On Covering Pairs by Quintuples: The Cases $v \equiv 3$ or 11 modulo 20

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ABSTRACT

Let $C(v)$ denote the least number of quintuples of a v-set V with the property that every pair of distinct elements of V occurs in at least one quintuple. It is shown, for $v \equiv 3$ or 11 modulo 20 and $v \geq 11$, that $C(v) = \lceil (v-1)/4 \rceil$ with the possible exception of $v \in \{83,131\}$.

1. Introduction.

A (k,t) cover of order v is a pair (V,F) where V is a v-set and F is a family of k-subsets, called *blocks*, of V which has the property that every t-subset of V occurs in at least one block. If C is a particular family such that no other (k,t) cover of V has fewer blocks, then we define the number $C(t,k,v)$ to be $|C|$, the number of blocks in C. (Clearly this number is independent of the v-set V chosen). The numbers $C(2,3,v)$ were determined by Fort and Hedlund [4], and the numbers $C(2,4,v)$ by Mills [15], [16]. In addition Mills [17] determined the numbers $C(3,4,v)$ for all $v \equiv 7$ mod 12, and the numbers $C(3,4,v)$ for $v \equiv 7$ mod 12, v large, were determined by Hartman, Mills, and Mullin [8]. Lamken, Mills, Mullin, and Vanstone [11] determined the values of $C(2,5,v)$ for $v \equiv 2$ mod 4 (with two possible exceptions), and for $v \equiv 1$ mod 4 for v large. In this paper we lay the foundations for the case $v \equiv 3$ mod 4; in particular we determine for $v \equiv 3$ and 11 modulo 20 covers of a specific type, which we call star covers, which appear to be fundamental to determining the numbers $C(2,5,v)$ in the remaining congruence classes of the case $v \equiv 3$ mod 4.

2. Star Covers.

For convenience, we henceforth abbreviate $C(2,5,v)$ to $C(v)$ since we will consider only (5,2) covers. It is also convenient to define $B(v) = \lceil v \lceil (v-1)/4 \rceil /5 \rceil$, where $\lceil x \rceil$ denotes the greatest integer not exceeding x. It is well-known that $C(v) \geq B(v)$. Let v be an integer greater than three which satisfies either $v \equiv 3$ modulo 20 or $v \equiv 11$ modulo 20. Then a *star cover* D of order v is a pair (V,F) where V is a v-set and F is a set of quintuples and one triple from V with the property that every pair of elements from V occurs in at least one block of D. We also admit one block consisting of the three elements of a 3-set as a star cover of order 3. A *star design* of order v is a star cover of order v in which there are $B(v) - 1$ quintuples (and one triple). Let $C^* = \{v: \exists$ a star design of order $v\}$. Clearly if $v > 3$ and $v \in C^*$, then $C(v) = B(v)$. A star cover $D' = (V',F')$ is a *sub-cover* of a star cover $D = (V,F)$ if $V' \subset V$ and $F' \subset F$. (Subdesigns of designs are defined analogously). Let $C = (V,F)$ be a (5,2) cover of order v. The *excess graph* $E = E(C)$ of C is a loop-free multigraph whose vertex set is V and whose edge set is defined as follows: the pair $\{x,y\}$, $x \neq y$, is an edge of multiplicity $e - 1$ in E if the pair xy occurs in exactly e blocks of C. The excess graph $E(D)$ of a star design is defined analogously.

A *star graph* is a loopless multigraph in which three vertices are of degree zero, and the remaining vertices are all of degree two.

