# PROPERTIES OF EDGE-MAXIMAL K-EDGE-CONNECTED D-CRITICAL GRAPHS

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## ABSTRACT

An undirected graph of diameter D is said to be D-critical if the addition of any edge decreases its diameter. The structure of D-critical graphs can be conveniently studied in terms of vertex sequences. Following on earlier results, we establish, in this paper, fundamental properties of K-edge-connected D-critical graphs for $K \geq 8$ and $D \geq 7$. In particular, we show that no vertex sequence corresponding to such a graph can contain an "internal" term less than 3, and that no two non-adjacent internal terms can exceed $K-\lceil 2\sqrt{K}\rceil+1$. These properties will be used in forthcoming work to show that every subsequence (except at most one) of length three of the vertex sequence contains exactly $K+1$ vertices, a result which leads to a complete characterization of edge-maximal vertex sequences.

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## 1. INTRODUCTION

All graphs considered in this paper are finite, loopless and have no multiple edges. Terminology is generally as given in [1]. A graph G of diameter D is said to be **D-critical** if the addition of any edge results in a graph having diameter less than D.

In [5] Ore characterizes general D-critical graphs on n vertices which for fixed n and D contain a maximum number of edges: we refer to such graphs as **edge-maximal** (or simply **maximal**). Ore characterizes edge-maximal D-critical graphs which are in addition constrained to be K-vertex-connected, and shows that they have a very simple structure. He does not, however, consider the problem of the structure of maximal D-critical K-edge-connected graphs. This problem is the subject of the present paper.

In [2], we presented a summary of Ore's main results, and provided a characterization of maximal D-critical K-edge-connected graphs G for each of the following cases:

(i) $2 \leq D \leq 5$ ;

(ii) $1 \leq K \leq 7$ ;

(iii) for $D \geq 6$, $K \geq 8$, provided that the order n of G is in a certain sense "minimal", consistent with diameter D and edge-connectivity K.

Even in these special cases, we find that the structure of the maximal graphs is much more varied than that of the maximal K-vertex-connected graphs studied by Ore. In this paper, we prove fundamental properties of maximal

D-critical K-edge-connected graphs; in forthcoming work, we shall make use of these properties, first to derive a characterization of such graphs [3], and then to provide a means of calculating, for given n, D, and K, the precise arrangement of their vertices and edges [4].

Section 2 reviews the main ideas and results for D-critical graphs conveniently expressed in terms of vertex sequences; we then state a sequence of Lemmas which easily yield Ore's main characterization theorem [5]. The new properties concerning maximal D-critical K-edge-connected graphs are established in Section 3.

## 2. D-CRITICAL GRAPHS

Let G be a graph of diameter D having vertex set V. Then a vertex $u \in V$ is said to be **peripheral** if there exists another vertex $v \in V$ such that $d(u,v) = D$. Imagine now that the vertices of G are arranged in **levels** $L_i = L_i(u)$, $i = 0, 1, \ldots, D$, where $L_i(u)$ consists of the vertices distance exactly $i$ from $u$. Then a **vertex sequence** $S_D = S_D(u)$ is given by

$$S_D = (n_0, n_1, \ldots, n_D) \qquad (2.1)$$

where $n_i = |L_i(u)|$, $i = 0, 1, \ldots, D$. Subsequences of $S_D$ of length $k \geq 1$ are referred to as **k-tuples**; in particular, for $k = 2$ and 3, as **doubles** and **triples**, respectively. A k-tuple $(n_i, n_{i+1}, \ldots, n_{i+k-1})$ is **internal** if $1 < i < D-k$, and a term $n_i$ of $S_D$ will be called **terminal** if $i = 0$ or $D$. The structure of D-critical graphs can, as noted in [2], be conveniently studied in terms of vertex sequences. The following results are due to Ore [5]:

**Lemma 1.** A graph G is D-critical if and only if every peripheral vertex gives rise to a vertex sequence (2.1) such that

(a) $n_0 = n_D = 1$ ;

(b) every vertex in $L_i$, $i = 0, 1, \ldots, D-1$, is adjacent to every other vertex in $L_i$ and $L_{i+1}$ . ▯

