COMPUTATIONAL COMPLEXITY OF INTEGRITY

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ABSTRACT

The integrity of a graph, $I(G)$, is given by $I(G) = \min_S(|S| + m(G - S))$ where $S \subseteq V(G)$ and $m(G - S)$ is the maximum order of the components of $G - S$. It is shown that, for arbitrary graph G and arbitrary integer k, the determination of whether $I(G) \leq k$ is NP-complete even if G is restricted to be planar. On the other hand, for every positive integer k it is decidable in time $O(n^2)$ whether an arbitrary graph G of order n satisfies $I(G) \leq k$. The set of graphs $\mathcal{G}_k = \{G | I(G) \leq k\}$ is closed under the minor ordering and by the recent results of Robertson and Seymour the set $\mathcal{O}_k$ of minimal elements of the complement of $\mathcal{G}_k$ is finite. The lower bound $|\mathcal{O}_k| > (1.7)^k$ is established for k large.

1. INTRODUCTION

The concept of integrity of a graph was introduced in [1] after examining other measures of the "vulnerability" of a graph, i.e., its resistance to fragmentation into components of small order by removal of a small number of vertices.

The *integrity* of a graph $G, I(G)$, is defined as follows. $I(G) = \min_S(|S| + m(G - S))$ where $S \subseteq V(G)$ and $m(G - S)$ is the maximum order of the components of $G - S$. The set S is called an *I-set* of G. In [1] the concept of edge-integrity of a graph was also introduced. In this paper, we are concerned only with vertex integrity.

Given k and $S \subseteq V(G)$ it is easy to decide whether $|S| + m(G - S) \leq k$, so determining the integrity of a graph is in NP. In the next section we will show

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that the vertex integrity problem is NP-complete but that for each fixed value of k it is decidable in time $O(n^2)$ whether an arbitrary graph G of order n satisfies $I(G) \leq k$. This last result is obtained by a simple application of the powerful results of Robertson and Seymour on graph minors (see the references of [6]; also, see Wilf [7] for a brief exposition of Robertson-Seymour's main theorem).

All graphs considered in this paper are simple, without loops or multiple edges. A simple graph H is a *minor* of the simple graph G iff H can be obtained from G by a sequence of operations of the following two kinds:

(a) replace a graph by a subgraph of itself or

(b) *contract* an edge uv (i.e., replace u and v by a single vertex w adjacent to those vertices to which u or v was adjacent).

It is important to note that the literature contains several variant definitions of *contraction* and *minor*. In particular, the results of Robertson and Seymour are established for general graphs (allowing loops and multiple edges) and for the minor operation with the contraction operation defined topologically. (The topological contraction of an edge may create loops or multiple edges.) For simple graphs the combinatorial minor ordering defined above coincides with the topological definition.

We see that all minors of a graph can be obtained by deleting edges, contracting edges and deleting isolated vertices. If $G \not\cong K_1$, isolated vertices play no role in determining $I(G)$ and so may be ignored in discussions of the integrity of minors of a graph. If the minor H of G satisfies $H \not\cong G$ then H is a *proper minor* of G.

We will use the following result of Robertson and Seymour [6].

Theorem A (Robertson and Seymour). Let F be a minor closed class of graphs such that some planar graph is not in F. Then there is an algorithm to determine membership of $G = (V, E)$ in F with running time $O(|V|^2)$.

We will also need the following result from [2]. Here P_n is the path on n vertices and $\lfloor x \rfloor$ and $\lceil x \rceil$ are the greatest integer and least integer functions, respectively.

Theorem B. (Barefoot, Entringer and Swart) If T is a tree of order n then $I(T) \leq I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$.

We will generally use the notation and terminology of Bondy and Murty [3] but require the following additional notions. A *suspended path* of a graph G is a path of G all internal vertices of which have degree 2 in G. The graph resulting from the contraction of an edge e will be denoted by $G \cdot e$ and $\langle V' \rangle$ will denote the subgraph of G induced by the subset V' of $V(G)$.

2. THE COMPUTATIONAL COMPLEXITY OF INTEGRITY

Consider the following decision problem.

VERTEX INTEGRITY

Input: A graph $G = (V, E)$ and an integer k.

Question: Is $I(G) \leq k$?