Lemma 2.1. *Let $E = E(D)$ be the excess graph of a star cover D. Then D is a star design if and only if E is a star graph.*

Proof. Let D be a star cover of order v whose excess graph is a star graph. Since star designs are only defined for $v \equiv 3$ or 11 modulo 20, we have $v - 1 \equiv 2$ modulo 4. Let x be any point of D. First assume that x is a point of degree zero in E. Then x occurs with all other points exactly once in the blocks of D. Since we have $v - 1 \equiv 2$ modulo 4, this implies that x occurs in the block of size 3 and in $\lceil (v-1)/4 \rceil - 1$ quintuples of D. For any point x of degree 2, a similar argument shows that x occurs in precisely $\lceil (v-1)/4 \rceil$ quintuples and no quadruples. Thus adding the frequencies $f(x)$ with which the points x of D occur in quintuples, we obtain

$$\sum_{x \in D} f(x) = v\lceil (v-1)/4 \rceil - 3.$$

Thus the number of blocks of size five in D is $N = (v\lceil (v-1)/4 \rceil - 3)/5$. Since we have either $v \equiv 3$ modulo 20 or $v \equiv 11$ modulo 20, we have $N = B(v) - 1$; so D is a star design. Conversely, if D is a star design, it is easily shown that E is a star graph. □

We use definitions of group divisible designs, transversal designs, balanced incomplete block designs, as given in [29], and definitions of flat and resolvable balanced incomplete block designs, as given in [25].

Lemma 2.2. *Suppose there exists a group divisible design G with s groups of size $g_1,g_2,\ldots,g_s$, and blocks of size 5 and that there exists an integer w such that for each i satisfying $1 \leq i \leq s-1$ there exists a star design of order $g_i + w$ which contains a star subdesign of order w. Suppose also that there exists a star design S of order $g_s + w$. Let $v = w + \sum_{i=1}^{s} g_i$. Then there exists a star design D of order v. If S contains a subdesign of order w, then D contains subdesigns of order w and $g_i + w$, $i = 1,2,\ldots,s$.*

Proof. Let $G_1,G_2,\ldots,G_s$ denote the groups of G of size $g_1,g_2,\ldots,g_s$ respectively. Let W be a set of cardinality w such that $W \cap G_i = \emptyset$, $i = 1,2,\ldots,s$. For each group G_i, $i = 1,2,\ldots,s-1$, form a star design D_i on $S_i = W \cup G_i$ such that D_i contains a star subdesign D'_i on the set W. Let D''_i denote the set of blocks complementary to the block set of D'_i in D_i. Now form the set of blocks consisting of the following three types.

Type 1. The blocks of G.

Type 2. The blocks of D''_i, $i = 1,2,\ldots,s-1$.

Type 3. The blocks of a star-design on the point set $G_s \cup W$.

It is clear that the above blocks form a star cover of the points of $W \cup \bigcup_{i=1}^{s} G_i$. Consideration of the associated excess graph shows it to be a star-graph, and thus the star-cover is in fact a star design. □

Henceforth, we use the notation $GDD(g_1^{a_1},g_2^{a_2} \cdots g_i^{a_s},K)$, (where $g_1,g_2,\ldots,g_s$, $a_1,a_2,\ldots,a_s$, are integers and K is a set of positive integers) to denote a group divisible design with a_i groups of size g_i, $i = 1,2,\ldots,s$, and blocks of sizes in K. If K contains only one element k, we abbreviate $\{k\}$ to k.

Unlike many recursive constrictions we cannot use $w = 0$ in the above, since the null design is not (a degenerate) star design. However every star design clearly has a subdesign of order 3.

3. Some Small Star Designs.

In this section, it is shown that there exist star designs of orders three, eleven, twenty-three and thirty-one. The design of order three is trivial, and a star design of order eleven is shown below.

	1 2 3	
1 4 5 9 10		2 5 7 8 10
1 6 7 8 11		3 4 7 8 9
2 4 6 9 11		3 5 6 10 11

A star design of order 23 can be constructed as follows. Let P denote the projective plane of order 4 defined on the set $\{1,2,...,21\}$. Clearly P contains three non-concurrent lines which, without loss of generality, can be assumed to be the lines $\{1,2,3,4,5\}$, $\{5,6,7,8,9\}$, $\{9,10,11,12,1\}$. Delete these lines from P to obtain a configuration P'. Take the lines of P' as blocks, and adjoin two points $\{22,23\}$ and the blocks

$$\begin{array}{ll}\{22,2,3,4,5\}, & \{13,14,15,22,23\},\\ \{1,2,3,4,23\}, & \{16,17,18,22,23\},\\ \{5,6,7,8,23\}, & \{19,20,21,22,23\},\\ \{22,6,7,8,9\}, & \{1,5,9\}.\\ \{9,10,11,12,23\}, & \\ \{22,10,11,12,1\}, & \end{array}$$

The resulting configuration is a star design of order 23.

