RAMSEY NUMBERS OF HYPERGRAPHS

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ABSTRACT

Some new lower bounds for higher Ramsey numbers are presented. Results concerning generalized hypergraph Ramsey numbers are also given.

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This note reports on an effort to obtain results on Ramsey numbers for uniform hypergraphs, the so called "higher" Ramsey numbers. This project is part of a larger effort to use standard techniques from Artificial Intelligence to achieve combinatorial constructions. Some of the ideas were described in more detail in [1].

Very little is known regarding specific values for these numbers. In [2] it is stated that

$$13 \leq R_3(4, 4) \leq 15$$

are the only known non-trivial bounds. The general upper bound for Ramsey numbers [see 2, p. 7] gives very large upper bounds for even small hypergraphs and it may be relatively weaker for hypergraphs than for graphs. For example, using this bound we get an upper bound for $R_3(4, 5)$ near 3000.

Some of the results given below pertain to noncomplete hypergraphs. It seems that, heretofore, little work has been done in generalized Ramsey theory for hypergraphs, in sharp contrast to the situation for graphs.

Our results for noncomplete hypergraphs all deal with the 3-uniform hypergraph on 4 vertices having 3 edges, which we denote by U. Other notation includes the use of K_s^r to denote the complete r-uniform hypergraph on s vertices. For a given coloring of a hypergraph H, we use $\deg_k(u_1, \ldots, u_k, c)$ to mean the number of edges in color c containing each of the vertices $u_1, \ldots, u_k$. Our notation and terminology relevant to Ramsey theory will generally follow [2].

Theorem 1. $r_3(U, U) = 7$

Proof. We first establish the upper bound $r(U, U) \leq 7$. To do so it will be sufficient to show the $deg_2(u, v, c) \leq 2$, for any pair of vertices, u, and v, and any color c. To prove this, assume the contrary and suppose we are given a U-free coloring of the edges K_n^3 wherein $deg_2(u, v, 1) \leq 3$. Let w_i, $1 \leq i \leq 3$, be third vertices in three of the color 1 edges containing u and v. If any edge of the form (v_i, w_j, w_k) is assigned color 1, then there is a monochromatic U in color 1. Hence all such edges must be assigned color 2. But then the vertex set (v_i, w_1, w_2, w_3) generates a monochromatic U in color 2. This contradiction proves $deg_2(u, v, c) \leq 2$, which implies that there are at most six vertices, as required.

To prove the lower bound we describe a coloring of K_6^3. Let $\{0, \ldots, 5\}$ be the vertex set. Assign color 1 edges:

$$(i, i + 1, 5), \qquad 0 \leq i \leq 4, \text{ and}$$

$$(i, i + 1, i + 3), \qquad 0 \leq i \leq 4$$

(In both cases, addition is performed modulo 5). The remaining edges are assigned color 2. Note that this coloring describes a self-complementary hypergraph.

Theorem 2. $r_3(U, K_4^3) = 8$.

Proof. First we prove the upper bound. Assume that we are given a good coloring of K_8^3. Let 1 be the U-free color and let color 2 be the K_4^3-free color. We consider upper bounds for $deg_2(u, v, 1)$. It is an easy matter to show $deg_2(u, v, 1) \leq 3$, we concentrate on establishing $\deg_2(u, v, 1) \leq 2$. If not, then let v_1, and v_2 share three color 1 edges, and let w_1, w_2, and w_3 be third vertices of these edges.

Since there is no U in color 1, all edges of the form (v_1, w_j, w_k) are assigned color 2. Therefore, (w_1, w_2, w_3) must be assigned color 1 or there will be a monochromatic K_4^3, in color 2. Let x be one of the remaining vertices. The following can be deduced:

a - (v_1, v_2, x) has color 2.

b - two edges of the form (v_1, w_j, x) have color 2, and without loss of generality, we take these to be (v_1, w_1, x) and (v_1, w_2, x).

c - (v_1, w_2, x) has color 1.

d - (w_1, w_3, x) and (w_2, w_3, x) have color 2.

e - (v_1, w_3, x) has color 1.

f - (v_2, w_3, x) has color 2.

g - (v_2, w_2, x) has color 1.

But, now whichever color is assigned to (v_2, w_1, x) we have a contradiction. Hence a given pair of vertices is contained in at most two edges in color. Hence there are at most 18 color 1 edges and at least 38 color 2 edges. But as reported by D. deCaen [personal communication], the Turan number T (7, 4, 2) = 37, yielding contradiction which proves the upper bound.

To prove the lower bound we used our computer construction algorithms and obtained a good 2-coloring of K_7^3. The edges in the

U-free color are listed in Table 1. Note that this is a 2-(7, 3, 2) design.

0 1 4	1 2 4
0 1 6	1 3 5
0 2 3	1 5 6
0 2 5	2 4 6
0 3 6	2 5 6
0 4 5	3 4 5
1 2 3	3 4 6

Table 1. The U-free color in a construction proving $r(U, K_4^3) \leq 8$.

Theorem 3.

a) $r_3(U, U, U) \geq 13$

b) $r_3(U, K_5) \geq 14$

c) $R_3(4, 5) \geq 30$

d) $R_3(3, 3, 3) \geq 56$

e) $R_4(5, 5) \geq 27$

Proof. The constructions for (a) and (b) are given in the tables 2 and 3 below. Those for (c), (d), and (e) are omitted for reasons of space.

