & \text{if } i = j \\ l & \text{if } i \ne j \end{cases}$$

where W_l is the partner of H_i when H_i plays against H_j. Then A is a SOLS(n). Further, define $B = (b_{ij})_{n\times n}$ by

$$b_{ii} = \begin{cases} i & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases} \qquad b_{ij} = k \quad (i \ne j)$$

where H_i plays against H_j in the kth round.

Conversely, given A and B as in the statement of the lemma, relabel so that $a_{ii} = i$ for each i. Then for the games in round k take games H_iW_l v H_jW_m $(i \neq j)$ where $b_{ij} = k$, $a_{ij} = l$, $a_{ji} = m$. □

It follows immediately from Lemmas 3.1 and 3.3 that if a $TWh(v)$ exists then $N(v) \geq 3$, where $N(v)$ denotes the maximum value of k for which it is true that there exist k mutually orthogonal latin squares (MOLS) of order v. But more can be said.

Lemma 3.4 [12]. *If a Z-cyclic $TWh(v)$ exists, $v \equiv 1 \pmod 4$, then $N(v) \geq 4$.*

Proof: We certainly have 3 MOLS, A, A^T and B. Now construct C orthogonal to each. If v (or 0) is the player omitted in the initial round, define $c_{ii} = v$ for each $i \leq v$. Now consider $i \neq j$. There is an unique game (i, k, j, l) in which i partners j, and if this is in the $(r+1)$th round then it is in fact the game $(i' + r, k' + r, j' + r, l' + r)$ where (i', k', j', l') is in the initial round. Define $c_{ij} = i'$. □

A similar result holds for $DWh(v)$. Baker in fact used his Z-cyclic $DWh(33)$ (Example 2.1) to establish that $N(33) \geq 4$; this remains the 'world record' for $N(33)$.

Very recently, Finizio [30] has shown how useful the existence of Z-cyclic triplewhist tournaments is in the actual explicit construction of SOLS-SOMs.

4 Z-cyclic $Wh(4n + 1)$

Any discussion of such designs involves the idea of a starter. The pairs $\{a_1, b_1\}, \ldots, \{a_{2n}, b_{2n}\}$ of nonzero elements of $\mathbf{Z}_{4n+1}$ are said to form a *starter* in $\mathbf{Z}_{4n+1}$ if

$$(\alpha) \cup_{i=1}^{2n} \{a_i, b_i\} = \mathbf{Z}_{4n+1} - \{0\},$$
$$(\beta) \cup_{i=1}^{2n} \{\pm(a_i - b_i)\} = \mathbf{Z}_{4n+1} - \{0\}.$$

Similarly, we shall say that the pairs $\{a_1, b_1\}, \ldots, \{a_{4n}, b_{4n}\}$ of $\mathbf{Z}_{4n+1}$ form a *2-fold starter* if

(γ) the elements a_i, b_i are the nonzero elements of $\mathbf{Z}_{4n+1}$ each occurring twice,

(δ) the elements $\pm(a_i - b_i)$ are all the nonzero elements of $\mathbf{Z}_{4n+1}$ each twice.

If, by convention, we take the initial round of a Z-cyclic $Wh(4n + 1)$ on $\mathbf{Z}_{4n+1}$ to be the round in which 0 does not play, then the condition for

$$(a_1, b_1, c_1, d_1), \ldots, (a_n, b_n, c_n, d_n) \tag{4.1}$$

to be the initial round games for a **Z**-cyclic $Wh(4n+1)$ are precisely [1]

(ε) the pairs $\{a_i, c_i\}$, $\{b_i, d_i\}$, $1 \le i \le n$, form a starter,

(ζ) the pairs $\{a_i, b_i\}$, $\{b_i, c_i\}$, $\{c_i, d_i\}$, $\{d_i, a_i\}$, $1 \le i \le n$, form a 2-fold starter.

The $Wh(p)$ constructed by Baker, and the one constructed by Bose and Cameron, both use the patterned starter $\{1, -1\}, \ldots, \{2n, -2n\}$ in (ε), but there are many other possibilities. We recall the well known Mullin-Nemeth starter.

Lemma 4.1 [43]. *Let $p = 2^k t + 1$ be prime, t odd, and let $d = 2^{k-1}$. Let θ be a primitive root of p. Then the pairs*

$$\{\theta^{2id+j}, \theta^{(2i+1)d+j}\}, \quad 0 \le i \le t-1, 0 \le j \le d-1$$

form a starter in $\mathbf{Z}_p$.