The following star design of order 31 was found by E.S. Kramer.

Let $S = \{1,2,...,28,\infty_1,\infty_2,\infty_3\}$, and let α denote the permutation $\alpha = (1,2,...,14)\,(15,16,...,28)\,(\infty_1,\infty_2)\,(\infty_3)$. Let B denote the set of blocks

$$\begin{array}{ll}\{1,9,11,14,17\} & \{1,8,19,20,\infty_1\}\\ \{2,15,17,25,26\} & \{1,8,15,22,\infty_3\}\\ \{\infty_1,\infty_2,\infty_3\}. & \end{array}$$

Then the distinct images of the above blocks under powers of α yield the required star-design.

The following star designs of order 63 and 71 were found by W.H. Mills.

$v = 63$. $C(2,5,63) = L(2,5,63) = 202$. We use the triple $(\infty_1,\infty_2,\infty_3)$ and the following 201 quintuples:

(0,1)	(3,1)	(6,1)	(9,1)	(12,1)	mod (15,-),	period 3
(0,2)	(3,2)	(6,2)	(9,2)	(12,2)	mod (15,-),	period 3
(0,1)	(1,1)	(0,2)	(1,2)	(0,4)	mod (15,-)	
(0,1)	(2,1)	(0,2)	(5,2)	(8,4)	mod (15,-)	
(0,1)	(4,1)	(11,2)	(5,4)	(13,4)	mod (15,-)	
(0,1)	(5,1)	(0,3)	(1,3)	(3,3)	mod (15,-)	
(0,1)	(7,1)	(9,3)	(4,4)	(10,4)	mod (15,-)	
(0,1)	(2,2)	(9,2)	(8,3)	(14,3)	mod (15,-)	
(0,1)	(8,2)	(10,2)	(4,3)	(12,3)	mod (15,-)	
(0,2)	(4,2)	(7,3)	(1,4)	(11,4)	mod (15,-)	
(0,2)	(8,3)	(13,3)	(4,4)	(6,4)	mod (15,-)	
(0,3)	(4,3)	(3,4)	(6,4)	(7,4)	mod (15,-)	
∞_1	(0,1)	(4,2)	(5,3)	(2,4)	mod (15,-)	
∞_2	(0,1)	(6,2)	(6,3)	(11,4)	mod (15,-)	
∞_3	(0,1)	(12,2)	(7,3)	(7,4)	mod (15,-)	

$v = 71$. $C(2,5,71) = L(2,5,71) = 256$. We use the triple $(\infty_1,\infty_2,\infty_3)$ and the following 255 quintuples:

(0,1)	(1,1)	(3,1)	(0,4)	(5,4)	mod (17,-)
(0,1)	(4,1)	(0,2)	(1,2)	(11,4)	mod (17,-)
(0,1)	(5,1)	(0,2)	(2,2)	(1,4)	mod (17,-)
(0,1)	(6,1)	(4,2)	(0,3)	(12,4)	mod (17,-)
(0,1)	(7,1)	(10,2)	(5,3)	(15,4)	mod (17,-)
(0,1)	(8,1)	(3,3)	(9,3)	(16,3)	mod (17,-)
(0,1)	(7,2)	(11,2)	(13,3)	(14,3)	mod (17,-)
(0,1)	(6,2)	(16,2)	(7,3)	(10,3)	mod (17,-)
(0,2)	(3,2)	(11,2)	(6,4)	(7,4)	mod (17,-)
(0,2)	(5,2)	(5,3)	(14,3)	(14,4)	mod (17,-)
(0,3)	(2,3)	(0,4)	(3,4)	(7,4)	mod (17,-)
(0,3)	(5,3)	(2,4)	(11,4)	(13,4)	mod (17,-)
∞_1	(0,1)	(5,2)	(4,3)	(3,4)	mod (17,-)
∞_2	(0,1)	(8,2)	(6,3)	(10,4)	mod (17,-)
∞_3	(0,1)	(9,2)	(2,3)	(9,4)	mod (17,-)