Theorem 1. VERTEX INTEGRITY is NP-complete, even for input restricted to planar graphs.

Proof. The problem is clearly in NP since it is easy to verify, given $S \subseteq V$, that $|S| + m(G - S) \leq k$.

To show that VERTEX INTEGRITY is NP-hard we reduce from the following decision problem which is known to be NP-complete [4].

VERTEX COVER

Input: A planar graph $G = (V, E)$ and an integer k.

Question: Is there a set $V' \subseteq V$ with $|V'| \leq k$ such that every edge of G is incident with some vertex of V'?

We can assume $k \leq |V| - 3$. (Indeed, by the Four-color Theorem G has an independent set of at least $\frac{1}{4}|V|$ vertices and so a vertex cover with at most $\frac{3}{4}|V|$ vertices.)

Given G and k we describe how to compute in polynomial time a graph G' and integer k' such that $I(G') \leq k'$ if and only if G has a vertex cover of cardinality at most k.

Let W denote the wheel with $2|V| - 2$ spokes. The order of W is then $2|V| - 1$. The graph G' is obtained from G by adding to each vertex v of G an edge to all the rim vertices of a copy of W. An example is shown in Fig. 1. Let $k' = 2|V| + k$. The graph G' is planar, since G is planar.

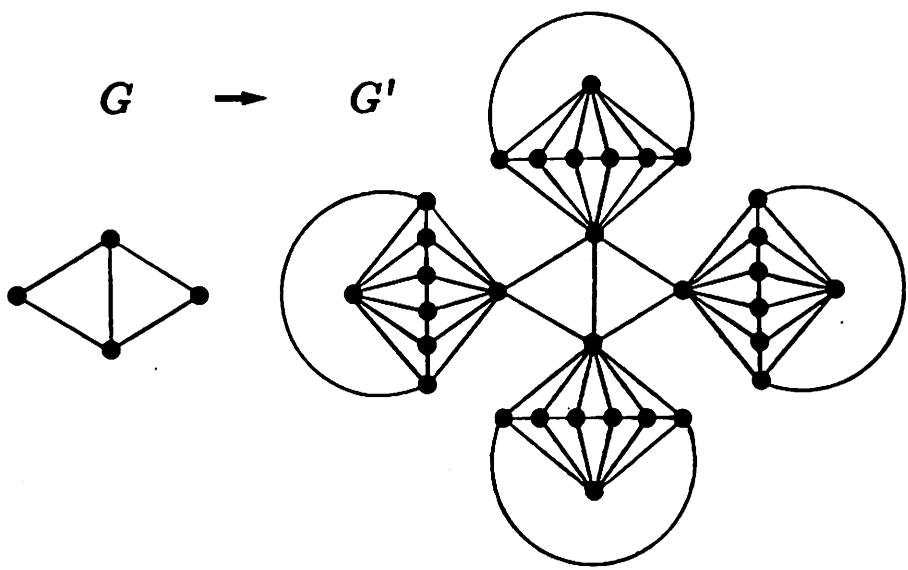


Figure 1. The construction of G' from G and W.

Suppose G has a vertex cover $S \subseteq V$ with $|S| \leq k$. Since every edge of G is incident with some vertex of S, $G - S$ has no edges and $m(G' - S) \leq 2|V|$. This implies $I(G') \leq k + 2|V| = k'$.

Conversely, suppose $I(G') \leq k' = k + 2|V|$. Then, for some set $S' \subseteq V'$, $|S'| + m(G' - S') \leq k + 2|V|$. In particular, $|S'| \leq k + 2|V| < 3|V|$ so some component C of $G' - S'$ has order at least $2|V| - 2$. This, in turn, implies that $|S'| \leq k + 2 < |V|$ and so, in fact, some component C of $G' - S'$ has order at least $2|V|$ which implies $|S'| \leq k$.

If any component C of $G' - S'$ contains two or more vertices of G (considered as a subgraph of G') then C has order at least $4|V| - k > 2|V| + k$, a contradiction. We may conclude that $G' - S'$ contains no edge of G. But this implies that $G - (S' \cap V)$ has no edges, i.e., $S' \cap V$ is a vertex cover of G of no more than k vertices. ■

In contrast to this result our next theorem shows that it is easy to decide where an arbitrary graph of order n has integrity at most k, for each fixed value of k.