**Lemma 2.** Suppose (2.1) is a vertex sequence for a D-critical graph G, and consider any 4-tuple $(n_{i-1}, n_i, n_{i+1}, n_{i+2})$ of (2.1), where $n_i > 1$ and $1 \leq i \leq D-2$. The transformation

$$(n_{i-1}, n_i, n_{i+1}, n_{i+2}) \rightarrow (n_{i-1}, n_i-1, n_{i+1}+1, n_{i+2})$$

changes the edge count by $n_{i+2} - n_{i-1}$. ▯

Let $G_e(n, D, K)$ ($G_v(n, D, K)$) denote the class of D-critical K-edge- (respectively, K-vertex) connected graphs on n vertices. A graph $G \in G_x(n, D, K)$, $x = e, v$, is said to be **edge-maximal** (or simply **maximal**) if no other graph in $G_x(n, D, K)$ has more edges than G. Similarly, the vertex sequence of a maximal graph is called a **maximal vertex sequence** (or simply a **maximal sequence**).

A consequence of Lemmas 1 and 2 is that for a maximal vertex sequence it may, without loss of generality, be assumed that

$$n_1 = n_{D-1} = K .$$

The next result follows immediately from Lemma 1 and the definition of connectivity.

**Lemma 3.** Let $S_D = (n_0, n_1, \ldots, n_D)$ be a vertex sequence of a D-critical graph G. Then

(a) $G \in G_e(n, D, K)$ if and only if:

(i) every triple of $S_D$ contains at least $K+1$ vertices;

(ii) the product of the terms of every double of $S_D$ is at least K.

(b) $G \epsilon G_v(n, D, K)$ if and only if $n_i \geq K$ for $1 \leq i \leq D-1$. ▯

Let $G \epsilon G_x(n, D, K)$, x=e, v, be a graph with vertex sequence S. We say that a vertex u of G is **removable** if $G-u \epsilon G_x(n-1, D, K)$. The sequence $T(S) = (n_0', n_1', \ldots, n_D')$ is said to be a **transformation** of S if $n_i' \geq 1$ for each i and $n_0' + n_1' + \cdots + n_D' = n$. T(S) is said to be **feasible** if the graph G' realized from it is a member of $G_x(n, D, K)$. That is, in a feasible transformation the criticality and connectivity properties are preserved; only the edge count can change. When writing down a transformation, we specify only those terms which change. An important property of maximal vertex sequences is given in the following lemma.

**Lemma 4.** Let $G_x \epsilon G_x(n, D, K)$, x = e, v, be a maximal graph with peripheral vertex u. If $v \epsilon L_i(u)$ and $w \epsilon L_j(u)$ are removable vertices in $G_x$, then $|i-j| \leq 1$.

**Proof:** Let $S = (n_0, n_1, \ldots, n_D)$ be the vertex sequence of $G_x$ and suppose that $|i-j| > 1$. Then the transformations

$$T_1: \quad n_i \to n_i - 1 \quad \text{and} \quad n_j \to n_j + 1$$

and

$$T_2: \quad n_i \to n_i + 1 \quad \text{and} \quad n_j \to n_j - 1 ,$$

are feasible and alter the edge count by:

$$(n_{j-1} + n_j + n_{j+1} - n_{i-1} - n_i - n_{i+1}) + 1$$

and

$$-(n_{j-1} + n_j + n_{j+1} - n_{i-1} - n_i - n_{i+1}) + 1$$

respectively. Hence, the edge count can be increased, contradicting the fact S is maximal. This proves that $|i-j| \leq 1$. ▯

**Corollary 1.** No maximal vertex sequence of $G_x(n, D, K)$, $x = e,v$, contains two non-adjacent terms greater than K. Moreover, there exists a maximal vertex sequence having at most one term greater than K. ▯

This Corollary together with Lemma 3 (b) yields Ore's main classification theorem:

