CIRCUIT CHARACTERIZATIONS OF UNIONS OF GRAPHS

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Abstract

It is shown that unlike the chromatic polynomial, which does not characterize unions of non-trivial graphs, the circuit polynomial characterizes the unions of many families of graphs. They include unions of chains, cycles and mixtures of these graphs, also unions of complete graphs. It is also shown that in general, if a hamiltonian graph is characterized by its circuit polynomial, then so also is the union of the graph with itself.

1. Introduction

The graphs considered here are finite and contain no loops nor multiple edges. Let G be such a graph. We define a circuit (cycle) with one and two nodes to be an isolated node and an edge respectively in G. Circuits with more than two nodes will be called **proper circuits**. A **circuit cover** (or simply a **cover**) of G is a spanning subgraph in which every component is a circuit.

Let us associate an indeterminate or **weight** w_r with every circuit with r nodes, and the monomial $w(S)=\prod w_r$ with every cover S, where the product is taken over all the components of S. Then the **circuit polynomial** of G is

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$$C(G;w) = \Sigma w(S),$$

where the sum is taken over all the covers of S , and w is the vector $(w_1, w_2, \ldots, w_p)$ of indeterminates . We define the circuit polynomial of the null graph (the graph with no nodes) as 1 .

If we restrict the components of the covers to be nodes and edges only , then the resulting polynomial is called the **matching polynomial** of G - denoted by M(G;w) . In this case , $w = (w_1, w_2)$.

The circuit and matching polynomials are special F-polynomials ; and were introduced in Farrell [3] . The basic properties of circuit polynomials are given in Farrell [1] and the introductory paper on matching polynomials is Farrell [2].

Definitions

Let G be a graph and P(G) a polynomial associated with G . We say that P(G) **characterizes** G if and only if for any graph H , P(H) = P(G) implies that $H \cong G$. In the case where P(G) is the circuit polynomial of G , we also say that the graph G is **circuit unique** .

It is of interest to determine whether or not a given polynomial characterizes a given graph . If it does , then the polynomial can be regarded to be a code for the graph- or as a representation of the graph . Such information can be useful for classification of graphs , among other things .

It has been shown (see Farrell and Guo [5] , [6]) that many families of connected graphs are characterized by their circuit polynomials . The problem of characterizations of disconnected graphs has not been looked into .

It is well known, from Read's "overlap" theorem (See Read [9]), that for any graph G with non-trivial components, G_1 and G_2, $\exists$ a graph H (not isomorphic to G) consisting of an isolated node and a component obtained by identifying a node of G_1 with a node of G_2, such that G and H are co-chromatic (i.e. G and H have the same chromatic polynomial). It is therefore not interesting to investigate the characterizations of disconnected graphs by the chromatic polynomial.

In the material which follows, we show that unlike the chromatic polynomial, the circuit polynomial characterizes many non-trivial families of disconnected graphs. These include unions of **chains** (trees with nodes of valencies 1 and 2 only), unions of cycles, unions of chains **and** cycles, unions of complete graphs and certain unions of hamiltonian graphs.

Since the same weight is used throughout the paper, we denote C(G;w) by C(G). Also, the graph G consisting of components G_1, G_2, .., G_k is denoted by $G_1 \cup G_2 \cup G_3 \ldots \cup G_k$. The valency of a node v in G is denoted by d(v) and the valency sequence of G, by $\Pi(G)$ (in which a^r denotes a, a, ...a (r-times)). If for all nodes v_i and v_j in G, $| d(v_i) - d(v_j) | \leq 1$, then we say that G is **nearly regular**. We denote the circuit with r nodes and the chain with r nodes, by Z_r and P_r respectively. Upper and lower limits of summations will be omitted when they are obvious from the context of the summand.

2. Some Preliminary Results

The following lemmas were established in [1] (Theorems 2 , 4, and 7) .

