THE CYCLE SPACE OF AN EMBEDDED GRAPH II

R. Bruce Richter
U.S. Naval Academy

Abstract: A basis is exhibited for the first homology space of a surface over a field. This basis is found by extending a basis of the boundary cycle space of an embedded graph to the cycle space of the graph.

1. Introduction

In homology texts, one common exercise is to find the rank of the first homology group of a surface. If the coefficient group is a field, then this homology group is a vector space. In this work, we exhibit a basis of this vector space, from within a basis for the cycle space of a graph embedded in the surface; this finds the dimension of the homology space in a new way. We proceed using the theory of embedded graphs developed by Hoffman and Richter [3] and the cycle space of a graph over an arbitrary field K.

Let $g:G \to \Sigma$ be an embedding of a connected graph G in the surface Σ so that each face is homeomorphic to the open unit disc. Let G^* denote the geometric dual of G. If d is a subset of the edge set E of G, then $\neg d$ denotes the set of edges of G^* that are dual to those edges of G in d. (Precise definitions will be given in the next section.)

Let $Z(g,K)$ denote the cycle space of G and let $Q(g,K)$ denote the subspace of $Z(g,K)$ spanned by the boundary cycles. In [7], the following results are established.

Theorem 1: a) If T is a spanning tree of G, then there is a subset T^* of $E-T$ such that $\neg T^*$ is a spanning tree of G^*.

b) For a spanning tree T of G, let $T^* \subseteq E-T$ be such that $\neg T^*$ is

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a spanning tree of G^*. For $e \notin T$, let $Z_{\cdot}e$ denote the fundamental cycle of e with respect to T. If q is any basis for $Q(g, \mathbb{Z}_{\cdot})$, then $q \cup \{Z_{\cdot}e \mid e \in E-(T \cup T^*)\}$ is a basis for $Z(g, \mathbb{Z}_{\cdot})$, where $\mathbb{Z}_{\cdot}$ is the set of integers modulo 2.

Part (b) of Theorem 1 asserts that the cosets $Z_{\cdot}e + Q$, for $e \in E-(T \cup T^*)$ form a basis for the first homology space $H = Z/Q$, when the underlying field is $\mathbb{Z}_{\cdot}$. Here, we generalize (b) as follows.

<u>Theorem 2</u>: Let G, G^*, T and T^* be as in (b) of Theorem 1 and let K be an arbitrary field. Orient the faces of G so that if $e \in T^*$, then e is traversed once in each direction. Let $E^{\sim}$ denote the subset of $E-T$ consisting of all edges traversed both times in the same direction. Let q be any basis for $Q(g, K)$. Then:

(a) If either $E^{\sim} = \emptyset$ or K has characteristic 2, then $q \cup \{Z_{\cdot}e \mid e \in E-(T \cup T^*)\}$ is a basis for $Z(g, K)$.

(b) Otherwise, for any member e^* of $E^{\sim}$, $q \cup \{Z_{\cdot}e \mid e \in E-(T \cup T^* \cup \{e^*\})\}$ is a basis for $Z(g, K)$.

In Section 2, the necessary notation and definitions are given. Section 3 develops the basic results concerning chain spaces and boundary maps; with these the cycle and boundary cycle spaces are defined. Section 4 is devoted to the proof of Theorem 2.

2. <u>Notation and Definitions</u>

We assume the reader has a basic familiarity with graph theory. We shall use the terminology of [1]; loops and multiple edges are allowed.

A _surface_ is a compact, connected 2-dimensional manifold without boundary; it may be non-orientable.

For a set A, 2^A denotes the set of all subsets of A. For a subspace A of a topological space B, cl(A,B) denotes the closure of A in B.

An _embedding_ of a graph G in a surface Σ is a function $g:(V \cup E) \to 2^{\Sigma}$ such that:

i) for $x, y \in V \cup E$, if $x \neq y$, then $g(x) \cap g(y) = \emptyset$;

ii) for each $v \in V$, $g(v)$ is a singleton;

iii) for each $e \in E$, $g(e)$ is homeomorphic to the open interval $(0,1)$;

and iv) if $e \in E$ has ends v and w, then $cl(g(e),\Sigma) = g(\{e,v,w\})$.