We first must show that the class of graphs $\mathcal{G}_k = \{G | I(G) \leq k\}$ is closed under the minor ordering. We remark that this is not true for the class of graphs with edge-integrity at most k.

Lemma 2. $\mathcal{G}_k$ is closed under the minor ordering.

Proof. It was observed in [1] and [2] that if H is any subgraph of G then $I(H) \leq I(G)$. It suffices then to show $I(G \cdot e) \leq I(G)$ where $G \cdot e$ is the resulting graph after contraction of the edge $e = uv$ of G.

Choose $S \subseteq V(G)$ such that $I(G) = |S| + m(G - S)$. If $\{u, v\} \subseteq S$ then $I(G \cdot e) < I(G)$ while if $\{u, v\} \cap S = \emptyset$ then u and v lie in the same componeent of $G - S$ so that $I(G \cdot e) \leq I(G)$. Finally, if $u \in S$ and $v \notin S$ we have $I(G \cdot e) \leq |S| + m(G - S - v) \leq I(G)$. ■

Theorem 3. For every positive integer k it is decidable in time $O(n^2)$ whether an arbitrary graph G of order n satisfies $I(G) \leq k$.

Proof. By the lemma, we can set F of Theorem A equal to $\mathcal{G}_k$ since, by Theorem B, $P_m \notin \mathcal{G}_k$ for $m > k^2/4 + k$. ■

This proof is nonconstructive because the proof of Theorem A establishes that the complement $\bar{\mathcal{G}}_k$ of $\mathcal{G}_k$ has a finite but unknown number of minimal elements in the minor ordering. Theorem 3 becomes constructive when these are identified.

3. OBSTRUCTIONS

Let O_k denote the set of minimal elements of $\mathcal{G}_k$ in the minor ordering. We will refer to the set of graphs O_k as the set of *obstructions* to integrity k or, briefly, as *k-obstructions*. We summarize some of the more important properties of O_k.

(i) O_k is finite,

(ii) If $I(G) > k$ then some minor of G is in O_k.

(iii) If $H \in O_k$ and H' is a proper minor of H then $I(H') \leq k$.

(iv) If $H \in O_k$ then $I(H) = k + 1$.

Property (iv) follows immediately from (iii) since $v \in V(H)$ and $H \in O_k$ implies $I(H - v) \leq k$ and hence $I(H) \leq k + 1$.

It is easy but tedious to identify O_k for small k; we do so for $k \leq 3$ presently. First, let us note that $H \in O_k$ iff $I(H) = k+1$ and both $I(H-e) \leq k$ and $I(H \cdot e) \leq k$ hold for all $e \in E(H)$.

So that we may systematically determine O_k we define $O_{k,i}$ to be the subset of O_k consisting of those graphs H of O_k satisfying $\min\{|S| : I(H) = |S| + m(H - S)\} = i$. If $I(H) = k + 1 = |S| + m(H - S)$ for some $S \subseteq V(H)$ and $|S| = k + 1$ then $m(H - S) = 0$ so that $k + 1 = |V(H)|$. But this implies $\min\{|S| : I(H) = |S| + m(H - S)\} = 0$ since $S = \emptyset$ is allowed in the definition of $I(H)$. Consequently, we have $O_{k,i} = \emptyset$ for $i > k$.

Theorem 4. The sets $O_{k,i}$, $0 \leq i \leq k \leq 3$, include those displayed in Table 1.

Proof. Any candidate H for membership in $O_{k,i}$ can be constructed as follows.

(i) Let $\langle S \rangle$ be any graph of order i,

(ii) Let $H - S$ consist of a arbitrary number of components each of which has order at most $k + 1 - i$ and at least one of which has order $k + 1 - i$.

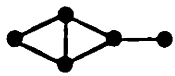
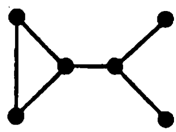
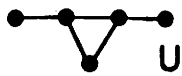
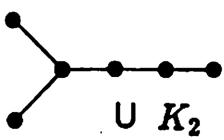
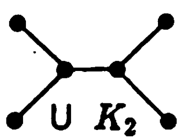
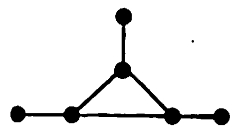
$k \backslash i$	0	1	2
0	K_1		
1	K_2		
2	K_3 $\quad P_2 \cup P_3$	P_4	
3	K_4 $\quad P_2 \cup C_4$ $P_3 \cup P_4$ $K_3 \cup K_{1,3}$ $P_2 \cup P_3 \cup K_{1,3}$	P_6    $\cup K_2$  $\cup K_2$  $\cup K_2$	C_5 

Table 1. Members of the classes $O_{k,i}$, $0 \le i \le k \le 3$.