**Theorem 1.** A graph $G \epsilon G_v(n, D, K)$, $D \geq 4$, is edge-maximal if and only if it has a vertex sequence

$$(1, K, n_2, n_3, \ldots, n_{D-2}, K, 1)$$

with $n_i = K$ for all $i \geq 2$ except possibly one or a consecutive pair. ▯

## 3. PROPERTIES OF MAXIMAL VERTEX SEQUENCES

Throughout this section $S = (1, n_1, n_2, \ldots, n_{D-1}, 1)$ will always denote a vertex sequence corresponding to a graph $G \epsilon G_e(n, D, K)$, where $D \geq 7$ and $K \geq 8$. A triple $(n_{i-1}, n_i, n_{i+1})$ of S is said to be a **minimum triple** if

$$n_{i-1} + n_i + n_{i+1} = K + 1 ,$$

achieving the lower bound allowed by Lemma 3(a); otherwise we shall call the triple fat. Observe that a fat triple does not necessarily include a removable vertex; for example, no vertex of the fat triple (1, K, 1) is removable.

**Lemma 5.** Let $G \epsilon G_e(n, D, K)$, $D \geq 7$, $K \geq 8$, be an edge-maximal graph with vertex sequence $S = (1, n_1, n_2, \ldots, n_{D-1}, 1)$. If $n_i < \sqrt{K}$ for some i, $3 \leq i \leq D-3$, then

$$n_i < \min\{n_{i-2}, n_{i+2}\} \tag{3.1}$$

**Proof:** Suppose that (3.1) is not true and that

$$\min\{n_{i-2}, n_{i+2}\} = n_i - x, \quad x \geq 0 .$$

Without loss of generality, we may suppose that $n_{i+2} = n_i - x$. S clearly contains the subsequence

$$(n_{i-1}, n_i, \ldots, n_{i+3}) = (\lceil K/n_i \rceil + a,\ n_i,\ \lceil K/n_i \rceil + b,\ n_i - x,\ \lceil K/(n_i - x) \rceil + c)$$

where a, b and c are non-negative integers.

Since

$$n_i < \lceil K/n_i \rceil$$

and

$$2n_i + \lceil K/n_i \rceil + b - x \geq K + 1 , \tag{3.2}$$

it follows that the triples $(n_{i-1}, n_i, n_{i+1})$ and $(n_{i+1}, n_{i+2}, n_{i+3})$ are both fat. Hence, if $b > 0$, the transformation

$$(n_{i+1}, n_{i+2}) \rightarrow (n_{i+1} - 1, n_{i+2} + 1)$$

is feasible and increases the edge count by

$$\lceil K/(n_i - x) \rceil + c - n_i > 0 .$$

Therefore $b = 0$ and so, since $n_{i+1}\, n_{i+2} \geq K$, $x = 0$. Now the only possible way (3.2) can hold is for $n_i = 1$. Lemma 4 then implies that at least one of a or c, say a, is zero. But then the transformation

$$(n_{i-1}, n_i, \ldots, n_{i+3}) = (K, 1, K, 1, K + c) \rightarrow (K, 2, K - 2, 2, K + c)$$

is feasible and has an additional

$$2K + c - 3$$

edges, contradicting the fact that $S$ is maximal. Hence, (3.1) must be satisfied. □

**Remark 1.** The above argument can be used to establish that if $n_2 < \sqrt{K}$, then $n_4 > n_2$.

**Remark 2.** If $n_i = \lceil K/n_i \rceil$ (this is so when $K = n_i^2$), then the above argument can be used to prove that $n_{i+2} > n_i$ unless $(n_{i-1}, n_i, n_{i+1})$ is a minimal triple.

**Lemma 6.** No internal term of an edge-maximal vertex sequence is one.

**Proof:** Suppose on the contrary that $S$ is a maximal vertex sequence with an internal term $n_i = 1$. Lemma 3 (a) together with Corollary 1 implies that

$$\min\{n_{i-1}, n_{i+1}\} = K .$$

First we consider the case $3 \leq i \leq D-3$. Lemma 5 implies that $n_{i-2} > 1$ and $n_{i+2} > 1$. Hence the transformations

$$T_1: \quad n_{i-1} \rightarrow n_{i-1} - 1 \quad \text{and} \quad n_i \rightarrow n_i + 1$$

and

$$T_2: \quad n_{i+1} \rightarrow n_{i+1} - 1 \quad \text{and} \quad n_i \rightarrow n_i + 1$$

are feasible and alter the edge count by $(n_{i+1} - n_{i-2})$ and $(n_{i-1} - n_{i+2})$, respectively. Hence, since $S$ is maximal, $n_{i-2} \geq n_{i+1}$ and $n_{i+2} \geq n_{i-1}$. Thus