Here, V and E are the vertex- and edge-sets of G, respectively. For simplicity, we shall write $g:G \to \Sigma$ rather than $g:(V \cup E) \to 2^{\Sigma}$.

A _face_ of an embedding $g:G \to \Sigma$ is a connected component of $\Sigma - g(G)$. Throughout this work, every face is assumed to be homeomorphic to the open unit disc.

For an edge e of G and an embedding $g:G \to \Sigma$, it can be shown that there is a continuous surjection $f_e:[0,1] \to cl(g(e),\Sigma)$ such that $f_e:(0,1) \to g(e)$ is a homeomorphism. Moreover, if e is not a loop, then $f_e:[0,1] \to cl(g(e),\Sigma)$ is a homeomorphism. (For details, see [6].) Let v and w be the vertices of G such that $f_e(0)=g(v)$ and $f_e(1)=g(w)$. Define the _head_ h(e) of e to be w and the _tail_ t(e) of e to be v.

From the results in [3], it is easily shown that, for a face F of an embedding $g:G \to \Sigma$, there is a continuous surjection $h_F:\bar{B}(0,1) \to cl(F,\Sigma)$ such that $h_F:B(0,1) \to F$ is a homeomorphism, where B(0,1) and

$\underline{B}(0,1)$ are, respectively, the open and closed unit discs in the plane. We parameterize the plane with polar coordinates (r,θ), so that, for example, $\underline{B}(0,1) = [0,1]\times[0,2\pi]$. Rotate h_F if necessary so that $h_F(1,0) \in g(V)$.

Let $0 = \theta_0 < \theta_1 < \ldots < \theta_n = 2\pi$ be those angles θ for which $h_F(1,\theta) \in g(V)$. For $j=0,1,\ldots,n$, let $h_F(1,\theta_j) = g(v_j)$. It can be shown that there is an edge e_j such that $h_F:\{1\}\times[\theta_{j-1},\theta_j] \to g(e_j)$ is a homeomorphism, $j=1,2,\ldots,n$. (See [3].) Thus, $(v_0,e_1,v_1,\ldots,e_n,v_n)$ is a closed walk of G, called the <u>boundary walk</u> of G induced by h_F.

Note that $h':\underline{B}(0,1) \to cl(F,\Sigma)$ defined by $h'(r,\theta) = h_F(r,2\pi-\theta)$ induces the inverse of the above walk.

Consider the edge $e = e_j$. The function $f_e^{-1}h_F:\{1\}\times[\theta_{j-1},\theta_j] \to [0,1]$ is a homeomorphism; in particular it is either an increasing or a decreasing function of θ. Define $\epsilon(e,h_F)$ to be $\epsilon^+(e,h_F) - \epsilon^-(e,h_F)$, where ϵ^+ is the number of indices k for which $e = e_k$ and $f_e^{-1}h_F$ is increasing on (θ_{k-1},θ_k) and ϵ^- is the number of such k for which the composition is decreasing.

Observe that $\epsilon(e,h_F) = -\epsilon(e,h')$, $\epsilon^+(e,h_F)+\epsilon^-(e,h_F) \in \{0,1,2\}$, for any edge e and any face F. Moreover, if D is the set of faces of g, then $\Sigma_{F\in D}(\epsilon^+(e,h_F)+\epsilon^-(e,h_F)) = 2$, for every edge e.

We conclude this section by describing the <u>geometric dual</u> of the embedding $g:G\to\Sigma$. This is the graph G^* whose vertices are the faces of g and, for each edge e of G, there is an edge $\neg e$ of G^*. The ends of $\neg e$ are the two faces of g in whose boundary walks the edge e appears. If e occurs twice in some boundary walk, then $\neg e$ is a loop. We shall use the following fact in this work.

<u>Lemma 3</u>: Let $g:G\to\Sigma$ be an embedding of a connected graph G in a sur-

face F. If T is a spanning tree of G, then $\gamma(E-T)$ contains a spanning tree of G^*.