Lemma 1 (The Fundamental Edge Theorem)

Let G be a graph and xy an edge in G . Let G' be the graph obtained from G by deleting xy , G" the graph obtained from G by removing nodes x and y and G* the graph G with the restriction that in every cover , xy must be part of a proper cycle . Then

$$C(G) = C(G') + w_2 C(G'') + C(G^*) .$$

Lemma 2

Let G be a graph consisting of components $G_1, G_2, ..., G_k$. Then

$$C(G) = \prod_{i=1}^{k} C(G_i)$$

The following Lemma is immediate from the definitions .

Lemma 3

Let C(G) be the circuit polynomial of a graph G with p nodes and q edges . Then

(i) The highest power of w_1 in C(G) is $w_1{}^p$ and this occurs with coefficient 1 .

(ii) The coefficient of $w_1{}^{p-2}$ is q .

(iii) The coefficient of w_p is the number of hamiltonian cycles in G .

(iv) The coefficient of $w_{r_1}, w_{r_2}, ... , w_{r_k}$ is the number of spanning subgraphs of G consisting of the disjoint cycles $Z_{r_1}, Z_{r_2}, ... , Z_{r_k}$.

The following lemma was proved in [6].

Lemma 4

Let G be a nearly regular graph and H a graph such that $C(H) = C(G)$. Then H is also nearly regular and $\Pi(H) = \Pi(G)$.

The circuit polynomial of the chain P_p coincides with its matching polynomial . The following result was established in [2] .

Lemma 5

$$C(P_p) = \sum_{k=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p-k}{k} w_1^{p-2k} w_2^{k} \quad .$$

The following result is given in Farrell and Grell [4] .

Lemma 6

$$\frac{\partial C(G)}{\partial w_r} = \sum C(G-Z_r) \quad .$$

where Z_r is a cycle with r nodes , $G-Z_r$ is the graph obtained from G by removing the nodes of Z_r and the summation is taken over all such cycles in G .

3. Unions of Chains

It has already been shown ([6]) that the circuit polynomial characterizes the union of two chains . We will not repeat the proof here. The interested reader can consult [6] . This result is stated formally in the following theorem .

Theorem 1

The circuit polynomial characterizes $P_p \cup P_q$, for all non-negative integers p and q .

For any positive integer s , we denote s copies of the graph G by $\cup^s G$. We will show that the circuit polynomial characterizes $\cup^s P_r$. But first of all , we prove the following lemma .

Lemma 7

Let $C(G) = C(P_{r_1} \cup P_{r_2} \cup \ldots \cup P_{r_s})$, where $r_i > 0$ $(i = 1,2,\ldots,s)$. Then

(i) G is the union of s chains .

(i) If $G \cong P_{t_1} \cup P_{t_2} \cup \ldots \cup P_{t_s}$, where $\Sigma_i t_i = \Sigma_i r_i$, , then the number of even (odd) elements in the set $\{ t_1, t_2, \ldots, t_s \}$ is equal to the number of even(odd) elements in the set $\{r_1, r_2, \ldots, r_s\}$.

Proof

(i) Let $H \cong P_{r_1} \cup P_{r_2} \cup \ldots \cup P_{r_s}$. Then H is nearly regular . By Lemma 4 , G is also nearly regular and $\Pi(G) = \Pi(H) = (1^{2s}, 2^{n-2s})$, where $n = \Sigma_i r_i$. It is clear that C(H) does not contain any term in w_r , for $r>2$. Therefore C(G) (=C(H)) does not contain any term in w_r , for $r>2$. Hence G has no proper cycles . It follows that G is a forest with nodes of valencies 1 and 2 only . ⇒ G is a union of chains . But from $\Pi(G)$ we see that G has 2s nodes of valency 1 . Therefore G is a union of s chains ,

(ii) Let $G \cong P_{t_1} \cup P_{t_2} \cup \ldots \cup P_{t_s}$ $(t_s > 0)$. Without loss in generality, we assume that the integers $r_1, r_2, \ldots, r_k$ $(1 \le k \le s)$ are odd ; also that $t_1, t_2, \ldots, t_m$ $(1 \le m \le s)$ are odd .The monomial containing the highest power of w_2 in C(H) is $w_1^k w_2^\alpha$, where

$$\alpha = \frac{1}{2}\left[\sum_{i=1}^{k}(r_i - 1) + \sum_{i=k+1}^{s} r_i\right]$$

