SYMPLECTIC EMBEDDINGS OF GRAPHS [1]

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ABSTRACT. The foundation of an analytic graph theory is developed.

1. Introduction

Graphs are important objects in computer science, in particular, in the areas of data structures and design of algorithms. The usual encoding of a graph by its adjacency matrix requires an arbitrary labeling of the graph vertices by, say, integers between 1 and n and an $n \times n$ matrix to represent their adjacencies. This data structure may not be the most efficient, for example, in case the graph has few edges compared to the number of vertices. Moreover, verifying whether a graph given by its adjacency matrix has a property which is nontrivial and monotone (preserved under the addition of edges) requires $\Omega(n^2/16)$ lookups in its adjacency matrix [13]. Thus checking connectivity requires $O(n^2)$, while it takes only linear time $O(n)$ if the graphs are represented by their adjacency lists. In general, determining properties of a graph or relations between two graphs given by their adjacency matrices or adjacency lists (for instance, whether a graph is hamiltonian, whether two graphs are isomorphic, or whether one contains an isomorphic copy of the other) cannot be answered easily. Representation of graphs by some sort of data structure is nevertheless necessary for automatic execution of graph-theoretic algorithms.

There have been various restricted coordinatizations of graphs so that adjacency among its vertices is *implicitely* determined by the coordinates assigned to them. For instance, [3] encodes graph vertices as strings in $\{0,1\}^*$, i.e. as vertices of some hypercube, so that two vertices are adjacent if and only if their Hamming distance does not exceed a threshold value. Although only possible for trees, the Prüfer correspondence encodes trees labeled with integers 1 through n as n-tuples of integers in the same range. In a slightly different

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direction, [14] encodes labeled and unlabeled planar graphs so that encoding and isomorphism testing of two graphs given by their encodings can be done efficiently (in time polynomial in the length of the encoding). On the other hand, [12] has shown that if graphs are encoded in the exponentially succinct way suggested by [7] using polylogarithmic sized circuits to quickly evaluate adjacency between two given binary representations of graph vertices (which is *not* possible for every graph), various trivial properties of a graph (such as having a triangle) become NP–complete while ordinarily NP–complete problems become (time or space) exponentially hard.

The adjacency matrix of a graph can be regarded as a method to encode into numbers arbitrary adjacencies among vertices in a graph. Thus it is natural to ask whether one can still have a general "analytic graph theory" in which arbitrary graphs are encoded as sets of tuples of suitable "numbers," just as cartesian analytic geometry encodes arbitrary continuous figures into sets of euclidean points which are tuples of real number coordinates. Such an encoding would then allow a uniform representation of graphs in a suitable "graphical space" and, as its cartesian analog, it would allow the reduction of graph-theoretical problems to questions about these numerical tuples, and vice versa, questions about these numerical tuples could be visualized as graph-theoretic questions.

The purpose of this paper is to propose a graphical coordinate system of this type. The graphical space is a vector space V over an arbitrary field, although the Galois field $GF(2)$ of 2 elements is particularly suitable for loopless simple graphs. A graph on n vertices is represented as a *set of tuples* – or *vectors* of V. *No specific adjacency relations are necessary* as they can be *implicitely* and *uniformly* encoded by endowing V with a suitable "dot product", here denoted $[u, v]$, so that two vertices u, v are adjacent exactly in case u is not orthogonal to v, i.e., in case $[u, v] \neq 0$. Of course, this makes the inner product unlike the ordinary one because the length of every vector is $[u, u] = 0$, but it is still nice, since with respect to an "orthonormal" basis (here called a symplectic basis) it takes the form of the ordinary sum of pointwise products after a cyclic permutation of the binary coordinates of v. This type of bilinear form is called *alternating* and the pair $\langle V, [\cdot, \cdot]\rangle$ is a classical object known as a *symplectic geometry*. For a given even dimension, a nondegenerate symplectic space of this type is uniquely determined up to isometry, i.e., up to a *linear* isomorphism preserving the bilinear form. This choice of coordinates naturally makes symplectic geometry a graphical geometry. It makes possible to reformulate arbitrary combinatorial graph problems as purely "numerical" *linear* problems about points in a space endowed with a geometric structure. In particular, it is possible to place important graph theoretic problems (e.g., graph data structures, graph traversal, graph isomorphism) in a different more geometric perspective that may prove valuable for graph theory in general.