$$(n_{i-2}, n_{i-1}, \ldots, n_{i+2}) = (K + a, K + b, 1, K + c, K + d)$$

with $a \geq c \geq 0$ and $d \geq b \geq 0$. It follows from Corollary 1 that

$$b = c = 0 \quad \text{and} \quad \min\{a,d\} = 0 .$$

Without any loss of generality, we take $a = 0$. Then the transformation

$$(n_{i-2}, n_{i-1}, n_i, n_{i+1}, n_{i+2}) = (K, K, 1, K, K+d)$$

$$\rightarrow (K, 2, K-2, K+1, K+d),$$

is feasible and increases the edge count by $K + d - 2 > 0$. This proves that

$$n_i > 1 \quad \text{for} \quad 3 \leq i \leq D-3 .$$

Consider next the case $n_2 = 1$. Because of symmetry, this is the only remaining case. Now we must have $n_4 \geq K$, since $n_1 = K$ and the transformation

$$n_2 \rightarrow n_2+1 \quad \text{and} \quad n_3 \rightarrow n_3-1$$

is feasible and alters the edge count by $K - n_4$. Without any loss of generality, we may take

$$(n_0, n_1, \ldots, n_4) = (1, K, 1, K, K+a)$$

with $a \geq 0$. Observing that $n_D = 1$, consider the sequence

$$S' = (n_2, n_3, \ldots, n_D, n_1, n_0)$$

formed by a rearrangement of S. This sequence is also edge-maximal. By Lemma 5, $n_{D-2} > 1$. Since the transformation

$$(n_{D-2}, n_{D-1}, n_D, n_1) = (n_{D-2}, K, 1, K) \rightarrow (n_{D-2}, K-1, 2, K)$$

is feasible and alters the edge count by $K-n_{D-2}$, we must have $n_{D-2} \geq K$. Now for $D \geq 7$ the transformations

$$T_1: \quad n_4 \rightarrow n_4 - 1 \quad \text{and} \quad n_{D-2} \rightarrow n_{D-2}+1$$

and

$$T_2: \quad n_4 \rightarrow n_4 + 1 \quad \text{and} \quad n_{D-2} \rightarrow n_{D-2}-1$$

are both feasible, since $n_{D-3} > 1$. Hence, the edge count can be increased, contradicting the maximality of $S$. The only remaining case is $D = 7$. In this case $S$ can, without any loss of generality, be taken to be

$$S = (1, K, 1, K, K, K+b, K, 1)$$

with $b \geq 0$. But the sequence

$$(1, K, 2, K-2, K, K+b+1, K, 1)$$

yields a graph with more edges, a contradiction. This completes the proof of the lemma. $\square$

**Remark 3.** For the above result, we need $D \geq 7$, since for $D = 6$ the sequence

$$(1, K, 1, K, K+a, K, 1)$$

will in fact be maximal for sufficiently large $a$.

Our next result establishes a lower bound of 3 for the internal terms of an edge-maximal vertex sequence.

**Lemma 7.** No internal term of an edge-maximal sequence can be two.

**Proof:** Suppose on the contrary that $S$ is an edge-maximal vertex sequence with an internal term $n_i = 2$. We may further suppose that $n_j > 2$ for $1 \leq j \leq i-1$. We distinguish two cases according to the value of $i$.

**Case 1.** $i = 2$.

Here $n_3 = \lceil \frac{1}{2}K \rceil + a$, with $a \geq 0$ and, by Remark 1, $n_4 > 2$. If $n_5 = 2$, then $n_4 \geq \lceil \frac{1}{2}K \rceil$ and $n_6 \geq \lceil \frac{1}{2}K \rceil$. In such a case, we may in fact assume, without any loss of generality, that $S$ contains the subsequence

$$(n_1, n_2, \ldots, n_6) = (K, 2, \lceil \tfrac{1}{2}K \rceil, \lceil \tfrac{1}{2}K \rceil, 2, \lceil \tfrac{1}{2}K \rceil + b)$$

with $b \geq 0$. But then the transformation

$$T_1: \quad n_2 \rightarrow n_2+1 \quad \text{and} \quad n_3 \rightarrow n_3-1 \tag{3.3}$$

