Combinatorial Results On Some Balanced Arrays With $\mu_2 = 1$

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Abstract. In this paper, we obtain results on the number of constraints m for some balanced arrays of strength 4 when the parameters μ_2, μ_3 assume the values 1 and 0 respectively. It is shown the maximum value of m is $\mu_1 + 4$, and the existence of such an array is established.

1. Introduction and Preliminaries

An array with m rows (constraints), N columns (runs, treatment combinations), and with two symbols is merely a matrix T of size $(m \times N)$ with two elements (say) 0, and 1. If $\underline{\alpha}$ is any column vector of T, the symbol $w(\underline{\alpha})$ will denote its weight (the weight being the number of 1's in $\underline{\alpha}$). These arrays become very interesting and useful in combinatorics and statistical design of experiments if some combinatorial structure is imposed on them. One such combinatorial structure leads us to the following definition:

Definition 1.1.. A balanced array (B-array) T of strength t with m constraints, N runs, and with two symbols is a matrix T $(m \times N)$ such that in every $(t \times N, 1 \leq t \leq m)$ submatrix T_0 of T, every $(t \times 1)$ column vector of weight i $(0 \leq i \leq t)$ occurs in T_0 with a constant frequency (say) μ_i.

The vector $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_t)$ is called the index set of the array T, and the B-array is sometimes denoted by $(m, N, t; \underline{\mu}')$

It is quite clear that

$$N = \sum_{i=0}^{t} \binom{t}{i} \mu_i$$

which, for $t = 4$, reduces to

$$N = \mu_0 + 4\mu_1 + 6\mu_2 + 4\mu_3 + \mu_4$$

Balanced arrays have been found to be quite useful in combinatorics, and in constructing factorial designs. If $\mu_i = \mu$ for each i, then the B-array is reduced to an orthogonal array (0-array), and μ is called the index of the 0-array. Also, the incidence matrix of a balanced incomplete block design corresponds to a B-array with $t = 2$. In what follows, the strength of a B-array is always assumed to be four.

Definition 1.2. An m-rowed B-array T is said to be a trim B-array if $X_0 = X_m = 0$, where X_i is the total number of columns of T each of weight i.

Remark: It is quite obvious that the existence of a trim array with $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ implies the existence of the non-trim array with $\underline{\mu}^{*\prime} = (\mu_0^*, \mu_1, \mu_2, \mu_3, \mu_4^*)$ where $\mu_0^* \geq \mu_0, \mu_4^* \geq \mu_4$.

The following can be easily established.

Theorem 1.1. Every B-array $T(m \times N)$ with $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ is also of strength $(4 - i)$ where $0 \leq i \leq 3$. The index sets for $i = 0, 1, 2,$ *and* 3 are given respectively by $(\mu_0 + \binom{i}{1}\mu_1 + \binom{i}{2}\mu_2 + \binom{i}{3}$ $mu_3 + \binom{i}{4}\mu_4, \mu_1 + \binom{i}{1}\mu_2 + \binom{i}{2}\mu_3 + \binom{i}{4}\mu_4, \ldots, \mu_{4-i} + \binom{i}{1}\mu_{5-i} + \ldots + \binom{i}{4}\mu_4)$ with the convention that $\binom{a}{b} = 0$ if $a < b$ or $b < 0$.

Remark. Every m_0-rowed subarray $(4 \leq m_0 \leq m)$ of an m-rowed T is also a B-array with the same index set as that of T.

Definition 1.3. A B-array $T(m \times N)$ with $t = m$ is said to be of full strength.

Remark: To obtain an array of full strength we write all the $\binom{m}{i}$ m-vectors of weight $i, 0 \leq i \leq m$. Considered as an array of strength four, its index set is $\{\binom{m-4}{i}, \binom{m-4}{i-1}, \binom{m-4}{i-2}, \binom{m-4}{i-3}, \binom{m-4}{i-4}\}$ with the convention that $\binom{a}{b} = 0$ if $b < 0$ or $a < b$,

Definition 1.4. An m-rowed B-array T is said to be row-complete (row saturated) if we cannot add another row to it.

Remark: It is not difficult to see that each array of full strength with $1 \leq i < m$ is row-complete.