Various other preliminary results are presented in this paper. For instance, theorem 3 provides an efficient algorithm that computes the least dimension of and a symplectic embedding of that dimension for an arbitrary given graph from any of its symplectic embeddings. Several applications of the approach, in particular a new general graph isomorphism test, will appear in [6].

2. Basic Constructions

The original motivation for this work was the relationship between group and graph isomorphism [5]. Later it became clear that the essential necessary properties of the groups involved lie in their central quotients, which are simply finite–dimensional vector spaces over the Galois field $GF(2)$ of 2 elements enriched with a notion of "projection" (a bilinear form).

For the convenience of the reader basic definitions and results of symplectic geometry are reproduced whenever necessary. S/he can consult [1], [11] and/or [8] for further background and proofs of these results.

Definition 1. An *alternating* (or *symplectic*) space over a field F is an F-vector V space together with a alternating bilinear form $[\cdot, \cdot] : V \times V \to V$, i.e., a two argument function $[\cdot, \cdot]$ satisfying the following three properties for all $x, y, z \in V$:

(a) $[x+y,z]=[x,z]+[y,z]$

(b) $[x,y]=-[y,x]$

(c) $[x,x]=0$

The form $[\cdot,\cdot]$ is *regular* if only the vector 0 is orthogonal to every other vector $x \in V$, that is, if for every $a \in V$,

$$\forall x \in V,\ [x,a]=0 \Rightarrow a=0.$$

More generally, the *perp* of a subset $A \subseteq V$ is the subset $A^{\perp}$ given by

$$A^{\perp} := \{x \in V \mid \forall a \in A, [x,a]=0\},$$

so V is regular if and only if its kernel $V^{\perp}=\{0\}$. In particular, the null space $V=\{0\}$ is a regular space of dimension 0. □

A bilinear form is completely determined by its values on a basis $\{x_1,\ldots,x_d\}$, i.e., by a matrix $(a_{ij})_{d\times d}$, where $a_{ij}=[x_i,x_j]$. Moreover, an alternating form is regular iff its matrix with respect to any basis is regular.

Theorem 1. [8, Theorem 1] An alternating form $[\cdot,\cdot]$ on V is regular if and only if its matrix with respect to any basis of V is invertible. □

Definition 2. Two alternating spaces V,W over a field F are *isometric* if there exists a linear isomorphism $\sigma : V \to W$ such that

$$\forall x,y \in V,\ [x,y]=[\sigma(x),\sigma(y)].$$

Any such mapping σ of V onto itself is called an *isometry* of V and the group of all isometries is called the *symplectic group* $Sp(V)$ of V, or more precisely, $Sp_d(V)$, where d is the dimension of the vector space V. □

The symplectic group $Sp(V)$ acts on elements of V by evaluation, and it is well known that this action is transitive on $V-\{0\}$ [1, p. 138].

Theorem 2. [8, theorem 19] Any regular alternating space V has a basis whose matrix of inner products is a direct sum of blocks of type $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Therefore any two regular alternating spaces of the same dimension are isometric and of even dimension. □

This basis will be denoted $\{u_1, \ldots, u_{d/2}, v_1, \ldots, v_{d/2}\}$, where $[u_i, v_j] = \delta_{ij}$, $[u_i, u_j] = [v_i, v_j] = 0$, and will be referred to as a *symplectic basis*. All vectors in V will be expressed in relation to this basis as strings of length d over the alphabet the field of V, unless explicitely stated otherwise. In particular, if the field has only 2 elements, vectors in V_d expressed with respect to a symplectic basis are represented as strings of length d over the binary alphabet $\{0, 1\}$. Symplectic bases are obtained from an arbitrary basis of V through an orthonomolization process so they can be easily computed. They have the computational advantage that calculation of the form at two vectors expressed in terms of symplectic coordinates $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ takes on a very simple form, which resembles the usual inner product in euclidean space:

$$[x, y] := \sum_{i=1}^{d/2} (x_i y_{i+d/2} - x_{i+d/2} y_i).$$

In particular, an alternating form over a finite field can be computed very fast (in linear time).