Liaw [37] has studied $Wh(p)$ arising from non-patterned starters. Consider, for example, the case $p = 4t+1$, i.e. when $k = 2$. Then the games

$$(1, \theta^{4e+1}, \theta^a, \theta^{4e+1+b}) \times 1, \theta^4, \theta^8, \ldots, \theta^{4(t-1)}$$

form the initial round of a **Z**-cyclic $Wh(p)$ provided

(η) $a, b \equiv 2 \pmod 4$ or $a \equiv 3$, $b \equiv 1 \pmod 4$,

(i) $(\theta^a - 1)\ (\theta^b - 1)$ is a square $\pmod p$,

(κ) precisely two of $\theta^{4e+1} - 1$, $\theta^{4e+1} - \theta^a$, $\theta^{4e+b+1} - 1$, $\theta^{4e+b+1} - \theta^a$ are squares.

Case (a). By choosing $e = 0$, $a = b = 2$ we get the initial games

$$(1, \theta, \theta^2, \theta^3) \times 1, \theta^4, \ldots, \theta^{4(t-1)}$$

(in which the partner pairs are the Mullin-Nemeth starter pairs) provided (from (κ)) that $\theta^2 + \theta + 1$ is not a square in $\mathbf{Z}_p$. Now we quote

Lemma 4.2 (Cohen [20]). *Let $p \ge 211$ be prime and let $g(x)$ be a quadratic polynomial over $\mathbf{Z}_p$ not of the form $a(x+b)^2$ where a is a square in $\mathbf{Z}_p$. Then $g(\theta)$ is a non square for some primitive root θ of p.* □

We can thus choose θ so that $\theta^2+\theta+1$ is not a square, provided $p > 211$. For $p < 211$, such a θ can be found by inspection. Indeed, we can take pairs (p, θ_p) given by (13,2), (29,8), (37,5), (53,5), (61,2), (101,2), (109,10), (149,3), (157,6), (173,2), (181,2), (197,3).

Case (b). Choose $a = b = 2t$; we obtain the games

$$(1, \theta^{4e+1}, -1, -\theta^{4e+1}) \times 1, \theta^4, \ldots, \theta^{4(t-1)}$$

provided (from (κ)) that precisely one of $\theta^{4e-1} \pm 1$ is a square. This gives Bose-Cameron types of tournament.

Case (c). Choose $a = b = 2$; we obtain initial round games

$$(1, \theta^{4e+1}, \theta^2, \theta^{4e+3}) \times 1, \theta^4, \ldots, \theta^{4(t-1)}.$$

Here the partner pairs are again the Mullin-Nemeth starter pairs, but this time they are out of phase. The construction will work if, for example, we can find e such that

$$\theta^{4e+1} - 1 = \square, \theta^{4e-1} - 1 \neq \square, \theta^{4e+3} - 1 \neq \square.$$

Many such examples exist

Example 4.1 $Wh(29)$. Take $\theta = 2$, $e = 3$ in case (c) to get

$$(1, 14, 4, 27), (16, 21, 6, 26), (24, 17, 9, 10), (7, 11, 28, 15),$$
$$(25, 2, 13, 8), (23, 3, 5, 12), (20, 19, 22, 18).$$

What about Z-cyclic $Wh(4n+1)$ when $4n+1$ is not a prime? Watson's results [46] cover all cases where $4n+1$ is a product of primes each of which is $\equiv 1 \pmod 4$. For other cases, Finizio [23] has studied small values of n. No Z-cyclic $Wh(9)$ exists, but there are many Z-cyclic $Wh(21)$.

Example 4.2 [23] Initial round of a Z-cyclic $Wh(21)$.

$$(1, 4, 2, 18), (3, 12, 5, 16), (6, 8, 11, 17), (7, 14, 15, 20), (9, 10, 19, 13).$$

We have already observed (Example 2.1) that a Z-cyclic $Wh(33)$ exists. Recently Finizio has obtained a Z-cyclic $Wh(45)$ [31] and a $Wh(49)$ [29], and has shown [29] how to construct a Z-cyclic $Wh(q^2 p_1^{\alpha_1} \ldots p_r^{\alpha_r})$ wherever a Z-cyclic $Wh(q^2)$ exists and where p_i, q are primes, $q \equiv 3 \pmod 4$, $q \geq 7$, $p_i \equiv 1 \pmod 4$. He also has results relating to $Wh(3qp^n)$ [28].

5 Z-cyclic $Wh(4n)$

For completeness, we begin by listing early examples of Z-cyclic $Wh(4n)$, $n \leq 10$. In each case the initial round games are given, and the elements (players) are in $\{\infty\} \cup \mathbf{Z}_{4n-1}$.

$Wh(4)$: $(\infty, 1, 0, 2)$.