4. Star designs of order less than 200.

In this and subsequence sections, we require many transversal designs (or equivalently, sets of mutually orthogonal latin squares). Unless otherwise stated, the reference for the existence of these is Brouwer [1]. In addition, in the proof of lemma 4.2, we use the fact that there is a group divisible design with six groups of size eight and blocks of size 5 (see [2]).

Lemma 4.1. *Let M, T and U be integers satisfying $u \geq 3$ and $0 \leq T \leq M$. Suppose that*

(i) there exists a star design of order $4M+U$ which contains a star subdesign of order U,

(ii) there exists a star design S of order $4T+U$, and

(iii) there exists a transversal design $TD(6,M)$.

Then there exists a star design S' of order $20M+4T+U$ which contains a star subdesign of order $4T+U$. Moreover, if S also contains a star subdesign of order U, then S' can be taken to have subdesign of order U.

Proof. Deleting a point from a projective plane of order 4 and an affine plane of order 5 yields group divisible designs P and A respectively, with blocks of size 5 and five and six groups of size 4 respectively. By deleting $M-T$ points from one group of the $TD(6,M)$ one obtains a group divisible design with blocks of sizes 5 and 6 having 5 groups of size M and one group of size T. Using Wilson's Fundamental Construction [29], from the above ingredients one obtains a group divisible design with five groups of size $4M$, one group of size $4T$ and blocks of size 5. Replacing the groups with star designs in a fashion analogous to that of lemma 2.2 establishes the result. □

Corollary 4.1.1. *Suppose that m and t are integers satisfying $0 \leq t \leq m$. If there exists a $TD(6,m)$ and if $\{4m+3,4t+3\} \subset C^*$, then $20m + 4t + 3 \in C^*$.*

Proof. Apply the above lemma with $U = 3$. □

Lemma 4.2. *Suppose that m and t are integers satisfying $0 \leq t \leq m$. If there exists a $TD(6,m)$ and if $\{8m+3,8t+3\} \in C^*$, then $40m + 8t + 3 \subset C^*$.*

Proof. The proof is that of the lemma 4.1 using the fact that there are group divisible designs with blocks of size 5 and six and five blocks of size 8, respectively. (The design with five groups follows from the fact that there exists a $TD(5,8)$ (see [2])).

Lemma 4.3. *Let v be an integer which satisfies $v \equiv 3$ or 11 modulo 20 and $0 \leq v \leq 200$. Then $v \in C^*$, with the possible exception of $v \in \{83,91,131\}$.*

Proof. For $v \leq 31$, the result follows from section 3. By applying Lemma 2.2 to the group divisible designs used in Lemma 4.2 and the star design of order 11, we have $\{43,51\} \subset C^*$. Since there is a $TD(6,5)$, by applying Corollary 4.1.1, we have $\{103,111,123\} \subset C^*$. Since there is a $TD(6,7)$, we have $\{143,151,163,171\} \subset C^*$. It remains to show that 183 and 191 belong to C^*. Since the star design of order 43 above contains a subsystem of order 11, and since $\{23,31\} \subset C^*$ and there exists a $TD(6,8)$, we have $\{183,191\} \subset C^*$ by Lemma 4.1. □

5. Star Designs of order less than 1200.

In this section, we require incomplete transversal designs. An incomplete transversal design $ITD(n,k,s)$ is a quadruple $D = (V,G,H,B)$ where V is an nk-set; G is a partition $G_1,G_2,...,G_k$ of V, where $|G_i| = n$, (the G_i are called groups of the design) $i = 1,2,...,k$; H is a collection $H_1,H_2,...,H_k$ of s-subsets such that $H_i \subset G_i$, $i = 1,2,...,k$; and B is a collection of k-element subsets (called blocks) of V which satisfies the following conditions

(i) Each block meets each group in precisely one point.