3. Homology

In this section, we review the basics of the homology of a surface. Let K be an arbitrary field and let $g:G\to\Sigma$ be an embedding. Define the following vector spaces over K:

1) $C_0 = C_0(g,K) = \{\Sigma_{v\in V}\alpha_v v \mid \alpha_v \in K\}$;

2) $C_1 = C_1(g,K) = \{\Sigma_{e\in E}\alpha_e e \mid \alpha_e \in K\}$;

and 3) $C_2 = C_2(g,K) = \{\Sigma_{F\in D}\alpha_F F \mid \alpha_F \in K\}$.

The sums are formal, with $(\Sigma_x\alpha_x x) + (\Sigma_x\beta_x x)$ and $\beta(\Sigma_x\alpha_x x)$ defined to be $\Sigma_x(\alpha_x+\beta_x)x$ and $\Sigma_x(\beta\alpha_x)x$, respectively.

Fix the functions f_e and h_F, for each $e \in E$ and each $F \in D$, as described in Section 2. Define the boundary maps $b_2:C_2 \to C_1$ and $b_1:C_1 \to C_0$ by:

1) $b_2(\Sigma_F\alpha_F F) = \Sigma_F\alpha_F(\Sigma_e(\epsilon(e,h_F)e)$; and

2) $b_1(\Sigma_e\alpha_e e) = \Sigma_e\alpha_e(h(e)-t(e))$.

<u>Lemma 4</u>: $b_1b_2:C_2 \to C_0$ is identically zero.

<u>Proof</u>: Since D is a basis for C_2 and both b_1 and b_2 are linear, it suffices to show that $b_1(b_2(F)) = 0$ for each face F of g.

Observe that $b_1(b_2(F)) = b_1(\Sigma_e\epsilon(e,h_F)e) = \Sigma_e\epsilon(e,h_F)(h(e)-t(e)) = \Sigma_e(\epsilon^+(e,h_F)-\epsilon^-(e,h_F))(h(e)-t(e)) = \Sigma_{i}(v_i - v_{i+1}) = v_0 - v_n = 0$. ∎

Another way to state Lemma 4 is to assert that $\mathrm{Im}(b_{\cdot}) \subseteq \mathrm{Ker}(b_{\cdot})$. The <u>cycle space</u> $Z(g,K)$ is defined to be $\mathrm{Ker}(b_{\cdot})$ and the <u>boundary cycle space</u> $Q(g,K)$ is defined to be $\mathrm{Im}(b_{\cdot})$. The <u>homology space</u> is the quotient space $H(g,K) = Z(g,K)/Q(g,K)$.

Evidently, the cycle space depends only on the oriented graph and not on the particular embedding. In Chapter 12 of [1] is an exposition of the basic facts about the cycle space of a graph. We summarize the relevant points here.

For a spanning tree T of G and an edge $e^{\cdot} \notin T$, $T+e^{\cdot}$ contains a unique polygon P. Let $P^{\cdot}$ be the set of edges from P whose orientations agree with that of $e^{\cdot}$ in P and let P^{-} be the remaining edges of P. It is readily verified that $\Sigma_{\cdot\cdot\cdot} e - \Sigma_{\cdot\cdot\cdot} e$ is in $Z = Z(g,K)$. This cycle is the <u>fundamental cycle</u> of $e^{\cdot}$ with respect to T and is denoted $Z_T e^{\cdot}$. It can be shown that $\{Z_T e \mid e \notin T\}$ is a basis for Z and if $z = \Sigma_{e \in E} \alpha_{\cdot} e$ is in Z, then $z = \Sigma_{e \notin T} \alpha_{\cdot} Z_T e$. Thus, $\dim(Z) = |E|-|V|+1$.

4. <u>Proof of Theorem 2</u>

For this section, let $g:G \to \Sigma$ be a fixed embedding of a connected graph G. Let T be a spanning tree of G and let $T^{\cdot} \subseteq E-T$ be such that $\neg T^{\cdot}$ is a spanning tree of the geometric dual $G^{\cdot}$ of g. It is easily shown that there are orientations $\{h_{\cdot} \mid F \in D\}$ of the faces of g such that if $e \in T^{\cdot}$, then $\Sigma_{\cdot} \epsilon(e,h_{\cdot}) = 0$. Fix these orientations.