is feasible and increases the edge count. This contradiction establishes that $n_5 > n_2 = 2$. Hence $(n_3, n_4, n_5)$ is a fat triple and so the transformation (3.3) is feasible and alters the edge count by $K-n_4$. Therefore, since $S$ is maximal, $n_4 \geq K$. Now since $n_5 > 2$, we may take $a = 0$ and so

$$(n_1, n_2, \ldots, n_5) = (K, 2, \lceil \tfrac{1}{2}K \rceil, K+b, n_5)$$

with $b \geq 0$. If $n_5 \leq K-2$, then the transformation

$$n_2 \rightarrow n_2 + n_5 + b \quad \text{and} \quad n_4 \rightarrow K-n_5$$

is feasible and increases the edge count by $2(n_5 + b)$. Hence $n_5 = K-1 + c$, $c \geq 0$. Now since $D \geq 7$ and the transformations

$$(n_4, n_5) \rightarrow (n_4 \pm 1, n_5 \mp 1)$$

are feasible, we must have $n_6 = \lceil \frac{1}{2}K \rceil$. But then the transformation

$$(n_1, n_2, \ldots, n_6) = (K, 2, \lceil \tfrac{1}{2}K \rceil, K+b, K-1+c, \lceil \tfrac{1}{2}K \rceil)$$

$$\rightarrow (K, K+b+c, \lceil \tfrac{1}{2}K \rceil, 2, K-1, \lceil \tfrac{1}{2}K \rceil)$$

is feasible and increases the edge count by $K+b+c-2 > 0$. This proves that $n_2 > 2$ and also (because of symmetry) that $n_{D-2} > 2$.

**Case 2.** $3 \leq i \leq D-3$.

Since $D \geq 7$ and we could consider $S$ in reverse order, we may take $i < D-3$. We must have

$$n_{i-1} = \lceil \tfrac{1}{2}K \rceil + a \quad \text{and} \quad n_{i+1} = \lceil \tfrac{1}{2}K \rceil + b \ ,$$

with $a, b \geq 0$. Lemma 5 implies that

$$n_{i-2} > 2 \quad \text{and} \quad n_{i+2} > 2 \ .$$

Since, by assumption, $n_j > 2$ for $2 \leq j \leq i-1$, the triple $(n_{i-3}, n_{i-2}, n_{i-1})$ is fat. Hence the transformation

$$n_{i-1} \rightarrow n_{i-1} - 1 \quad \text{and} \quad n_i \rightarrow n_i + 1$$

is feasible and alters the edge count by $(n_{i+1} - n_{i-2})$. We must therefore have

$$n_{i-2} = \lceil \tfrac{1}{2}K \rceil + c \ ,$$

with $c \geq b$.

We now prove that $n_{i+3} \geq 3$. Suppose that $n_{i+3} < 3$. Then, since $i < D-3$, $n_{i+3} = 2$ and hence

$$(n_i, n_{i+1}, \ldots, n_{i+4}) = (2, \lceil \tfrac{1}{2}K \rceil + b, \lceil \tfrac{1}{2}K \rceil + x, 2, \lceil \tfrac{1}{2}K \rceil + y)$$

with $x, y \geq 0$. Without loss of generality, we may take $b = 0$. Now if $x > 0$, the transformation

$$n_{i-2} \rightarrow n_{i-2} + x \quad \text{and} \quad n_{i+2} \rightarrow n_{i+2} - x$$

is feasible and alters the edge count by

$$x(n_{i-3} + a + c - 2) \ .$$

If $i = 3$, then $a = [\tfrac{1}{2}K]$. Hence, since $n_{i-3} \geq 3$ for $i \geq 4$, $n_{i-3} + a + c > 2$ and so $x = 0$. But then the edge count can be increased via the transformation

$$(n_i, n_{i+1}, n_{i+2}, n_{i+3}) = (2, \lceil \tfrac{1}{2}K \rceil, \lceil \tfrac{1}{2}K \rceil, 2)$$

$$\rightarrow (3, \lceil \tfrac{1}{2}K \rceil - 1, \lceil \tfrac{1}{2}K \rceil - 1, 3).$$

This proves that $n_{i+3} > 2$ and hence that $(n_{i+1}, n_{i+2}, n_{i+3})$ is a fat triple.