(ii) Let x_i and x_j denote any pair of points of V which occur in groups G_i and G_j of D_1 where $i \neq j$. If $x_i \in H_i$ and $x_j \in H_j$, then no block contains both x_i and x_j; otherwise there is a unique block in B which contains the pair $x_i x_j$.

Clearly an incomplete transversal design is equivalent to a set of $k - 2$ mutually orthogonal latin squares which is missing a set of common subsquares of order s, or equivalently an incomplete array in the sense of [9].

The following construction is due to D.R. Stinson.

Lemma 5.1. *Suppose that m and t are positive integers satisfying $0 \leq t \leq m - 2$. Suppose that there is a star design of order $4m + 3$ which contains a subdesign of order 11, and that $4t + 3$ is in C^*. Suppose further that there exists an $ITD(m,6,2)$. Then there exists a star design of order $20m + 4t + 3$ which contains subdesigns of order 43 and $4t + 3$.*

Proof. The proof involves a modification of Wilson's fundamental construction [29]. Let $D = (V,G,H,B)$ be the $ITD(m,6,2)$, and let P and A denote a $GDD(4^5,5)$ and $GDD(4^6,5)$. In D delete $m - t$ points from the group G_1, including both points of H_1, to obtain a new system D' which contains blocks of sizes 5 and 6. In analogy with Wilson's construction, inflate the points of D' by a factor of four to obtain a configuration D'' with one group G'_1 of size $4t$, and five groups $G'_2,G'_3,...,G'_6$, of size $4m$, (each having a distinguished subset H' of size 8), and blocks of size 5. Note that D'' contains all pairs of element from different groups except for those both of whose members come from distinguished subsets. Replace the groups G'_i, $i = 2,3,\dots,6$, by star design covers S_i on the sets $G'_i \cup W$ (where W is a three-element set disjoint from the groups G'_i, $i = 1,2,\dots,6$) in such a fashion that S_i contains a star subdesign T_i of order 11 on the set $H'_i \cup W$. Replace the group G'_1 by a star design S_1 on the set $G'_i \cup W$ in such a fashion that the block T_1 of size 3 contains the elements of W. Now replace the blocks of $\bigcup_{i=1}^{6} T_i$ by a star design of order 43 on the set $W \cup \bigcup_{i=2}^{6} H'_i$, again in such a fashion that the block of size 3 contains the elements of W. It is easily verified that the resulting set of blocks is a star design of order $20m + 4t + 3$ on the set $W \cup \bigcup_{i=1}^{6} G'_i$ with subdesigns on $W \cup \bigcup_{i=2}^{6} H'_i$ and $W \cup G'_1$. This establishes the lemma. □

Lemma 5.2. *Suppose that there exists a star design of order $4m + w$ which contains a subdesign of order w, and that there exists a transversal design $TD(5,m)$. Then there exist star designs of order $20ms + w$ and $20ms + 4m + w$ which contain subsystems of orders w and $4m + w$, for any $s \geq 1$.*

Proof. It is well-known [7] that there exist *BIBD*s with $k = 5$, $\lambda = 1$, $v = 20s + 1$ and $20s + 5$, $s = 1,2,\dots$. Deleting a point from such designs yields a $GDD(4^{5s},5)$ or $GDD(4^{5s+1},5)$. Inflating these by the transversal design yields a $GDD(4m^{5s},5)$ or a $GDD(4m^{5s+1},5)$. Applying Lemma 2.2 establishes the lemma. □

Corollary 5.2.1. $\{100s+3,100s+23,140s+3,140s+31,160s+11,160s+43\} \subset C^*$, *for any* $s \geq 1$.