3. Symplectic embeddings

Definition 3. A subset $X \subseteq V$ is a *symplectic embedding* of a graph G if G is isomorphic to the graph with vertex set X and edge set

$$\{(x, y) \in V \times V \mid [x, y] \neq 0\}.$$

In this case, $dim(V)$ is the *dimension* of the embedding X. □

Thus, $dim(G) \leq d$ if and only if G is isomorphic to the subgraph of the graph defined by the symplectic space V_d of dimension d induced by some *subset* of V_d. This means that the vertices of G can be labelled by d-tuples of 0's and

1's so that adjacency can be determined for two vertices by computing their inner product. In particular, there is a "universal" graph of given dimension d, namely V_d, which contains as vertex induced subgraphs all (finitely many) graphs of dimension d. These spaces (graphs) form an infinite ascending chain

$$V_0 \subset V_2 \subset \cdots \subset V_d \subset \cdots,$$

where $V_0 := 0$, and each space will be identified with its image in the next space in the chain whenever convenient (for instance, in the proof of theorem 3 below). By definition, the isometries $\sigma \in Sp(V_d)$ are exactly the bijections of V_d that preserve adjacency, i.e., $Sp(V_d)$ is precisely the automorphism group of the graph V_d. Hence a symplectic embedding X of a graph G into a space V_d is far from unique, since the action of any $\sigma \in Sp(V_d)$ that does not set-stabilize X will produce another embedding $\sigma(X)$ of G. And conversely, by Witt's theorem[11, 1.1.18], every two embeddings of G of least dimension into V_d are isomorphic by a (partial) isometry of V which can be extended to an element $\sigma \in Sp(V_d)$. Thus, all optimal embeddings of G are equivalent under the action of $Sp(V_d)$ induced on subsets (subgraphs) of V_d. This fact can be used as the basis of a general graph isomorphism test [6].

Lemma 1. Every graph on n vertices has a symplectic embedding of dimension at most $2n$ which can be found in $O(n^2)$−time.

Proof. By doubling the number of vertices in a graph G and then connecting the additional vertices to corresponding vertices of the original graph one can assume without loss of generality that the adjacency matrix (a_{ij}) of G is invertible. Given an invertible (a_{ij}) matrix of size $n \times n$, let V be the alternating space generated by a basis $B = \{b_1, \cdots, b_n\}$ of n elements whose bilinear form at basis elements is given by $[b_i, b_j] =: a_{ij}$. Since A is invertible, V is regular by theorem 1. Thus, the set of elements in the original basis is a symplectic embedding of G in V.

If the original graph G is totally disconnected, the vertices of the extended graph constructed in the previous paragraph is in fact a symplectic basis of

V and an embedding has been found. Assume therefore that G is not a totally disconnected graph. By theorem 2, there exists a symplectic basis of V. Finding the coordinates of the original vertex set with respect to the symplectic basis, and thus a specific embedding of G, requires essentially an orthogonalization process that proceeds recursively as follows (*cf.* [8, proof of Theorem 2]).

Identifying vertices of G and the original basis elements in B so that, say, $[b_1, b_2] \neq 0$, replace each of the remaining basis elements b_i $(i \geq 2)$ by

$$b_i' = b_i + [b_i, b_2]b_1 + [b_i, b_1]b_2,$$

so that, as is easily verified, all elements b_i' $(i \geq 2)$ are orthogonal to both b_1 and b_2. These operations can be done in linear time and produce the first pair of vectors $u_1 := b_1$ and $v_1 := b_2$ in a symplectic basis of V. Recursively, find a symplectic basis of the regular subspace generated by $B' := B - \{u_1, v_1\}$ and join it to u_1, v_1 to obtain a symplectic basis of V. The entire process clearly takes quadratic time. □

From now on all graphs on n vertices will be regarded as subsets of some regular symplectic space V of dimension $\leq 2n$. Lemma 1 makes it possible to define the dimension of the smallest such space over any field.