$Wh(8)$: $(\infty, 4, 0, 5), (1, 2, 3, 6)$.

$Wh(12)$: $(\infty, 4, 0, 5), (1, 2, 10, 8), (3, 6, 7, 9)$.

$Wh(16)$: $(\infty, 5, 0, 10), (1, 4, 2, 8), (3, 12, 11, 14), (6, 7, 9, 13)$.

$Wh(20)$: $(\infty, 11, 0, 12), (3, 16, 9, 1), (4, 13, 14, 18), (6, 5, 8, 2), (7, 10, 15, 17)$.

$Wh(24)$: $(\infty, 22, 0, 6), (1, 20, 15, 18), (2, 9, 5, 14), (3, 13, 11, 19),$
$(7, 10, 8, 21), (12, 4, 16, 17)$.

$Wh(28)$: $(\infty, 4, 0, 5), (1, 6, 16, 25), (23, 9, 2, 14), (17, 21, 15, 3),$
$(20, 26, 24, 13), (18, 7, 8, 10), (19, 22, 12, 11)$.

$Wh(32)$: $(\infty, 28, 0, 20), (30, 23, 12, 3), (27, 9, 17, 25), (19, 13, 18, 1),$
$(29, 10, 24, 14), (11, 7, 8, 5), (22, 6, 16, 15), (2, 4, 26, 21)$.

$Wh(36)$: $(\infty, 25, 0, 5), (15, 20, 10, 30), (11, 12, 9, 8), (14, 18, 6, 2),$
$(26, 7, 29, 13), (4, 33, 16, 22), (21, 32, 34, 23), (19, 28, 1, 27),$
$(17, 24, 3, 31)$.

$Wh(40)$: $(\infty, 26, 0, 13), (14, 38, 16, 10), (15, 24, 19, 7), (17, 35, 25, 1),$
$(21, 18, 37, 28), (29, 23, 22, 4), (6, 33, 31, 34), (20, 30, 32, 9),$
$(27, 3, 12, 5), (2, 3, 11, 36)$.

The $Wh(8)$, $Wh(12)$, $Wh(16)$ above are due to Safford and appear in [41]. The others are due to Whitfeld, as described in the second edition of Mitchell's book, apart from $Wh(32)$ which is Mitchell's own.

In general, if we wish to construct a Z-cyclic $Wh(4n)$ we use $\{\infty\} \cup \mathbf{Z}_{4n-1}$ where $4n - 1 \equiv 3 \pmod 4$. We can write

$$4n - 1 = QP \tag{5.1}$$

where Q consists only of primes $q \equiv 3 \pmod 4$ and P consists only of primes $p \equiv 1 \pmod 4$. Anderson and Finizio have developed a strategy which produces a Z-cyclic $Wh(QP + 1)$ from a Z-cyclic $Wh(Q + 1)$. The main stumbling block so far has been the difficulty in constructing examples of Z-cyclic $Wh(Q + 1)$. The strategy (an example of which is described in detail in the next theorem) utilizes the existence of an Z-cyclic $Wh(P)$ (as in section 4) and a Z-cyclic $Wh(Q + 1)$, and we denote it by the

$$P, Q + 1 \rightarrow PQ + 1$$

strategy.

Theorem 5.1 [9]. *Let p, q be primes, $p \equiv 1 \pmod 4$, $q \equiv 3 \pmod 4$, $q \geq 7$. Suppose that a Z-cyclic $Wh(q+1)$ exists. Then a Z-cyclic $Wh(qp+1)$ exists.*

Proof: On $Z_{pq} \cup \{\infty\}$ we

(a) construct the initial round of a Z-cyclic $Wh(q+1)$ on $\{\infty\} \cup \{$multiples of $p\}$,

(b) construct the initial round of a Z-cyclic $Wh(p)$ on the nonzero multiples of q,

(c) construct initial round games on the reduced set of residues $E \pmod{pq}$.

Note that $|E| = (p-1)(q-1)$. Take a common primitive root θ of p and q; then $ord_p\theta = p-1$, $ord_q\theta = q-1$,

$$ord_{pq}\theta = l.c.m.(p-1, q-1) = \frac{(p-1)(q-1)}{2e} = 4t$$

where $g.c.d.(p-1, q-1) = 2e$ for some odd e. Let

$$H = \{\pm 1, \pm\theta, \dots, \pm\theta^{4t-1}\}.$$

Then we can partition E into cosets:

$$E = x_0 H \cup x_1 H \cup \cdots \cup x_{e-1} H$$

where the 'representatives' x_i are chosen to satisfy $x_0 = 1$, $x_i \notin \cup_{j<i} x_j H$. Since $xH = \{\pm x, \pm\theta, \dots, \pm x\theta^{4t-1}\}$ where $4t \geq p-1$, each x_i can be chosen so that $x_i \equiv 1 \pmod 4$.