Now the transformation

$$n_i \rightarrow n_i + 1 \quad \text{and} \quad n_{i+1} \rightarrow n_{i+1} - 1$$

is feasible and alters the edge count by $(n_{i-1} - n_{i+2})$. We must therefore have

$$n_{i+2} = \lceil \tfrac{1}{2}K \rceil + d \ ,$$

with $d \geq a$. Since $n_{i+3} \geq 3$, we must in fact have $n_{i+1} = \lceil \tfrac{1}{2}K \rceil$; that is $b = 0$. Thus

$$(n_{i-2}, n_{i-1}, \ldots, n_{i+2}) = (\lceil \tfrac{1}{2}K \rceil + c, \lceil \tfrac{1}{2}K \rceil + a, 2, \lceil \tfrac{1}{2}K \rceil, \lceil \tfrac{1}{2}K \rceil + d) \ .$$

Our next task is to prove that $a = 0$. If $a > 0$, then since $(n_{i-3}, n_{i-2}, n_{i-1})$ is a fat triple and $d \geq a$, the edge-maximal graph G corresponding to S would have removable vertices in levels i-1 and i+1, unless $(n_{i+2}, n_{i+3}, n_{i+4})$ is a minimal triple. Hence $(n_{i+2}, n_{i+3}, n_{i+4})$ must be minimal. We must therefore have

$$(n_{i+2}, n_{i+3}, n_{i+4}) = (\lceil \tfrac{1}{2}K \rceil + d,\ 2 + e,\ \lceil \tfrac{1}{2}K \rceil - f) ,$$

with $e > 0$, $f > 0$ and

$$e + d - f = K - 2\lceil \tfrac{1}{2}K \rceil - 1 .$$

Note that

$$f > e + d . \qquad (3.4)$$

Now the transformation

$$(n_{i-1}, n_i, n_{i+1}) = (\lceil \tfrac{1}{2}K \rceil + a,\ 2,\ \lceil \tfrac{1}{2}K \rceil)$$

$$\rightarrow (\lceil \tfrac{1}{2}K \rceil + a + f - e,\ 2 + e,\ \lceil \tfrac{1}{2}K \rceil - f)$$

is feasible and increases the edge count by

$$(f - e)(c + f) + f(a - d)$$

$$> d(c + f) + f(a - d)$$

$$= dc + fa$$

$$> 0 . \qquad \text{(by (3.4))}$$

This proves that $a = 0$.

In fact, the above argument shows that $(n_{i+2}, n_{i+3}, n_{i+4})$ is a fat triple. An analogous argument shows that for $i \geq 4$ $(n_{i-4}, n_{i-3}, n_{i-2})$ is a fat triple. But then, since $n_{i-3} \geq 3$ and $n_{i+3} \geq 3$, our edge-maximal graph G has removable vertices in levels $i-2$ and $i+2$, a contradiction. This establishes the lemma for $i \geq 4$.

The only remaining case is $i=3$. Since $n_{i+3} \geq 3$ and $(n_{i+2}, n_{i+3}, n_{i+4})$ is a fat triple, our graph G has a removable vertex in level $i+2$. Hence $n_{i+4} \leq n_{i+1}$. Let

$$n_{i+3} = 2 + e, \quad n_{i+4} = \lceil \tfrac{1}{2}K \rceil - f ,$$

with $e \geq 1$ and $f \geq 0$. Further, let

$$t = 2\lceil \tfrac{1}{2}K \rceil + d + e - f + 2 - K - 1 . \tag{3.5}$$

Then G has

$$r = \min\{d+1, t\}$$

removable vertices in level $i+2$. The transformation

$$n_{i-1} \rightarrow n_{i-1} + r \quad \text{and} \quad n_{i+2} \rightarrow n_{i+2} - r$$

alters the edge count by

$$r(\lfloor \tfrac{1}{2}K \rfloor - d - e + r) \leq 0 .$$

Hence

$$e + d - r \geq \lfloor \tfrac{1}{2}K \rfloor . \tag{3.6}$$

If $D=7$, then $n_{i+3} = K$ and the edge count can be increased via the transformation

$$(1,\ K,\ \lceil \tfrac{1}{2}K \rceil,\ 2,\ \lceil \tfrac{1}{2}K \rceil,\ \lceil \tfrac{1}{2}K \rceil + d,\ K,\ 1)$$

$$\rightarrow (1,\ K,\ \lceil \tfrac{1}{2}K \rceil - 2,\ 3,\ \lceil \tfrac{1}{2}K \rceil,\ \lceil \tfrac{1}{2}K \rceil + d + 1,\ K,\ 1)\ .$$

Therefore $D > 7$. We now consider two subcases according to the value of $r$.