Proof. Apply Lemma 5.2 with the pairs $(4m+w,w)$ in the set $\{(23,3)(31,3)(43,11)\}$. □

Lemma 5.3. *Suppose there exists a resolvable $BIBD(20s+5,5,1)$, and a star design of order $4t+3$, where $0 \leq t \leq 5s$. Then there exists a star design of order $80s + 4t + 23$.*

Proof. Let D denote the resolvable $BIBD$. Then D contains $5s+1$ resolution classes $R_0, R_1, ..., R_{5s}$. Let $X = \{\infty_1, \infty_2, \ldots, \infty_t\}$ be a set of t points disjoint from the point set of D. Adjoin ∞_i to each block of R_i, $i = 1,2,...,t$, and a block $B_0 = X$. Taking the blocks of R_0 and B_0 as groups, the result is $GDD(5^{4s+1}t^1, \{5,6\})$. Inflating with the GDD's P and A of Lemma 5.1 and applying Lemma 2.2, we obtain the result. □

Lemma 5.4. *Suppose there exists a $BIBD(v,6,1)$ with a flat of order s. If there exists a star design of order $4t+3$ where $0 \leq t \leq s-1$, then there exists a star design of order $4(v-s+t)+3$.*

Proof. By replacing the flat of order s by a single block B_0 (if it is not such initially), one obtains a pairwise balanced design D with all blocks of size 6 with the possible exception of the distinguished block B_0. Let ∞ denote a fixed point of B. Let $X = \{B - \{\infty\}: B$ is a block of D which contains $\infty\}$. Delete $s-t-1$ further points of B_0 from D. Then the remaining part of B_0 together with X form the groups of a GDD with blocks of size 5 and 6. Proceed as in Lemma 5.3. □

In most applications of the above, the flat in question consists of a single block. As a source of $BIBD(v,6,1)$ we use Mills [19], [20]. Henceforth

$$B(k) = \{v: \exists \text{a } BIBD(v,k,1)\}.$$

Lemma 5.5. *There exist star designs for $v \in \{411,431,443,471,483,611\}$.*

Proof. From [20], $\{106,111,156\} \in B(6)$. Apply Lemma 5.4. □

Lemma 5.6. *There exist star designs for $v \in \{303,311,323,331,403,423,451,491,683,871\}$.*

Proof. Use Corollary 5.2.1. Consider the following equations.

$303 = 3 \cdot 100 + 3$	$423 = 4 \cdot 100 + 23$
$311 = 2 \cdot 140 + 31$	$451 = 3 \cdot 140 + 31$
$323 = 3 \cdot 100 + 23$	$491 = 3 \cdot 160 + 11$
$331 = 2 \cdot 160 + 11$	$683 = 4 \cdot 160 + 43$
$403 = 4 \cdot 100 + 3$	$871 = 6 \cdot 140 + 31$

These establish the lemma. □

Lemma 5.7. *Suppose that, for some integer $m \geq 2$, both $20m + 3$ and $20m + 11$ belong to C^*. If there exists a $TD(6,5m)$ and a $TD(6,5m+2)$, and if $100m \leq v \leq 100(m+1)$, where $v \equiv 3$ or 11 modulo 20 then v belongs to C^*.*

Proof. Apply Corollary 4.1.1, using the fact that $\{3,11,23,31,43,51\} \subset C^*$. □

Lemma 5.8. *Suppose the $v \equiv 3$ or 11 modulo 20. If $200 < v < 1100$, then $v \in C^*$.*

Proof. For $203 \leq v \leq 231$, use Lemma 5.1, noting that Brouwer [3] has constructed an $ITD(10,6,2)$.

For $243 \leq v \leq 291$, use Corollary 4.1.1 with $m = 7$.

For $303 \leq v \leq 331$, use Lemma 5.6.

For $343 \leq v \leq 391$, use Lemma 5.3, noting that there exists an $RBIBD(85,5,1)$ (see [12]).

For $v = 403 \leq v \leq 491$, $v \neq 463$, use Lemmas 5.5 and 5.6.

For $v = 463$, note that the star design of order 111 constructed in section 4 contains a star subdesign of order 23. Since there exists a $T = TD(5,88)$, applying Lemma 4.1 shows that $463 \in C^*$, since $463 = 5 \cdot 88 + 23$.