Consider the games

$$(x_i, x_i\theta, -x_i, -x_i\theta) \times 1, \theta^2, \theta^4, \dots, \theta^{4t-2}$$

$0 \leq i \leq e-1$. The *partner* differences are $\pm 2x_i, \pm 2\theta x_i \times 1, \theta^2, \dots$, i.e. they are all the elements of E once. The *opponent* differences are

$$\pm(\theta - 1)x_i, \pm(\theta + 1)x_i \times 1, \theta^2, \dots, \theta^{4t-2} \qquad (5.2)$$

(twice). We have only to prove that the differences (5.2) are all distinct. To achieve this, we choose θ suitably. By Cohen's lemma 4.2, if $p > 211$ we can choose a primitive root θ_p of p such that $\theta_p^2 - 1 \neq \square$. Then $\theta_p - 1 = \theta_p^\alpha$, $\theta_p + 1 = \theta_p^\beta$ where *one* of α, β is odd. So we choose θ so that $\theta \equiv \theta_p \pmod p$. Then suppose

$$x_i(\theta - 1) \equiv \pm x_j(\theta + 1)\theta^{2u} \pmod{pq}$$

for some u. Then

$$\theta_p - 1 \equiv \pm(\theta_p + 1)\theta_p^{2u} \pmod p \text{ i.e. } \theta_p^{\alpha-\beta} \equiv \pm\theta_p^{2u} \pmod p$$

where $\alpha - \beta$ is odd. Since $p \equiv 1 \pmod 4$, this is impossible; so the differences (5.2) do indeed give every element of E once.

For $p < 211$ we can check that there indeed exists θ_p such that $\theta_p^2 - 1 \neq \Box$, except when $p = 13$; this case is dealt with by a separate construction. $\Box$

For application of Theorem 5.1 we need primes $q \equiv 3 \pmod 4$, $q \geq 7$, for which a Z-cyclic $Wh(q+1)$ is known to exist. The cases $q = 7, 11, 19, 23, 31$ are "classical"; apart from these the only cases known so far are $q = 43, 47, 59$, due to Finizio [31].

A more general result of $P, Q+1 \Rightarrow PQ+1$ type can also be obtained.

Theorem 5.2 [9]. *Let $q, p_1, \ldots, p_m$ be primes, $q \equiv 3 \pmod 4$, $q \geq 7$. If there exists a Z-cyclic $Wh(q+1)$ then there exists a Z-cyclic $Wh(qp_1^{\alpha_1} \ldots p_m^{\alpha_m} + 1)$.*

We make a few remarks about the proof of Theorem 5.2. Let $N = qp_1^{\alpha_1} \ldots p_m^{\alpha_m}$ and, for convenience, denote q by $p_{m+1}^{\alpha_{m+1}}$ where $\alpha_{m+1} = 1$. Partition Z_N into 'layers', one layer being the set P_1 of multiples of p_1, and the remainder being of the form

$$I(\beta_2, \ldots, \beta_{m+1}) = \{x \in \mathbf{Z}_N : p_1 \not| x; p_i^{\beta_i} | x, p_i^{\beta_i+1} \not| x \text{ if } \beta_i < \alpha_i\},$$

where $(\beta_2, \ldots, \beta_{m+1})$ is an m-tuple satisfying $0 \leq \beta_i \leq \alpha_i$ for each i, $2 \leq i \leq m+1$. Then a collection of games is constructed on each $I(\beta_2, \ldots, \beta_{m+1})$ such that the partner (opponent) differences give every element of $I(\beta_2, \ldots, \beta_{m+1})$ once (twice). We then take all of these games, along with the games of the initial round of a Z-cyclic $Wh(N/p_i + 1)$ on $P_1 \cup \{\infty\}$, to obtain the initial round of a Z-cyclic $Wh(N+1)$. For full details see [9]. This decomposition into layers is also used in [6, 7, 8, 10].

All of the above examples have $4n - 1 = PQ$ where $Q = q \geq 7$. The case $q = 3$ has to be treated separately (since one of $\theta \pm 1$ is not in the reduced set here). Z-cyclic $Wh(3p^n + 1)$ are constructed in [4], but we will return to this when constructing triple whist tournaments in Section 6. Recently some progress has been made with other Q. Anderson, Finizio and Odoni [11] deal with the case $Q = q^2 q_1$ in a forthcoming paper. The method is quite general, but is limited at present by the lack of many examples of Z-cyclic $Wh(q+1)$ and $Wh(q^2)$.