**(a)** $r = t < d + 1$

Equation (3.5) gives

$$d - t + e - f = K - 2\lceil \tfrac{1}{2}K \rceil - 1 < 0\ .$$

Therefore, since $t \leq d$, $e < f$. The transformation

$$(n_2,\ n_3,\ \ldots,\ n_5) = (\lceil \tfrac{1}{2}K \rceil,\ 2,\ \lceil \tfrac{1}{2}K \rceil,\ \lceil \tfrac{1}{2}K \rceil + d)$$

$$\rightarrow (\lceil \tfrac{1}{2}K \rceil + f + t - e,\ 2 + e,\ \lceil \tfrac{1}{2}K \rceil - f,\ \lceil \tfrac{1}{2}K \rceil + d - t)$$

is feasible and increases the edge count by

$$(t + f - e)(\lfloor \tfrac{1}{2}K \rfloor + f + t) - d(t + f)$$

$$\geq (d + 1)(\lfloor \tfrac{1}{2}K \rfloor + f + t) - d(t + f)$$

$$> 0\ .$$

Consequently $t \geq d + 1$.

**(b)** $r = d + 1$

Then, by (3.6)

$$n_{i+3} = 2 + e \geq \lfloor \tfrac{1}{2}K \rfloor + 3\ .$$

Consequently, we can take

$$n_{i+3} = \lceil \tfrac{1}{2}K \rceil + x\ ,$$

with $x \geq 3$. Now since $n_{i+4} \geq 2$, our graph G has removable vertices in level i + 3 whenever $(n_{i+3}, n_{i+4}, n_{i+5})$ is a fat triple. We may therefore, without loss of generality, assume that $(n_{i+3}, n_{i+4}, n_{i+5})$ is a minimal triple. Then

$$n_{i+5} = \lceil \tfrac{1}{2}K \rceil - g, \quad g \geq 0 ,$$

$$n_{i+6} \geq n_{i+3} , \tag{3.7}$$

and

$$x - f - g = K + 1 - 3\lceil \tfrac{1}{2}K \rceil . \tag{3.8}$$

In view of (3.7), we may assume that $n_{i+4}$ is as small as possible. That is

$$(\lceil \tfrac{1}{2}K \rceil - f - 1)(\lceil \tfrac{1}{2}K \rceil - g + 1) < K .$$

Since $n_{i+4}\, n_{i+5} \geq K$, it thus follows that $f \geq g$ and hence

$$(\lceil \tfrac{1}{2}K \rceil - g)(\lceil \tfrac{1}{2}K \rceil + x) \geq K .$$

Thus the transformation

$$(n_2, n_3, \ldots, n_5) = (\lceil \tfrac{1}{2}K \rceil, 2, \lceil \tfrac{1}{2}K \rceil, \lceil \tfrac{1}{2}K \rceil + d)$$

$$\rightarrow (\lceil \tfrac{1}{2}K \rceil + \lambda, \lceil \tfrac{1}{2}K \rceil + x, \lceil \tfrac{1}{2}K \rceil - f, \lceil \tfrac{1}{2}K \rceil - g)$$