(iii) Add edges e to $\langle S \rangle \cup \langle H - S \rangle$ so that exactly one end vertex of e is in S.

Entries of Table 1 can be progressively determined by identifying those graphs H constructed according to (i), (ii) and (iii) which satisfy $I(H) = k + 1$ and for which deletion or contraction of any edge of $E(H)$ decreases integrity and which do not belong to $O_{k,j}$ for $j < i$. Let us, for example, determine the members of $O_{2,1}$.

We have $\langle S \rangle = K_1$ with, say, vertex v. Then, for an arbitrary candidate H for membership in $\mathcal{O}_{2,1}$ we see that $H - S$ consists of some number, a, of copies of K_1 and some number, $b+c$, of copies of K_2, exactly c of which form a triangle with v. We first note that $b+c>0$ since $I(K_{1,n-1}) < 3$.

If $c > 0$ then H has a triangle, say uvw, and, since $K_3 \in \mathcal{O}_{2,0}$, v is adjacent to some vertex $x \notin \{u,w\}$. But $I(H \cdot ux) \geq 3$ since $H \cdot ux$ contains a triangle. We conclude that $b > c = 0$.

Now $I(P_4) = 3$ so that H cannot contain P_5 as a subgraph. Thus $b = 1$ and, since $I(P_3) = 2$, we must have $a \geq 1$. We cannot have $a \geq 2$ for then H contains P_4 as a proper minor so that H is not an obstruction. We conclude that $H \cong P_4$. ■

Certain of the graphs of Table 1 suggest the following infinite classes of obstructions.

Theorem 5. The following graphs are obstructions as indicated:

i) $K_{k+1} \in \mathcal{O}_k$, $k \geq 0$,

ii) $\bigcup_{r=1}^{k} K_{1,r} \in \mathcal{O}_k$, $k \geq 1$,

iii) $P_k \in \mathcal{O}_{\lceil 2\sqrt{k+1}\rceil - 3}$ for $k = m^2$ or $k = m(m+1)$, $m \geq 1$ and

iv) $K_k^* \in \mathcal{O}_k$, $k \geq 1$, where K_k^* is the complete graph with vertex set $\{v_1, \ldots, v_k\}$ together with the vertices u_i, $1 \leq i \leq k$ and edges u_iv_i, $1 \leq i \leq k$.

Proof. i) It is known [1] that $I(K_k) = k$. It is vacuously true that any minor of K_1 has integrity 0; we assume $k \geq 1$ and let $e = uv$ be any edge of K_k. Since $K_k \cdot e \cong K_{k-1}$ we have $I(K_k \cdot e) = k-1$. Also, letting $S = V(K_k) \setminus \{u,v\}$ we see that $I(K_k - e) = k - 1$.

ii) Let $G \leq \bigcup_{r=1}^{k} K_{1,r}$. It is easily seen that $I(G) \leq k+1$. Now let S be a minimal I-set for G and recall, from [1], that S contains no end vertex of G. Label a center of $K_{1,r}$ as v_r, $1 \leq r \leq k$, and let j be the largest index such that $v_j \notin S$ (if no such index exists then $I(G) = k+1$). Then $m(G-S) \geq j+1$ and $|S| \geq k-j$ so that $I(G) = k+1$.

Let G' be the graph obtained from G by contracting or deleting an arbitrary edge of $K_{1,i}$ for some i, $1 \leq i \leq k$. We set $S = \{v_{i+1}, v_{i+2}, \ldots, v_k\}$ and have $m(G-S) = i$ so that $I(G') \leq k$.

iii) In [?] it was shown that $I(P_k) = \lceil 2\sqrt{k+1} \rceil - 2$. But for any edge e of P_k we have $I(P_k \cdot e) = I(P_{k-1}) = \lceil 2\sqrt{k} \rceil - 2 < \lceil 2\sqrt{k+1} \rceil - 2$.