In C(G) , the monomial with the highest power of w_2 is $w_1^m w_2^\beta$, where

$$\beta = \frac{1}{2}\left[\sum_{i=1}^{k}(t_i - 1) + \sum_{i=m+1}^{s} t_i\right]$$

But C(G) = C(H) . Therefore $w_1^k w_2^\alpha = w_1^m w_2^\beta$. ⇒ $m = k$. Hence the result follows . □

Theorem 2

Let $H \equiv \cup^s P_r$. Then C(H) characterizes H , for all non-negative integers r and s .

Proof

Let G be a graph such that C(G) = C(H) . Then , by Lemma 7 , $G \cong P_{t_1} \cup P_{t_2} \cup \dots \cup P_{t_s}$ ($t_s > 0$) , where $\Sigma_i t_i = rs$. Also , if r is even , then all the t_i's are even and if r is odd , all t_i's are odd .

<u>Case 1 (r - odd)</u>

Let r = 2m+1 , for some non-negative integer m . From Lemma 5 , the term with the highest power of w_2 in $C(P_r)$ is $(m+1)w_1 w_2{}^m$. It follows that the term in C(H) containing the highest power of w_2 is $(m+1)^s w_1{}^s w_2{}^{ms}$. The comparative term in C(G) is $\Pi_i(n_i+1)w_1{}^s w_2{}^N$, where $t_i = 2n_i+1$, and $N = \Sigma_i n_i$,for non-negative integers n_i (i = 1, 2, ...,s) .

It follows that

$$(m+1)^s = \Pi_i (n_i + 1) \qquad \dots (1)$$

and N = ms . $\Rightarrow (m+1)s = \Sigma_i (n_i + 1)$ . ...(2)

It can be easily verified (by elementary algebra) that these simultaneous equations have the unique solution $n_i = m$, for i=1,2, ...,s .

Hence $2n_i + 1 = 2m + 1$. $\Rightarrow t_i = r$, for i = 1, 2, ... , s .

<u>Case 2 (r-even)</u>

Let r = 2m and $t_i = 2n_i$, for i = 1, 2, ... ,s . From Lemma 5 , the term in $w_1{}^2$ in the polynomial $C(P_r)$ is $\binom{m+1}{2} w_1{}^2 w_2{}^{m-1}$. It follows that the term in $w_1{}^2$ in C(H) is $s\binom{m+1}{2} w_1{}^2 w_2{}^{ms-1}$. By comparing coefficients , we get

$$s\binom{m+1}{2} = \Sigma_i \binom{n_i+1}{2} .$$

$\Rightarrow$ [$s(m+1)(m)$]/2 $= 1/2$ [$n_1^2 + n_2^2 + \ldots + n_s^2 + n_1 + n_2 + \ldots + n_s$] ... (3)

But $\Sigma_i\ t_i = 2\Sigma_i\ n_i = rs = 2ms$. $\Rightarrow \Sigma_i\ n_i = ms$ . .... (4)

Therefore Equation (3) yields

$$\Sigma_i\ n_i^2 = sm^2 . \qquad \text{... (5)}$$

Equations (4) and (5) have the unique solution $n_i = m$, for $i = 1, 2, \ldots, s$.

$\Rightarrow t_i = 2m = r$, for $i = 1, 2, \ldots, s$.

Thus , in both cases ,

$G \cong P_{t_1} \cup P_{t_2} \cup \ldots \cup P_{t_s} \equiv \cup\ P_r \equiv H$. Hence the result follows . □

4 Unions of Cycles

Theorem 3

The circuit polynomial characterizes the union of any finite number of cycles .