6 Z-cyclic triplewhist tournaments

Triplewhist tournaments $TWh(v)$ were defined in Section 2; here we consider Z-cyclic $TWh(v)$

For a Z-cyclic $TWh(4n+1)$ we have to find an initial round

$$(a_1, b_1, c_1, d_1), \ldots, (a_n, b_n, c_n, d_n)$$

such that

(i) the partner pairs $\{a_i, c_i\}$, $\{b_i, d_i\}$ form a starter in $\mathbf{Z}_{4n+1}$;

(ii) the first opponent pairs $\{a_i, b_i\}$, $\{c_i, d_i\}$ form a starter in $\mathbf{Z}_{4n+1}$;

(iii) the second opponent pairs $\{a_i, d_i\}$, $\{b_i, c_i\}$ form a starter in $\mathbf{Z}_{4n+1}$.

Example 6.1 [Baker [12])

(i) Initial round of a $TWh(29)$.

$$(1,2,9,27), (16,3,28,26), (24,19,13,10), (7,14,5,15,),$$
$$(25,21,22,8), (23,17,4,12), (20,11,6,18).$$

(ii) Initial round of a $TWh(37)$.

$$(1,2,17,4), (16,32,13,27), (34,31,23,25), (26,15,35,30), (9,18,5,36),$$
$$(33,29,6,21), (10,20,22,3), (12,24,19,11), (7,14,8,28).$$

Until recently these were the only $\mathbf{Z}$-cyclic $TWh(p)$ in the literature. Finizio [23] has verified that there is no $\mathbf{Z}$-cyclic $TWh(p)$ for primes $p < 29$, although there exist $\mathbf{Z}$-cyclic $TWh(21)$, $TWh(25)$, $TWh(33)$.

Example 6.2 (Finizio)

(i) $TWh(21)$ [23] $(1,12,2,15)$, $(5,6,3,18)$, $(4,7,11,20)$, $(8,13,19,17)$, $(14,10,9,16)$.

(ii) $TWh(25)$ [30] $(5,3,1,14)$, $(9,17,16,20)$, $(15,6,21,7)$, $(10,4,8,13)$, $(2,12,19,22)$, $(11,18,24,23)$.

(iii) $TWh(33)$ [28] $(1,3,5,20)$, $(4,17,28,29)$, $(31,25,26,2)$, $(9,12,22,30)$, $(13,27,24,19)$, $(6,16,7,23)$, $(32,11,18,14)$, $(15,8,21,10)$.

The set of values of n for which a $\mathbf{Z}$-cyclic $TWh(4n+1)$ is known to exist has been considerably expanded by recent work of Finizio [26], Anderson, Cohen and Finizio [2] and Liaw [37]. We illustrate with the case $p = 4t+1$, t odd, i.e $p \equiv 5 \pmod 8$. For such a p, let θ be a primitive root, and note that

$$\theta^{2t} \equiv -1 \pmod p, \quad 2t \equiv 2 \pmod 4.$$

Construction A. Take initial round games

$$(1, \theta, -\theta, \theta^{-2}) \times 1, \theta^4, \theta^8, \ldots, \theta^{4(t-1)}$$

The partner differences are pairs $\pm(\theta+1)$, $\pm\theta^{-2}(\theta^3-1) \times 1, \theta^4, \ldots$, and so the partner pairs form a starter provided $(\theta^3-1)(\theta+1) \neq \square$. Similarly the first kind opponent pairs form a starter provided $(\theta^3+1)(\theta-1) \neq \square$ and the second kind opponent pairs form a starter provided $\theta^2 - 1 \neq \square$ (here we use the fact that 2 is a nonsquare since $p \equiv 5 \pmod 8$). So Construction A yields a $\mathbb{Z}$-cyclic $TWh(p)$ provided

$$\theta^2 - 1 \neq \Box, \quad \theta^2 + \theta + 1 = \Box, \quad \theta^2 - \theta + 1 = \Box. \tag{6.1}$$

Alternatively, we can use

Construction B. $(1, \theta, -\theta, -\theta^4) \times 1, \theta^4, \ldots, \theta^{4(t-1)}$. This works provided

$$\theta^4 + 1 \neq \Box, \quad \theta^2 + \theta + 1 = \Box, \quad \theta^2 - \theta + 1 = \Box. \tag{6.2}$$

Further, we could use

Construction C. $(1, \theta, \theta^3, -\theta^4) \times 1, \theta^4, \ldots, \theta^{4(t-1)}$. which works provided

$$\theta^2 - 1 = \Box, \quad \theta^4 + 1 = \Box, \quad \theta^6 - 1 = \Box. \tag{6.3}$$