Again, let $e = uv$ be any edge of P_k and let S be an I-set for $P_k \cdot e$. In $P_k \cdot e$ let x denote the identification of u and v. If $x \notin S$ then $m(P_k - e - S) \leq m(P_k \cdot e - S)$ and $I(P_k - e) \leq |S| + m(P_k \cdot e - S) < I(P_k)$. If $x \in S$ we set $S' = S \setminus x$ and have $m(P_k - e - S') \leq m(P_k \cdot e - S) + 1$. Thus $I(P_k - e) \leq |S| - 1 + m(P_k \cdot e - S) + 1 < I(P_k)$.

iv) Let S be a smallest I-set for K_k^*. Then $u_i \notin S$, $1 \leq i \leq k$, and $m(K_k^* - S) = 2(k - |S|)$ so that $2k - |S| = I(K_k^*) \leq k + 1$. Thus $|S| \geq k - 1$. If $|S| = k - 1$ then $m(K_k^* - S) = 2$ while if $|S| = k$ then $m(K_k^* - S) = 1$ so that, in any case, $I(K_k^*) = k + 1$.

If any edge $u_i v_i$, $1 \leq i \leq k$, is contracted or deleted from K_n^* to form a graph G then, taking $S = \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k\}$ we have $m(G-S) = 1$ so that $I(G) \leq k$.

If any edge $v_i v_j$, $1 \leq i < j \leq k$, is contracted or deleted we argue as in the proof of i) that the integrity of the resulting graph is strictly less than that of K_k^*. ■

It follows from results of [?] that $I(P_{n-1}) = I(P_n)$ unless $n = m^2$ or $m(m+1)$ for some $m \geq 2$. Thus the paths described in iii) are the only obstructions that are paths.

4. A LOWER BOUND FOR $|O_k|$

In this section we will show that $|O_k|$, although known to be finite by the results of Robertson and Seymour, is at least an exponential function of k.

In [2] we showed that of all trees of order n the path, P_n, has maximum integrity. The key idea of the proof was contained in the proof of Lemma 4 there and is exploited again in the proof of the following result.

Lemma 6. Let $v_k v_{k-1}$ and $v_k v_{k+1}$ be two edges of a graph G for which $G - v_k v_{k-1} -$

$v_k v_{k+1} \cong P_{k-1} \cup G_1 \cup G_2$ for some graphs G_1 and G_2 and path $P_{k-1} = (v_1 v_2 \cdots v_{k-1})$ with $v_k \in V(G_1)$ and $v_{k+1} \in V(G_2)$ (see Fig. 2). Then $I(G) \leq I(G - v_k v_{k+1} + v_1 v_{k+1})$.

Proof. Set $H = G - v_k v_{k+1} + v_1 v_{k+1}$ and let S be on I-set for H. If $S \cap \{v_k, v_{k+1}\} \neq \emptyset$ then $I(G) \leq I(H)$ since $G - v_{k+1} \cong H - v_{k+1}$ and $G - v_k$ is a subgraph of $H - v_k$. Furthermore, we may assume v_k and v_{k+1} are in different components of $H - S$, say H_k and H_{k+1}, for otherwise we obviously have $I(G) \leq I(H)$. Thus there must exist $v_{i_1}, \ldots, v_{i_m}$, $m \geq 1$, in S with $1 \leq i_1 < \cdots < i_m \leq k - 1$.

Let S' be S with v_{i_j} replaced by $v_{i_j+k-i_m}$, $1 \leq j \leq m$. If $m(G - S') \leq m(H - S)$ we have $I(G) \leq I(H)$. Thus we may assume $m(G - S') > m(H - S)$. But this can hold only if the (path) component $v_1 \cdots v_{i_1+k-i_m-1}$ of $G - S'$ has order greater than $m(H - S)$, i.e., $i_1 + k - i_m - 1 \geq m(H - S) + 1$.