The problem concerning the maximum possible constraints for 0-arrays of various index values has been studied, among others, by Bose and Bush (1952), Rao (1947), Seiden (1955), and Seiden and Zemach (1966). A similar problem for B-arrays of strength two has been studied by Rafter and Seiden (1974), and for B-arrays of strength four by Chopra (1985). The study of B-arrays of strength four with $\mu_2 = 1$ becomes very important because a great majority of the 2^m fractional factorial designs of resolution V for $m \geq 7$ and lower values of N are obtained from such arrays. (See for example, Chopra, 1977).

2. Balanced Arrays With $\mu_2 = 1$.

We may remark here that there exists always a 4-rowed B-array of strength four with $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$. It can be obtained by writing each of the 4-vector of weight i exactly μ_i times, $i = 0, 1, 2, 3, 4$.

Lemma 2.1. Let T be a trim B-array $(m, N; \underline{\mu}')$ with $m \geq 5$ and $\mu_3 = 0$. Then $\mu_4 = 0$.

Proof: We can always write a $(4 \times N)$ B-array T with $\mu_4 > 0$. In order to obtain a trim array from T, we must place a 0 under every column of weight 4 which contradicts $\mu_3 = 0$.

Remark. In view of the above result, we do not include vectors of weight four in the $(4 \times N)$ array T. It is also evident, because of $\mu_3 = 0$, that every 4-vector of weight $(2 - i)$ can have at most i 1's added $(i = 0, 1, 2)$ to it in the process of attaching additional rows to T. The final array obtained from T be such that none of its column contains more than two 1's in it. Each 4-vector of weight one appears μ_1 times, and it is easy to check that we must place an identity matrix I of size μ_1 under each such a set of μ_1 columns, otherwise the condition $\mu_2 = 1$ is not satisfied for some 4-rowed subarray.

Theorem 2.1. Let T be a trim B-array $(m, N; \underline{\mu}')$ with $m \geq 5$ and $\mu_3 = 0$. Then $\mu_0 = \frac{\mu_1(\mu_1 - 1)}{2}$, the maximum value of m is $\mu_1 + 4$, and that the array $T(\mu_1 + 4, N; \underline{\mu}')$ exists.

Proof. That the maximum value of m is $(\mu_1 + 4)$ follows from the discussion in the remark above. Let the various subarrays be denoted by T_i as shown below.

T_0 = made up of 4 vectors of weight 0	T_1 = vectors of weight 1	T_2 = vectors of weight 2
T_3	T_4 = vectors of weight 1	T_5 = vectors of weight 0

Since T has to be of strength 1 also, therefore every row must have $(\mu_1 + 3)$ 1's in it. Thus the number of 1's in every row of $T_3 = (\mu_1 + 3) - 4 = \mu_1 - 1$. The number of rows in T_3 is μ_1, and each column in it must contain two 1's to satisfy the condition $\mu_2 = 1$. Thus T_3 corresponds to the incidence matrix of a BIB design with $v = \mu_1, r = \mu_1 - 1, \lambda = 1, b = \mu_0, k = 2$ which gives us $b = \mu_0 = \frac{\mu_1(\mu_1 - 1)}{2}$. The array so obtained with these parameters is such that it has every distinct $(\mu_1 + 4)$-vector of weight 2 in it, therefore, it is of full strength and row-complete.

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REFERENCES

Bose, R.C. and Bush, K.A. (1952). Orthogonal arrays of strength two and three. Ann. Math. Statist. 23, 508-524.

Cheng, C.S. (1980). Optimality of some weighing and 2^n fractional designs. Ann. Statist. 8, 436-446.

Chopra,D.V. (1977). Trace-Optimal balanced 2^9 reduced designs of resolution V with 46 to 54 runs. Jour. Indian Statl. Association 15, 179-186.

Chopra, D.V. (1985). On balanced arrays with two symbols. Ars Combinatoria 20A, 59-63.

Longyear, J.Q. (1984). Arrays of strength s on two symbols. Jour. Statl. Plan. and Inference 10, 227-239.

Rafter, J.A. and Seiden, E. (1974). Contributions to the theory and construction of balanced arrays. Ann. Statist. 2, 1256-1273.

Rao, C.R. (1947). Factorial experiments desirable from Combinatorial arrangement of arrays. J. Roy. Statl. Soc. Suppl. 9, 128-139.

Seiden, E. (1955). On the maximum number of constraints of an orthogonal array. Ann. Math. Statist. 26, 132-135.

Seiden, E. and Zemach, R. (1966). On orthogonal arrays. Ann. Math. Statist. 27, 1355-1370.