Proof

Let $G \cong Z_{n_1} \cup Z_{n_2} \cup \ldots \cup Z_{n_k}$ ($n_k \geq 1$). Let H be a graph such that $C(H) = C(G)$. Also , let $p = \Sigma_i\ n_i$. Now, the circuit polynomial of the cycle Z_{n_i} contains the term w_{n_i} arising from the cover consisting of Z_{n_i} itself . It follows that C(G) contains the term $\Pi_i\ w_{n_i}$. Therefore C(H) has the term $\Pi_i w_{n_i}$. Hence H contains a cover consisting of the disjoint cycles Z_{n_i} ($i = 1, 2, \ldots, k$) . Also , the coefficient of $w_1^{n_i - 2} w_2$ (from Lemma 4(ii)) in $C(Z_{n_i})$ is n_i . $\Rightarrow$ the coefficient of $w_1^{p-2} w_2$ in C(G) (= C(H)) is p . Therefore H has p edges . But the cover $Z_{n_1} \cup Z_{n_2} \cup \ldots \cup Z_{n_k}$ of H has p edges . Therefore $H \cong Z_{n_1} \cup Z_{n_2} \cup \ldots \cup Z_{n_k}$. Hence the result follows . □

When $n_i = 1$, for $i = 2, 3, .., k$, we obtain the folowing corollary .

Corollary 3.1

The circuit polynomial characterizes the union of a cycle together with any finite number of isolated nodes .

The following is the analogous result for matching polynomials.

Theorem 4

Let G be a graph consisting of the cycle Z_n ($n > 1$), together with an isolated node i.e. $G \cong Z_n \cup Z_1$. Let H be the graph consisting of the chain P_n with an edge attached to a node of valency 2 which is adjacent to an endnode of P_n . Then $M(G) = M(H)$ i.e. G and H are comatching .

Proof

Apply Lemma 1 to G by deleting an edge of Z_n . This yields

$$M(G) = w_1 M(P_n) + w_1 w_2 M(P_{n-2}) .$$

Apply Lemma 1 to H by deleting the edge incident to the node of valency 3 . This yields

$$M(G) = w_1 M(P_n) + w_1 w_2 M(P_{n-2}) .$$

Therefore $M(G) = M(H)$. $\square$

Theorem 4 implies that matching polynomial does not characterize the union of a proper cycle and an isolated node . Corollary 3.1 shows that the circuit polynomial does .

5. Unions of Mixtures of Cycles and Chains

In this section , we show that the circuit polynomial characterizes all disconnected graphs whose components are either cycles or chains .

Theorem 5

Let $G \cong Z_{n_1} \cup Z_{n_2} \cup ... \cup Z_{n_k} \cup^j P_r$ ($n_k \geq 1$ and $r \geq 1$) . Then C(G) characterizes G .

Proof

Let H be a graph such that C(H) = C(G) . It can be easily verified , by using Lemmas 3 , 4 and 5 , that

(i) G has $p = \Sigma n_i + jr$ nodes

(ii) G has $q = \Sigma n_i + j(r-1)$ edges

(ii) $\Pi(G) = \Pi(H) = (1^{2j}, 2^{p-2j})$.

From the component Theorem we have

$$C(G) = [C(P_r)]^j \Pi^k C(Z_{n_i}) . \quad ... (6)$$

Therefore , by taking the term in $w_1{}^r$ from the j polynomials $C(P_r)$ and the spanning cycle term w_n from each of the k polynomials $C(Z_{n_i})$, we get that C(G) contains a term in $w_1^{jr} w_{n_1} w_{n_2} ... w_{n_k}$, with non-zero coefficient . It follows that H has a cover S consisting of jr isolated nodes and the k cycles $Z_{n_1}, Z_{n_2}, ..., Z_{n_k}$. This cover contains Σ ni edges . Therefore j(r-1) edges must be added to S in order to obtain H .

The j(r-1) edges must be added to S , subject to the property (ii) above for G . Clearly then , the edges must join nodes of valency 0 in S , since H had on nodes of valency 3 . Also , no new cycles should be created , since this will yield a cover consisting of the k cycles , together with other proper cycles and C(G) has no term representing such a cover . Therefore all the new edges must form chains . But $\Pi(H)$ contains the parts 1^{2j} ; Therefore exactly j chains must be formed .