Now θ remains to be chosen suitably. Using character sums we can show [2] that, except when $p = 61$, there exists a primitive root θ of $p \geq 29$ such that

$$\theta^2 + \theta + 1 = \Box, \quad \theta^2 - \theta + 1 = \Box.$$

Choose such a θ. If $\theta^2 - 1 \neq \Box$, use Construction A. If $\theta^2 - 1 = \Box$, then $\theta^6 - 1 = (\theta^2 - 1)(\theta^2 - \theta + 1)(\theta^2 + \theta + 1) = \Box$ and so we can use Construction C if $\theta^4 + 1 = \Box$. Finally, if $\theta^4 + 1 \neq \Box$, we use Construction B. We deal with $p = 61$ separately.

Example 6.3 [26]. The games $(1, 2, 35, 4) \times 1, 2^4, 2^8, \ldots, 2^{56}$ are the initial round games of a $\mathbb{Z}$-cyclic $TWh(61)$.

Thus we have

Theorem 6.1. *If $p \equiv 5 \pmod 8$ is prime, $p \geq 29$, there exists a $\mathbb{Z}$-cyclic $TWh(p)$.*

We note that Baker's examples of $TWh(29)$ and $TWh(37)$ given in Example 6.1 are in fact

$$(1, 2, 2^7, 2^2) \times 1, 2^4, 2^8, \ldots, 2^{24}$$

and

$$(1, 2, 2^{10}, -2) \times 1, 2^4, \ldots, 2^{32}.$$

The following is also proved in [2].

Theorem 6.2. *If the primes p_i are all $\equiv 5 \pmod 8$, $p_i \geq 29$, then there exists a $\mathbb{Z}$-cyclic $TWh(p_1^{\alpha_1} \ldots p_r^{\alpha_r})$.*

The cases of $p = 2^k t + 1$, t odd, $k > 2$ have been studied by Liaw [37]. First consider $k = 3$: $p = 8t + 1$, t odd, $t \geq 3$. Consider the games

$$(1, \theta, -\theta, \theta^4), (\theta^2, \theta^3, -\theta^3, \theta^6) \times 1, \theta^8, \ldots, \theta^{8(t-1)}. \tag{6.4}$$

It is straightforward to check that the starter conditions are satisfied provided θ^2+1, θ^2-1, $\theta^2+\theta+1$, $\theta^2-\theta+1$ are either all squares or all non-squares, i.e. provided $\theta^4-1=\Box$, $(\theta^3-1)(\theta+1)=\Box$, $(\theta^3+l)(\theta-1)=\Box$. It follows from the ideas in [20] that such a θ will exist provided p is sufficiently large.

Example 6.4 A Z-cyclic $TWh(89)$. $89=8.11+1$. Take $\theta=6$ to obtain initial round games

$$(1,6,83,38)\times 6^{8i+2j}\quad (0\le i\le 10, 0\le j\le 1).$$

The construction (6.4) does not provide solutions for $p=41$ or 73, but appears to work thereafter. The case $p=73$ falls to a generalization of this method: see Example 6.6 below. We can deal with $p=41$ by many similar constructions, e.g. the following.

Example 6.5 A Z-cyclic $TWh(41)$. Take $\theta=7$ to obtain initial round games

$$(1,7^9,-7,7^{-4})\times 7^{8i+2j}\quad (0\le i\le 4, 0\le j\le 1).$$

i.e.

$$(1,13,34,25),(8,22,26,36)\times 1,37,16,18,10.$$

We can generalize (6.4) to deal with all primes $p=2^k t+1$, $k\ge 3$, t odd, $t\ge 3$. The games

$$(1,\theta,-\theta,\theta^{1+a})\times\theta^{di+2j}\quad (0\le i\le t-1, 0\le j\le n)\qquad (6.5)$$

where $d=2^k$, $n=2^{k-2}-1$, from the initial round of a Z-cyclic $TWh(p)$ provided

(i) $a\equiv 2^{k-1}-1 \pmod{2^k}$,

(ii) $\theta^{a+1}-1=\Box$,

(iii) $(\theta+1)(\theta^a-1)=\Box$,

(iv) $(\theta-1)(\theta^a+1)=\Box$.

By this approach, Liaw has obtained the following result.