Let $E^{\sim} = \{e \in E-T \mid \Sigma_{\cdot} \epsilon(e,h_{\cdot}) = \pm 2\}$, so $E^{\sim}$ is the subset of $E-(T \cup T^{\cdot})$ consisting of those edges traversed both times in the same direction by the boundary walks induced by the $h_{\cdot}$'s. (Note that the surface Σ is orientable if and only if we can orient the faces so that, for each edge e, $\Sigma_F \epsilon(e,h_F) = 0$. It is straightforward to see that this is equivalent to $E^{\sim} = \emptyset$.)

We now restate Theorem 2 in a slightly different form. We write Z for Z(g,K) and Q for Q(g,K).

<u>Theorem 5</u>: 1) If either $E^{\sim} = \emptyset$ or K has characteristic 2, then $\dim(Q) = |D| - 1$ and $\{Z_e + Q \mid e \in E-(T \cup T^*)\}$ is a basis for $H(g,\Sigma)$.

2) If $E^{\sim} \neq \emptyset$ and K does not have characteristic 2, then $\dim(Q) = |D|$ and, for any $e^* \in E^{\sim}$, $\{Z_e + Q \mid e \in E-(T \cup T^* \cup \{e^*\})\}$ is a basis for $H(g,\Sigma)$.

This result is proved in Propositions 7 and 8. First, we require the following result.

<u>Lemma 6</u>: If $\Sigma_{F \in D} \alpha_F F \in \mathrm{Ker}(b_2)$, then there is an α such that $\alpha_F = \alpha$ for all $F \in D$.

<u>Proof</u>: If $b_2(\Sigma_F \alpha_F F) = 0$, then $\Sigma_F \alpha_F (\Sigma_e \epsilon(e,h_F) e) = 0$. Reversing the order of summation, this is the same as $\Sigma_e (\Sigma_F \alpha_F \epsilon(e,h_F)) e = 0$. But E is a basis for C_1, so we must have $\Sigma_F \alpha_F \epsilon(e,h_F) = 0$ for each edge e. If $e \in T^*$, then there are distinct faces F and F' such that $\epsilon(e,F) = 1$ and $\epsilon(e,F') = -1$. For every other face F'', $\epsilon(e,F'') = 0$, so $\alpha_F = \alpha_{F'}$. Since this is true for every edge of T^*, and T^* is a spanning tree of G^*, the result follows. ∎

Now for the dimension statements of Theorem 5.

Proposition 7: 1) If either $E^{-} = \emptyset$ or K has characteristic 2, then $\dim(Q) = |D| - 1$.

2) Otherwise, $\dim(Q) = |D|$.

Proof: Suppose $\Sigma_F \alpha_F F \in Ker(b_*)$. By Lemma 6, there is an α such that $\alpha_F = \alpha$, for all F. Hence, $\Sigma_F \alpha_F F = \alpha \Sigma_F F$ and either $\alpha = 0$ or $\Sigma_F F \in Ker(b_*)$.

Now $b_*(\Sigma_F F) = \Sigma_e(\Sigma_F \epsilon(e,h_F))e = \Sigma_{e\in T}(\Sigma_F \epsilon(e,h_F))Z_t e = \Sigma_{e\in E^{-}}(\Sigma_F \epsilon(e,h_F))Z_t e$, since $e \in T \cup E^{-}$ implies $\Sigma_F \epsilon(e,h_F) = 0$.

For $e \in E^{-}$, $\Sigma_F \epsilon(e,h_F) = \pm 2$, so $b_*(\Sigma_F F) = 0$ if and only if either $E^{-} = \emptyset$ or K has characteristic 2. It follows from Lemma 6 that if $E^{-} \neq \emptyset$ and K does not have characteristic 2, then $\{b_*(F) \mid F \in D\}$ is a basis for Q. Otherwise, the only dependency among the $b_*(F)$ is $\Sigma_F b_*(F) = 0$, which implies that $\{b_*(F) \mid F \in D-\{F^*\}\}$ is a basis for Q, where F^* is any member of D.∎

Finally, we prove the assertions about the basis for H.