Let the chains be $P_{t_1}, P_{t_2}, ..., P_{t_j}$. Then

$$H \equiv Z_{n_1} \cup Z_{n_2} \cup ... \cup Z_{n_k} \cup P_{t_1} \cup P_{t_2} \cup ... \cup P_{t_j} .$$

Since C(H) = C(G) , it follows from the Component Theorem for H , and Equation (6) , that

$C(P_{t_1} \cup P_{t_2} \cup ... \cup P_{t_j}) = C(\cup^j P_r) . \Rightarrow t_i = r \ (i = 1, 2, ..., j)$ by Theorem 2 .

Hence $H \cong Z_{n_1} \cup Z_{n_2} \cup \ldots \cup Z_{n_k} \overset{j}{\cup} P_r$. The result therefore follows . □

Corollary 5.1

The circuit polynomial characterizes the graph $Z_{n+1} \cup P_n$.

It has been shown (Farrell and Wahid [8]) , that the graphs P_{2n+1} and $Z_{n+1} \cup P_n$ are comatching . Therefore Corollary 5.1 does not hold for matching polynomials .

6. Unions of Complete Graphs

It has been shown ([5]) that the circuit polynomial characterizes complete graphs . It has also been shown (Farrell , Guo and Constantine [7]) that the matching polynomial characterizes the union of any finite number of copies of a complete graph . It is clear from the definition of the matching polynomial , that all the terms of M(G) belong to C(G) , for any graph G . Therefore any graph that is characterized by its matching polynomial is also characterized by its circuit polynomial . We therefore have the following result .

Theorem 6

The circuit polynomial characterizes $\cup^s K_r$, for all $r,s \geq 0$.

7. Unions of Hamiltonian Graphs

In this section , we will prove the general result that the circuit polynomial characterizes the union of any hamiltonian graph with itself . First of all , we establish the following lemma .

Lemma 8

Let G be a graph with 2p nodes and containing $t(>1)$ p-cycles (cycles with p nodes) . Then

(i) if t is even i.e. $t = 2n$, for some positive integer n , then the 2n p-cycles can form at most n^2 ($=t^2/4$) pairs of disjoint p-cycles ; and this maximum is attained if and only if there are exactly n p-cycles spanning the same p nodes of G and exactly n other p-cycles spanning the remaining p nodes of G .

(ii) If t is odd i.e. t=2m+1 , for some positive integer m ; then the 2m+1 p-cycles can form at most m(m+1) $[=(t^2-1)/4]$ pairs of disjoint p-cycles ; and this maximum is attained if and only if there are exactly m p-cycles spanning the same p nodes of G and the remaining p-cycles , spanning the remaining p odes of G .

Proof

We will prove the result by induction on t . For t=2 , the two p-cycles can from at most one pair of disjoint p-cycles ; and this can occur if and only if the two p-cycles span disjoint sets of p nodes in G . Therefore the result holds for t=2 .

When t=3 , m=1 . There are two possibilities .

(1) All three p-cycles share a set of common nodes .

(2) ∃ a pair of p-cycles with no nodes in common .i.e a disjoint pair .

In Case(1) , the three p-cycles form 0 (<2) pairs of disjoint p-cycles .

(Note that the three p-cycles form exactly 2 pairs of disjoint p-cycles if and only if exactly one p-cycle is disjoint from the other two .) . In Case(2) , let the disjoint p-cycles be $C_p^{(1)}$ and $C_p^{(2)}$. Then G consists of two disjoint subgraphs with p nodes ; H_1 containing the spanning cycle $C_p^{(1)}$ and H_2 containing the spanning cycle $C_p^{(2)}$. The third cycle $C_p^{(3)}$ must either be (i) a spanning subgraph of H_1 or a spanning subgraph of H_2 or (ii) containing nodes or both H_1 and H_2 . In Case(i) , there will be two pairs of disjoint p-cycles , which is the maximum possible . In case (ii) , there will be 1 (<2) pair of disjoint p-cycles . Hence the result holds for t=3 .