Color 1

0	1	3	0	1	7	0	1	9	0	1	10	0	2	4
0	2	7	0	2	9	0	2	11	0	3	4	0	3	5
0	3	11	0	4	8	0	4	10	0	5	7	0	5	9
0	5	10	0	7	8	0	8	9	0	8	11	0	10	11
1	2	3	1	2	4	1	2	10	1	2	11	1	3	6
1	3	8	1	4	7	1	4	8	1	4	9	1	6	7
1	6	9	1	6	10	1	7	11	1	8	10	1	8	11
1	9	11	2	3	5	2	3	7	2	3	9	2	4	5
2	4	6	2	5	10	2	5	11	2	6	7	2	6	9
2	6	11	2	7	10	2	9	10	3	4	6	3	4	7
3	4	9	3	5	6	3	5	8	3	6	11	3	7	8
3	7	11	3	8	9	3	9	11	4	5	7	4	5	8
4	5	9	4	6	8	4	6	10	4	7	10	4	9	10
5	6	7	5	6	9	5	6	10	5	7	11	5	8	10
5	8	11	5	9	11	6	7	8	6	8	9	6	8	11
6	10	11	7	8	10	7	10	11	8	9	10	9	10	11

Color 2

0	1	5	0	1	6	0	2	5	0	2	8	0	2	10
0	3	6	0	3	10	0	4	6	0	4	11	0	5	11
0	6	8	0	6	9	0	7	9	0	7	10	0	7	11
1	2	6	1	2	7	1	2	8	1	3	5	1	3	10
1	4	5	1	4	11	1	5	8	1	5	9	1	6	11
1	7	9	1	7	10	1	10	11	2	3	6	2	3	8
2	3	11	2	4	7	2	4	8	2	4	10	2	5	6
2	5	7	2	5	9	2	6	10	2	7	11	2	8	9
2	9	11	2	10	11	3	4	10	3	4	11	3	5	7
3	5	11	3	6	7	3	7	9	3	8	10	3	9	10
4	5	6	4	5	10	4	6	7	4	7	9	4	8	11
4	9	11	5	6	8	5	6	11	5	7	8	5	7	10
5	9	10	5	10	11	6	7	10	6	7	11	6	8	10
6	9	10	6	9	11	7	8	9	7	8	11	8	10	11

Color 3

0	1	2	0	1	4	0	1	8	0	1	11	0	2	3
0	2	6	0	3	7	0	3	8	0	3	9	0	4	5
0	4	7	0	4	9	0	5	6	0	5	8	0	6	7
0	6	10	0	6	11	0	8	10	0	9	10	0	9	11
1	2	5	1	2	9	1	3	4	1	3	7	1	3	9
1	3	11	1	4	6	1	4	10	1	5	6	1	5	7
1	5	10	1	5	11	1	6	8	1	7	8	1	8	9
1	9	10	2	3	4	2	3	10	2	4	9	2	4	11
2	5	8	2	6	8	2	7	8	2	7	9	2	8	10
2	8	11	3	4	5	3	4	8	3	5	9	3	5	10
3	6	8	3	6	9	3	6	10	3	7	10	3	8	11
3	10	11	4	5	11	4	6	9	4	6	11	4	7	8
4	7	11	4	8	9	4	8	10	4	10	11	4	7	9
5	8	9	6	7	9	7	9	10	7	9	11	8	9	11

Table 2. A U-free 3-coloring of K_{12}^{3}.

0	1	2
0	1	6
0	1	9
0	2	5
0	2	7
0	3	4
0	3	5
0	3	6
0	4	8
0	4	12
0	5	11
0	6	11
0	6	10
0	7	8
0	7	10
0	9	10
0	9	12
0	10	11
0	11	12
1	2	4
1	2	11
1	3	5
1	3	8
1	3	11
1	4	12
1	5	6
1	5	9
0	5	10
1	6	7
1	7	10
1	7	12
1	8	9
1	8	12
1	10	11
2	3	4
2	3	6
2	3	8
2	4	7
2	5	10
2	5	12
2	6	10
2	6	11
2	7	9
2	8	11
2	8	12
2	9	10
2	9	11
2	9	12
3	4	9
3	5	12
3	6	7
3	6	12
3	7	8
3	7	11
3	9	10
3	9	11
3	10	12
4	5	7
4	5	9
4	5	10
4	6	9
4	6	10
4	6	11
4	7	11
4	8	9
4	8	11
4	10	12
5	6	8
5	6	11
5	7	8
5	7	12
5	8	10
5	9	11
6	7	9
6	8	12
6	9	12
6	11	12
7	11	12
8	9	10
8	10	11
8	10	12

Table 3. A U-free hypergraph, with a K_5^3-free complement, showing $r_3(U, K_5^3) \geq 14$.

REFERENCES

[1] G. Exoo, Constructing Ramsey graphs with a computer. Proceedings of the 18th Southeastern Conference on Combinatorics, Graph Theory and Computing. (to appear)

[2] R.L. Graham, B.L. Rothschild, J.H. Spencer. Ramsey Theory, John Wiley & Sons, New York, (1980).