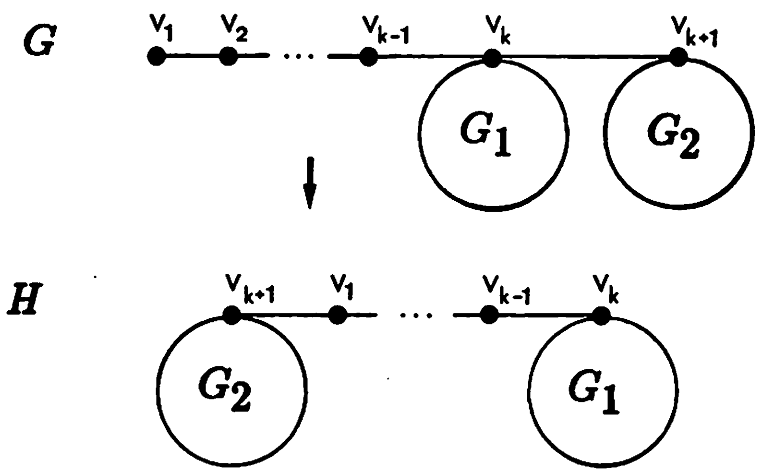


Figure 2. The construction of the graph H.

Let h be the number of vertices in H_{k+1} that are not in the set $\{v_1, \ldots, v_{i_1-1}\}$ and note that

$$1 \leq h \leq m(H - S) - (i_1 - 1) \leq k - i_m - 1 .$$

Finally, let S^* be the set S with v_{i_j} replaced with v_{i_j+h}, $1 \leq j \leq m$, and consider the components of $G - S^*$. By the constructions of S and S^* all components of $G - S^*$, with the possible exceptions of those containing v_1 or v_k (and so v_{k+1}), have at most $m(H - S)$ vertices. But the vertex set of the component of $G - S^*$ containing v_1 is obtained from the corresponding vertex set of $H - S$ by deleting the h vertices of $H_{k+1} - \{v_1, \ldots, v_{i_1-1}\}$ and appending the vertices $v_{i_1}, \ldots, v_{i_1+h-1}$ for no net change in the number of vertices. Similarly, the vertex set of the component of $G - S^*$ containing v_k is obtained from H_k by deleting the vertices $v_{i_m+1}, \ldots, v_{i_m+h}$ and appending the h vertices of $H_{k+1} - \{v_1, \ldots, v_{i_1-1}\}$ again for no net change in the number of vertices. Thus $I(G) \leq |S^*| + m(G - S^*) \leq |S| + m(H - S) = I(H)$. ∎

We note for use in the next result that neither G_1 nor G_2 was required to be connected in the lemma.

Theorem 7. If T is a tree of order $n = m^2$ or $n = m(m+1)$, $m \geq 1$, and $I(T) = I(P_n)$ then $T \in \mathcal{O}_{\lceil 2\sqrt{n+1} \rceil - 3}$.

Proof. By Theorem B, for any such tree T we have $I(T) = I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$ and for any edge e of T, $I(T \cdot e) \leq I(P_{n-1}) \leq \lceil 2\sqrt{n+1} \rceil - 3$ where the last inequality follows as in the proof of part iii) of Theorem 5. Hence if the theorem is not true it is only because there exists a nonempty set $\mathcal{T}$ of trees T of order n for which $I(T) = I(P_n)$ and $I(T - e) = I(P_n)$, also, for some edge e in T. Let T be a member of $\mathcal{T}$ that has maximum diameter, say d, and note that, by part iii) of Theorem 5, $T \not\cong P_n$.

Let $v_1 v_2 \cdots v_{d+1}$ be a longest path of T and let k be the smallest index for which $d(v_k) \geq 3$. Since $1 < k < d+1$ there are subtrees T_1 and T_2 of T containing v_k and v_{k+1}, respectively, such that $T - v_{k-1}v_k - v_k v_{k+1} = P_{k-1} \cup T_1 \cup T_2$ where $P_{k-1} = (v_1 \cdots v_{k-1})$. We set $H = T - v_k v_{k+1} + v_1 v_{k+1}$ and note that H has diameter greater than d so that by our choice of T and Lemma 6 we have $I(H) \geq I(T) = I(P_n)$ and $I(H - e') < I(P_n)$ for all $e' \in E(H)$.

Since $T - v_k v_{k+1} \cong H - v_1 v_{k+1}$ we have $I(T - e) < I(P_n)$ for $e = v_k v_{k+1}$. If

e is an edge of T_1 or T_2 we apply Lemma 6 to $T - e$ and have $I(T-e) < I(P_n)$ in this case also. Finally, suppose $e = v_{i-1}v_i$ for some i, $1 < i \leq k$. We relabel the path $v_{d+1}v_d \cdots v_1$ as $v_1v_2 \cdots v_{d+1}$ and apply the above arguments to conclude $I(T-e) < I(P_n)$ in this case also.