Theorem 6.3 [37]. *If p is a prime, $p\equiv 1 \pmod 4$, $29\le p<1000$, $p\ne 257$, there exists a Z-cyclic $TWh(p)$.*

Example 6.6

(i) A Z-cyclic $TWh(73)$. $73=2^3.9+1$. In (6.5) take $\theta=31$, $a=11$ to obtain initial round games

$$(1,31,42,65)\times 31^{8i+2j}\quad (0\le i\le 8, 0\le j\le 1).$$

(ii) A $\mathbb{Z}$-cyclic $TWh(97)$. $97 = 2^5.3 + 1$. Take $\theta = 7$, $a = 79$ to obtain initial round games

$$(1, 7, 90, 62) \times 7^{32i+2j} \quad (0 \le i \le 2, 0 \le j \le 7).$$

(iii) A $\mathbb{Z}$-cyclic $TWh(113)$. $113 = 2^4.7 + 1$. Take $\theta = 21$, $a = 7$ to obtain

$$(1, 21, 92, 64) \times 21^{16i+2j} \quad (0 \le i \le 6, 0 \le j \le 3)$$

We now consider $\mathbb{Z}$-cyclic $TWh(v)$ where $v \equiv 0 \pmod 4$. The first such designs were the $TWh(3p + 1)$ obtained by Moore [42].

Example 6.7 Moore's $TWh(16)$.

Round 1	$(\infty, 5, 0, 10)$	(1,4,2,8)	(6,13,9,7)	(11,12,3,14)
Round 2	$(\infty, 6, 1, 11)$	(2,5,3,9)	(7,14,10,8)	(12,13,4,0)
Round 3	$(\infty, 7, 2, 12)$	(3,6,4,10)	(8,0,11,9)	(13,14,5,1)
Round 4	$(\infty, 8, 3, 13)$	(4,7,5,11)	(9,1,12,10)	(14,0,6,2)
Round 5	$(\infty, 9, 4, 14)$	(5,8,6,12)	(10,2,13,11)	(0,1,7,3)
Round 6	$(\infty, 10, 5, 0)$	(6,9,7,13)	(11,3,14,12)	(1,2,8,4)
⋮				
Round 11	$(\infty, 0, 10, 5)$	(11,14,12,3)	(1,8,4,2)	(6,7,13,9)
⋮				
Round 15	$(\infty, 4, 14, 9)$	(0,3,1,7)	(5,12,8,6)	(10,11,2,13).

Example 6.8 Moore's $TWh(40)$. Initial round is

$(\infty, 0, 13, 26), (1, 8, 25, 5), (2, 16, 11, 10), (4, 32, 22, 20), (14, 38, 18, 21),$
$(15, 24, 23, 29), (17, 35, 33, 6), (27, 31, 34, 12), (28, 36, 3, 37), (30, 7, 19, 9).$

Anderson and Finizio extended Moore's method to obtain $\mathbb{Z}$-cyclic TWh $(3p^n + 1)$ for all $p \equiv 1 \pmod 4$ in [4], and then later [6,7] obtained TWh $(3p_1^{\alpha_1} \ldots p_n^{\alpha_n} + 1)$ whenever the p_i are *compatible*, i.e. whenever each $p_i - 1$ is divisible by the same power of 2. The resulting TWh designs all possess a remarkable property shown in Example 6.7. Note there that the elements of $B = \{1, 4, 2, 8\}$ are all distinct mod 5, so that this block, along with its translates $B + 5$, $B + 10$ can be taken as other blocks in the initial round; they are permuted so as to cause the partner differences in B to become first opponent differences in $B + 10$ and second opponent differences in $B + 5$. Rounds 6 and 11 contain the same *blocks* as round 1, but arranged differently. Thus if we take only the first 5 rounds, we obtain a 1-rotational resolvable (16,4,1) design in which the resolution classes are translates of the first class. This all goes through in the general case of $3p_1^{\alpha_1} \ldots p_n^{\alpha_n} + 1$, so we obtain:

Theorem 6.4 [7]. *If the $p_i \equiv 1 \pmod 4$ are compatible, then there exists a cyclically resolvable $(3p_1^{\alpha_1} \dots p_n^{\alpha_n} + 1, 4, 1)$ design.*

In the case when the p_i are not compatible, Liaw [36] has shown that a 1-rotational design still exists (although not necessarily resolvable).

We now consider the $P, Q+1 \to PQ+1$ strategy for triplewhist tournaments when $Q \neq 3$. The success of such a construction will of course be governed by the availability of Z-cyclic $TWh(q+1)$ designs when $q \equiv 3 \pmod 4$ is prime. So far only a handful of such designs are known. In Example 1.1 we dealt with $q = 7$.

Example 6.9 Initial rounds for Z-cyclic $TWh(q+1)$, $q = 19, 23, 31$. All are due to Finizio.

(a) $TWh(20)$ [23] $(\infty, 13, 0, 17)$, (8,2,10,1), (3,18,15,4), (7,12,16,9), (11,14,5,6).