For $503 \leq v \leq 591$, use Lemma 5.7.

For $603 \leq v \leq 611$, use Lemmas 5.5 and 5.6.

For $623 \leq v \leq 691$, we note that since there exists $T = TD(6,31)$ and $D = BIBD(31,6,1)$, we obtain a $BIBD(186,6,1)$ with a flat of order 31 by replacing the groups of T by copies of D. Applying Lemma 5.4 yields the required star designs.

For $703 \leq v \leq 791$, use Lemma 5.7.

For $803 \leq v \leq 871$, apply Corollary 4.1.1 with $m = 40$.

For $883 \leq v \leq 891$, use Corollary 4.1.1 with $m = 37$.

For $903 \leq v \leq 951$, use Corollary 4.1.1 with $m = 45$.

For $963 \leq v \leq 991$, star covers can be constructed as follows. It is shown in [12] that there is a $RBIBD(205,5,1)$. Applying Lemma 5.3, noting that $\{143,151,163,171\} \subset C^*$, yields the required star designs.

For $1003 \leq v \leq 1051$, apply Corollary 4.1.1 with $m = 40$.

For $1063 \leq v \leq 1083$, apply Corollary 4.1.1 with $m = 45$.

For $v = 1091$, the final case of this lemma, proceed as follows.

It is shown in [20] that there exists $BIBD(136,6,1)$ which we denote by D. Considering each point of D as a group yields a $GDD(1^{136},6)$. Inflate by a factor of 8 to obtain a $GDD(8^{136},6)$ and apply Lemma 2.2. Since $1091 = 8 \cdot 136 + 3$, the required star design exists. This completes the proof of the lemma. □

Henceforth let $X = \{83,91,131\}$. The foregoing can be summarized as follows.

Lemma 5.9. *Let v be an integer congruent to 3 or 11 modulo 20. If $3 \leq v \leq 1091$, then v belongs to C^* with the possible exception of $v \in X$.*

6. The covering numbers $C(v)$.

In this section, it is shown there is a star design of order $v > 0$ for all $v \equiv 3$ or 11 modulo 20 with the possible exception of $v \in X$. Moreover, for $v \geq 11$, if v is congruent to 3 or 11 modulo 20, then $C(v) = B(v)$ with the possible exception of $v \in X'$ where $X' = \{83,131\}$.

Theorem 6.1. *Let v be positive integer satisfying $v \equiv 3$ or 11 modulo 20. Then v belongs to C^* with the possible exception of $v \in X$.*

Proof. For all $m \geq 53$, there exists a $TD(6,m)$. For $v \geq 1203$, a straightforward induction (using Lemma 5.7 for $m \geq 12$) establishes the result. □

Lemma 6.2. *Suppose that for $v \equiv 19$ modulo 20, $C(v) = B(v)$. Then $C(5v-4) = B(5v-4)$.*

Proof. For $s \equiv 18$ modulo 20, there exists a $TD(5,s)$. Let T denote a $TD(5,v-1)$. Let ∞ denote a point which does not occur in T. Replace each group G_i of T by a copy of a (5,2) cover of v points with $B(v)$ blocks defined on the set $G_i \cup \{\infty\}$. The resulting set of blocks is easily shown to be a minimal cover of $5v - 4$ points. □

Corollary 6.2.1. $C(91) = B(91)$.

Proof. It is shown in Gardner [6] that $C(19) = B(19)$. The corollary follows. □

The results of the foregoing are brought together in the following theorem.

Theorem 6.3. *Let v be an integer which is congruent to* 3 *or* 11 *modulo* 20, $v \geq 11$. *Then* $C(v) = B(v)$ *with the possible exception of* $v \in \{83,131\}$.

Proof. Recall that, for any v satisfying the above conditions, if $v \in C^*$, then $C(v) = B(v)$. This, combined with the above corollary, establishes the theorem. □

Star designs form the basis for an investigation of minimal (5,2) covers for $v \equiv 3$ modulo 4. Results on this problem will be reported elsewhere.

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