where

$$\lambda = d + g + f - \lceil \tfrac{1}{2}K \rceil - x + 2$$

$$\geq d + 1 , \qquad \text{(by (3.7))}$$

is feasible and increases the edge count by

$$(g + d + f)(\lambda - f) + \lambda\lfloor \tfrac{1}{2}K \rfloor$$

$$> \lambda(\lambda + \lfloor \tfrac{1}{2}K \rfloor - f) > 0 \ .$$

This contradiction completes the proof of the lemma. □

We recall now some useful notation introducted in [2]. For given $K \geq 8$,

$$\alpha = \lceil 2\sqrt{K} \rceil \ .$$

Let $\alpha_1$ be the least integer $\geq 3$, such that $\alpha(\alpha - \alpha_1) \geq K$, and let $\alpha_2 = \alpha - \alpha_1$. Then $\alpha$ is the least order of a double compatible with K-edge-connectivity. Further, defining $K' = K + 1$ and $K_x = K' - x$ we observe that every permutation of the minimum triple $(K_\alpha, \alpha_1, \alpha_2)$ satisfies K-edge-connectivity. Our next result establishes an upper bound for the internal terms of an edge-maximal vertex sequence. This upper bound is more refined than that given by Corollary 1.

**Lemma 8.** No two non-adjacent terms of an edge-maximal vertex sequence can exceed $K_\alpha$.

**Proof:** Suppose on the contrary that $S$ is an edge-maximal vertex sequence with

$$n_i = K_\alpha + a \ ,$$

$$n_j = K_\alpha + b \ ,$$

where $a > 0$, $b > 0$, $|i - j| > 1$, $2 \leq i \leq D-2$ and $2 \leq j \leq D-2$.

Our graph G corresponding to S cannot have removable vertices in each of the level sets $i$ and $j$. Suppose without any loss of generality that G has no removable vertex in level i. Then at least one of the following conditions must hold:

(i) $(n_{i-2}, n_{i-1}, n_i)$ is minimal;
(ii) $(n_{i-1}, n_i, n_{i+1})$ is minimal;
(iii) $(n_i, n_{i+1}, n_{i+2})$ is minimal;
(iv) $\{\min n_{i-1}, n_{i+1}\} = \lceil K/(K_\alpha + a) \rceil$ .

Condition (i) cannot hold since it would imply that

$$n_{i-2} + n_{i-1} \leq \alpha - a$$

and hence

$$n_{i-2}\, n_{i-1} < K \ .$$

Similarly, condition (iii) cannot hold. Now suppose (iv) is true. It suffices to consider only the case when $n_{i-1} \leq n_{i+1}$. Let

$$f(K, a) = \frac{K}{K_\alpha + a - 1} \ .$$

Now for fixed K, f(K, a) attains its maximum value at $a = 1$. We have

$$n_{i-1} < f(K, a) \leq f(K, 1) \ ,$$

and hence, by Lemma 7

$$f(K, 1) > 3 \ .$$

That is

$$K < \frac{3}{2} \{\lceil 2\sqrt{K} \rceil - 1\} \ .$$

This inequality does not hold for any $K \geq 8$. Hence condition (iv) cannot hold.

The only remaining possibility is condition (ii). Since (i) and (iii) are not possible, we have

$$n_{i-2} > n_{i+1}$$

and

$$n_{i+2} > n_{i-1} .$$

The transformation

$$n_{i-1} \rightarrow n_{i-1} + 1 \quad \text{and} \quad n_i \rightarrow n_i - 1$$

is feasible, since (iv) is not satisfied, and yields a higher edge count. This contradiction completes the proof of the lemma. □

## 4. DISCUSSION

Many of the proofs of Section 3 are unfortunately lengthy and detailed. However, the properties themselves are simply stated and will be seen in [3] to be important for the characterization of the edge maximal graphs of the class $G_e(n, D, K)$. The property stated in Lemma 8 in particular allows us to assert that, for fixed $D$ and $K$ and sufficiently large $n$, "additional' vertices will all belong to the same level set; but in order to show that this result holds also for values of $n$ close to the "minimal" values considered in [2], it will be necessary essentially to establish the following:

**Lemma 9.** In an edge-maximal vertex sequence, every internal fat triple contains a removable vertex. □

This property is the subject of forthcoming work [3]. From it one may immediately establish the main characterization theorem:

**Theorem 2.** The vertex sequence of an edge-maximal graph $G \in G_e(n, D, K)$, $D \geq 6$, $K > 8$, takes the form

$$(1, K, n_2, n_3, \ldots, n_{D-2}, K, 1)$$

with every internal triple, except possibly one, being minimal. The exceptional triple contains $n_2$ or $n_{D-2}$. $\square$


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