Consequently $\mathcal{T} = \emptyset$ and the theorem follows. ∎

Corollary 8. If T is a tree of order $n = m^2$ or $n = m(m+1)$, $m \geq 2$, and T has a suspended path of length at least $n - \sqrt{n+1} + \frac{5}{4}$ then $T \in O_{\lceil 2\sqrt{n+1}\rceil - 3}$.

Proof. In view of Theorem 7 it suffices to prove $I(T) = I(P_n)$. We label T so that $uv \cdots wx$ is the suspended path with $d(u) \neq 2$, $d(x) \neq 2$ and $T_u(T_x)$ is the component of $T - v$ $(T - x)$ containing $u(x)$ (respectively). We define $T_1 = T - uv + xu$ and note that, by Lemma 6, $I(T) \geq I(T_1)$. Now consider the tree $T_{n,k}$ defined in [1] where $k = |V(T_u)| + |V(T_x)| \leq \sqrt{n+1} - \frac{5}{4}$ (see Fig. 3). We label $T_{n,k}$ so that x is the vertex satisfying $d(x) \geq 3$, $v \cdots x$ is the longest path in $T_{n,k}$ with x as an end vertex and x is adjacent to k end vertices. We next show that $I(T_1) \geq I(T_{n,k})$.

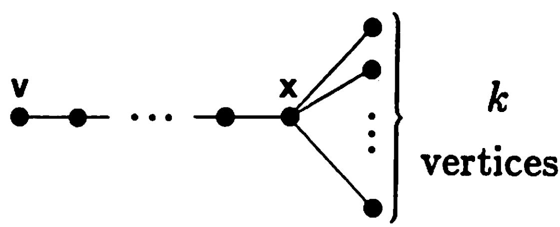


Figure 3. The graphs $T_{n,k}$.

Let S be an I-set for T_1 and let S' be the set of those vertices of S that are on the path $v \cdots x$. If $S \setminus S' = \emptyset$ then $m(T_1 - S) \geq m(T_{n,k} - S)$ so that $I(T_1) \geq I(T_{n,k})$. If $S \setminus S' \neq \emptyset$ set $S'' = (S \setminus S') \cup \{x\}$ so that now $m(T_1 - S) \geq m(T_{n,k} - S'')$ and again $I(T_1) \geq I(T_{n,k})$. But, by Fact 5 of [1] we have $I(T_{n,k}) = \lceil 2\sqrt{n+1}\rceil - 2$ so that $I(T) = I(P_n)$ and $T \in O_{\lceil 2\sqrt{n+1}\rceil - 3}$. ∎

Corollary 9. $|O_k| > (1.7)^k$ for all sufficiently large k.

Proof. Fix k and define

$$n = \begin{cases} m^2 & \text{if } k = 2m-2 \\ m(m+1) & \text{if } k = 2m-1 \end{cases}$$

so that, in both cases, $k = \lceil 2\sqrt{n+1} \rceil - 3$. Let $v \cdots x$ be a path P of length $\lceil n - \sqrt{n+1} + \frac{5}{4} \rceil$ and let T' be any tree rooted at x but otherwise disjoint from P and having order $r = n - \lceil n - \sqrt{n+1} + \frac{5}{4} \rceil$ so that the tree T formed by P and T' has order n. From Corollary 8 it follows that $T \in O_k$.

Now

$$r = \left\lfloor \sqrt{n+1} - \frac{5}{4} \right\rfloor = \begin{cases} m-2 & , n = m^2 \\ m-1 & , n = m(m+1) \end{cases} = \begin{cases} k/2 & , k = 2m-2 \\ (k+1)/2 & , k = 2m-1 \end{cases}$$

so that, in any case, $r \geq k/2 - 1$.

Otter [5] has shown that if A_n is the number of nonhomeomorphic rooted trees of order r then $A_r \sim 0.4399237(2.95576)^r r^{-3/2}$. From this it follows that $|O_k| \geq (1.7)^k$ for sufficiently large k. ∎

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