Let us assume that the result holds for all $t < k$. We show that it also holds for $t = k$. Suppose that no pair of the p-cycles is disjoint . Then obviously there will be 0 pairs of disjoint p-cycles and the result follows trivially as in Case (1) above . We therefore assume that there is at least one pair of disjoint p-cycles . In this case , G will be of the form described in Case (2) above .

<u>Case 1 (k-even)</u>

Let $k = 2n$, for some positive integer n . Clearly , if there are n p-cycles in H_1 and n p-cycles in H_2 , then G has n^2 pairs of disjoint p-cycles .

Suppose that there are a_1 p-cycles in H_1 , a_2 in H_2 and a_3 with nodes both in H_1 and H_2 ; where $a_2 \geq 1$, $a_2 \geq 1$, $a_3 \geq 0$ and $a_1 + a_2 + a_3 = k$. Since $a_3 < k$, by the induction hypothesis , the a_3 p-cycles form at most $[a_3^2/4]$ (if a_3 is even) or $[(a_3^2-1)/4]$ (if a_3 is odd) pairs of disjoint p-cycles . Since a_1 and a_2 p-cycles are disjoint , the total number of pairs of disjoint p-cycles is

$$h \leq \begin{cases} a_1 a_2 + a_3^2/4 & \text{, if } a_3 \text{ is even} \\ a_1 a_2 + (a_3^2 - 1)/4 & \text{, if } a_3 \text{ is odd .} \end{cases}$$

Suppose that a_3 is even . Then

$$n^2 - h \geq (1/4)(a_1 + a_2 + a_3)^2 - (a_1 a_2 + a_3^2/4) = (1/4)[(a_1 - a_2)^2 + 2a_3(a_1 + a_2)] \geq 0 .$$

Also , the second equality holds if and only if $a_1 = a_2$ and $a_3 = 0$ (since a_1 and a_2 are positive) .

When a_3 is odd,

$$n^2 - h \geq (1/4)(a_1 + a_2 + a_3)^2 - (a_1 a_2 + (a_3^2 - 1)/4) =$$

$$(1/4)[(a_1 - a_2)^2 + 2a_3(a_1 + a_2 + 1)] > 0 .$$

Therefore we conclude that $h = n^2$ if and only if $a_1 = a_2 = n$; otherwise $h < n^2$. Hence the result holds if k is even .

Case 2 (k-odd)

Let $k=2m+1$, for some positive integer m . Suppose that m p-cycles are in H_1 and m+1 are in H_2 . Then the number of pairs of disjoint p-cycles is $m(m+1) = (k^2-1)/4$.

We now consider the difference $[(k^2-1)/4] - h$.

When a_3 is even ,

$$(1/4)(k^2-1)-h = (1/4)[(a_1+a_2+a_3)^2 -1] - (a_1 a_2+a_3{}^2/4) =$$
$$(1/4)[(a_1-a_2)^2+2a_3(a_1+a_2)-1] .$$

Since k is odd and a_3 is even , $a_1+a_2 = k-a_3$ is odd . $\Rightarrow |a_1-a_2| \geq 1$.

$\therefore$ $(1/4)(k^2-1)-h \geq 0$. Also , the difference is 0 , if and only if $|a_1-a_2| = 1$ and $a_3 = 0$.

When a_3 is odd ,

$$(1/4)(k^2-1)-h = (1/4)[(a_1+a_2+a_3)^2 -1] - (a_1 a_2+(a_3{}^2-1)/4) =$$
$$(1/4)[(a_1-a_2)^2+2a_3(a_1+a_2)] > 0 .$$

Hence we conclude that $h=m(m+1)$ if and only if $|a_1-a_2| = 1$ and $a_3 = 0$; otherwise , $h < m(m+1)$. Hence the result holds when k is odd . The proof is therefore completed by the Principle of Induction . $\square$

Theorem 7

Let H be a hamiltonian graph that is characterized by its circuit polynomial . Then $H \cup H$ is also characterized by its circuit polynomial .