Proposition 8: Let q be any basis for Q. Then:

1) If either $E^{-} = \emptyset$ or K has characteristic 2, then $q \cup \{Z_t e \mid e \in E-(T \cup T^*)\}$ is a basis for Z.

2) If $E^{-} \neq \emptyset$ and K does not have characteristic 2, then $q \cup \{Z_t e \mid e \in E-(T \cup T^* \cup \{e^*\})\}$ is a basis for Z, where e^* is any member of E^{-}.

Proof: Using Proposition 7, we see that in each case the indicated set has $|E|-|V|+1$ elements, so it suffices to prove that it spans Z. To this end, let $z = \Sigma_e \alpha_e e \in Z$.

Claim: There are $\beta_\bullet \in K$ such that, for each $e \in T^\bullet$, $\Sigma_\bullet \beta_\bullet \epsilon(e, h_\bullet) = \alpha_\bullet$.

Proof: Let $A = (a_{\bullet,\bullet})$ be the $(|D|-1) \times |D|$ matrix whose rows are indexed by the edges in $T^\bullet$ and whose columns are indexed by the faces of g, such that $a_{\bullet,\bullet} = \epsilon(e, h_\bullet)$. The set-up is such that each row of A has exactly two non-zero entries, one of which is 1 and the other is -1. Thus $A\underline{1} = \underline{0}$, where $\underline{1}$ and $\underline{0}$ are the vectors of all 1's and 0's respectively.

Lemma 6 implies $\mathrm{Ker}(A) = \{\alpha\underline{1} \mid \alpha \in K\}$, so that A has rank $|D|-1$. Thus, if $\underline{\alpha} = (\alpha_\bullet)$ is the given $(|D|-1) \times 1$ vector, there is a solution $\underline{\beta}$ to $Ax = \underline{\alpha}$, as claimed. ∎

To prove Proposition 8, observe that $z - b_2(\Sigma_F \beta_\bullet F) = \Sigma_{e \in E} \alpha_\bullet e - (\Sigma_{e \in T^\bullet} \alpha_\bullet e + \Sigma_{e \notin T^\bullet} \mu_\bullet e)$, for some $\mu_\bullet \in F$. Thus, $z = \Sigma_{e \notin T^\bullet} (\alpha_\bullet - \mu_\bullet) e + b_\bullet(\Sigma_F \beta_\bullet F) = \Sigma_{e \notin (TUT^\bullet)} (\alpha_\bullet - \mu_\bullet) Z_\bullet e + b_\bullet(\Sigma_F \beta_\bullet F)$, which proves (1).

To complete the proof of (2), we know that $b_\bullet(\Sigma_F F) = \Sigma_{e \in E^-} (\Sigma_F \epsilon(e, h_\bullet)) Z_\bullet e$. Therefore, $Z_\bullet e^\bullet = (\Sigma_F \epsilon(e^\bullet, h_\bullet))^{-1} (b_\bullet(\Sigma_F F) - \Sigma_{e \in E^- - \{e^\bullet\}} (\Sigma_F \epsilon(e, h_\bullet)) Z_\bullet e)$. Plugging this into our previous expression for z shows z to be a linear combination of the elements $Z_\bullet e$, for $e \in E - (TUT^\bullet U\{e^\bullet\})$ and members of q, as required. ∎

5. Concluding Remarks

The decomposition of Σ described in Sections 2 and 3 shows Σ to be a normal CW-complex of dimension 2 (see [4]). Since the homology of

CW-complexes is the same as simplicial or singular homology. the space $H(g,K)$ is the first homology space of Σ; it is independent (up to isomorphism) of the choices of g and G.

Since Σ is orientable if and only if $E^- = \emptyset$, if K does not have characteristic 2, then dim(H) distinguishes between the orientable and non-orientable surfaces having the same Euler characteristic $|V| - |E| \cdot |D|$.

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