(b) $TWh(24)$ [10] $(\infty, 22, 0, 10)$, (14,20,7,12), (19,18,15, 8), (4,6,13,9), (11,3,17,5), (21,1,16,2).

(c) $TWh(32)$ [10] $(\infty, 18, 0, 25)$, (12,4,2,5), (6,28,9,20), (23,11,8,24), (15,13,29,19), (1,14,3,10) (17,16,22,27), (21,7,30,26).

Theorem 6.5 [10]. *If a Z-cyclic $TWh(q+1)$ exists and a Z-cyclic $TWh(p)$ exists, $q \equiv 3 \pmod 4$, $p \equiv 1 \pmod 4$, $q \geq 7$, then a Z-cyclic $TWh(qp+1)$ exists.*

Proof: The proof is similar to that of Theorem 5.1. For the reduced set of residues take the games $(x_i, x_i\theta, -x_i\theta, -x_i\theta^2) \times 1, \theta^2, \dots, \theta^{4t-4}$. The proof goes through provided we choose $\theta \equiv \theta_p \pmod p$ where θ_p is a primitive root of p such that $2(\theta_p^2+1)$ is a square in $\mathbf{Z}_p$. Once again, Cohen's theorem guarantees the existence of such θ_p for $p > 211$; for $p < 211$ we can find a suitable θ_p in each case.

Note that the existence of a Z-cyclic $TWh(36)$ cannot be confirmed by this theorem since no Z-cyclic $TWh(5)$ exists. Finizio [30] has however constructed Z-cyclic $TWh(36)$ (and $TWh(28)$).

Example 6.10 Z-cyclic $TWh(36)$.

$(\infty, 16, 0, 32), (28, 14, 3, 27), (31, 13, 4, 2), (20, 24, 34, 33), (26, 1, 19, 6),$
$(7, 12, 8, 15), (17, 23, 21, 29), (18, 9, 30, 11), (22, 10, 5, 25).$

The proof of Theorem 6.5 can be generalized along the lines of Theorem 5.2.

Theorem 6.6 [10]. *Let $q, p_1, \dots, p_n$ be primes, $p_i \equiv 5 \pmod 8$, $q \equiv 3 \pmod 4$, $q \geq 7$, $p_i \geq 29$, and suppose that a Z-cyclic $TWh(q+1)$ exists. Then a Z-cyclic $TWh(qp_1^{\alpha_1} \dots p_r^{\alpha_r} + 1)$ exists.*

The natural generalization of this result to the primes p_i of Liaw's Theorem 6.3 can also be established [37].

7 Existence of $TWh(4n)$

According to the survey [17] by Bennett and Zhu, a resolvable SAMDRR(n) exists for all positive integers n except possibly for $n \in \{10, 14, 46, 54, 58, 62, 66, 70, 74, 82, 98, 102, 118, 142, 174, 194, 202, 214, 230, 258, 278, 282, 394, 398, 402, 422, 1322\}$. The SOLSSOMs in [38] and [49] do not appear to have mates with the properties required in Lemma 3.3. Thus a $TWh(4n)$ certainly exists for all n except those listed above. Many of the remaining values of n can be dealt with by the results of this survey: e.g. $40 = 3.13 + 1$, $184 = 3.61 + 1$, $232 = 8 \times 29$, $264 = 8 \times 33$, $328 = 8 \times 41$, $776 = 8 \times 97$, $1688 = 7.241 + 1$. However, very recently B. Du [21] has shown that the above list of values of n can be reduced to $\{10, 14, 46, 54, 58, 62, 66, 70\}$. As a result, it appears that the only values of $4n$ for which it is not known if a $TWh(4n)$ exists are 56,216,248,280.

8 Some open problems

(1) Find a method of constructing **Z**-cyclic $Wh(q+1)$ for primes $q \equiv 3 \pmod 4$, $q \geq 67$.

(2) Find a method of constructing **Z**-cyclic $TWh(q+1)$ for primes $q \equiv 3 \pmod 4$, $q \geq 43$.

(3) Find a **Z**-cyclic $TWh(257)$.

(4) Construct **Z**-cyclic $TWh(3p_1^{\alpha_1} \ldots p_r^{\alpha_r} + 1)$ when the primes $p_i - 1 \pmod 4$, are not compatible.

(5) Find (not necessarily **Z**-cyclic) $TWh(56)$, $TWh(216)$, $TWh(248)$, $TWh(280)$.

(6) Find a method of constructing **Z**-cyclic $Wh(q^2)$ where $q \equiv 3 \pmod 4$ is prime.

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