Proof

Let G be a graph such that $C(G) = C(H \cup H) = [C(H)]^2$. Also, let $C(H) = w_1{}^p+qw_1{}^{p-2}w_2+ \dots + tw_p$, where p and q are the numbers of nodes and edges respectively in H . By examining the terms of C(G) in accordance with Lemma 3 , we deduce that G has 2p nodes and 2q edges . Also , C(G) has the terms $2tw_1{}^pw_p$ and $t^2w_p{}^2$. These terms indicate that G has 2t p-cycles and t^2 pairs of disjoint p-cycles . It follows by Lemma 8 , that G has two disjoint subgraphs H_1 and H_2 with t p-cycles in H_1 and t p-cycles in H_2 .

We will show that no edge of G join a node of H_1 to a node of H_2 . Suppose the contrary , i.e. G is connected . By using Lemma 6 , we get

$$\frac{\partial C(G)}{\partial w_p} = \sum_{i=1}^{2t} C(G-Z_p^{(i)}) = 2t\,(w_1^p + qw_1^{p-2}w_2 + \dots + tw_p) \qquad \dots (7)$$

Since G is connected , the removal of the nodes of any p-cycle in H_1 will not only remove all the edges of H_1 , but also all the link edges which join the nodes in H_1 to the nodes in H_2 . The same is true for the removal of a p-cycle in H_2 . Therefore the number of link edges is $2q-| E(G-Z_p{}^{(i)}) | + | E(G-Z_p{}^{(j)}) |$, where $Z_p{}^{(i)}$ and $Z_p{}^{(j)}$ are p-cycles in H_1 and H_2 respectively and E(G) is the edge set of G .

$\Rightarrow$ $2q > | E(G-Z_p{}^{(i)}) | + | E(G-Z_p{}^{(j)}) |$, for every pair of disjoint p-cycles . It follows that $\Sigma^{2t} | E(G-Z_p{}^{(i)}) | < 2qt$.

Therefore the coefficient of $w_1{}^{p-2}w_2$ in $\Sigma^{2t}\ C(G-Z_p{}^{(i)}) \neq 2qt$.

This contradicts Equation (7) . Therefore G is not connected and has two components .

Let the components of G be H_1 and H_2 . Then

$$C(G) = C(H_1)\ C(H_2) = [\ C(H)]^2 . \qquad \dots (8)$$

From Equation (7) , we get

$$\sum_{i=1}^{2t} C(G-Z_p^{(i)}) = 2t\, C(H) = \sum_{Z_p^{(i)} \subseteq H_1} C(G-Z_p^{(i)}) + \sum_{Z_p^{(i)} \subseteq H_2} C(G-Z_p^{(i)})$$

$$= t\, C(H_2) + t\, C(H_1) .$$

$$\Rightarrow 2C(H) = C(H_2) + C(H_1) .$$

$$\Rightarrow C(H_2) = 2C(H) - C(H_1) .$$

By substituting for $C(H_2)$ in Equation (8) , we get

$$C(H_1)\, [2C(H) - C(H_1)] = [\, C(H)\,]^2 .$$

$$\Rightarrow [C(H)]^2 - 2C(H_1)\, C(H) + [C(H_1)]^2 = 0 .$$

$$\therefore [C(H) - C(H_1)\,]^2 = 0 . \Rightarrow C(H) = C(H_1) .$$

Hence $C(H_1) = C(H_2) = C(H)$. But (by data) H is characterized by its circuit polynomial . Therefore $H_1 \equiv H_2 \equiv H$. Hence $G \equiv H \cup H$. This completes the proof . []

In [5] , it is shown that the complete graph K_n , the regular complete bipartite graph $K_{n,n}$ and the wheel W_n are all circuit unique . Since these graphs are also hamiltonian , we obtain the following corollary .

Corollary 7.1

For all positive integers n ,

$$K_n \cup K_n \,,\, K_{n,n} \cup K_{n,n} \text{ and } W_n \cup W_n$$

are characterized by their circuit polynomials .

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