Definition 4. The *(symplectic) dimension* of a graph G over a field F, herein denoted $dim_F(G)$, is the smallest dimension of a symplectic embedding of G into a regular symplectic space over F. □

Although this dimension is defined over any field, from a computational viewpoint the inner product and other operations are simplest when F is the field of two elements $GF(2)$. Therefore, in the remainder of this paper $F = GF(2)$ and mention of F will be omitted.

The symplectic embedding of a graph G provided by lemma 1 is usually of dimension $> dim(G)$ since it contains n linearly independent vertices. To find a more efficient embedding one can substitute a vertex $x_n \in X$ by a linear combination $y := \sum_{i<n} \lambda_i x_i$ of the remaining vertices x_i so that the adjacencies

are preserved. This can be done efficiently since such a substitution is possible if and only if the linear system of equations

$$[x_i, y] = [x_i, x_n], (1 \leq i < n)$$

in the $n-1$ unknowns $\lambda_i (i < n)$ has a solution y which is distinct of all the remaining vertices of X. The result is an embedding containing only $n-1$ linearly independent vectors. By repeated substitutions of this type in the remaining subset $X - \{x_n\}$ one arrives at *linear core* of G as defined next.

Definition 5. A *linear core* of a graph G is an induced subgraph G_0 of G of smallest order such that some embedding X_0 of G_0 generates an embedding X of G (in the sense that every other vertex in X is a linear combination of elements in X_0.) □

Lemma 2. If G has no isolated vertices, the number of vertices in any linear core G_0 of G equals $dim(G)$ and hence is uniquely determined by G. Moreover, in any optimal embedding of G into V_d, any linear core of G is embedded as a basis of V_d and $dim(G) = dim(G_0)$.

Proof. Let X be an embedding of G, X_0 any linear core of X, $Y \subseteq V_d$ be an optimal embedding of G of dimension $d = \dim(G)$, and $f : Y \to X$ a graph isomorphism. If $Y_0 := f^{-1}(X_0)$ were linearly dependent then some of its elements y would be a linear combination of the other elements in Y_0. Substituting $f(y)$ by the corresponding linear combination of the corresponding elements in X_0 would yield a linearly independent subset of X properly contained in a linear core of X. Therefore Y_0 is linearly independent. If Y_0 generates a proper subspace of V_d of dimension $d' < d$, this subspace is a direct sum of its radical $R \cap R^{\perp}$, a subspace of totally isolated vertices, and a regular subspace R'. Since no vertex in Y_0 is isolated, $Y_0 \subseteq R'$, i.e., G could be embedded in a regular symplectic space of dimension $d' < d$. Therefore Y_0 is a basis of V_d with d elements. □

In order to find an optimal symplectic embedding of a linear core it will be necessary to determine first the structure of the universal graphs V_d of dimension d, of interest in itself. These graphs can be obtained by means of a new graphical construction and an auxiliary result given in lemma 3 below.

Definition 6. The *s-power* of a graph G consists of the union of 4 disjoint copies of G_i of G so that $G_1 \cup G_i$ $(i \geq 2)$ are isomorphic to the cartesian product $K_2 \times G$ while the remaining pairs $G_i \cup G_j$ $(2 \leq i < j \leq 4)$ have edge set $E(G_i) \cup E(G_j) \cup \{(x_i, y_j) \mid (x, y) \notin E(G)\}$, where x_i, y_j denote the vertices of G_i, G_j corresponding to vertices $x, y \in G$, respectively.

Fig. 1 shows the structure of the s-power of a graph G. Parallel lines indicate incidences of corresponding vertices and cross lines indicate incidences of vertices of the copies in G and the corresponding nonadjacent vertices of the other copy of G. Thus, V_2 consists of a triangle and an isolated vertex.

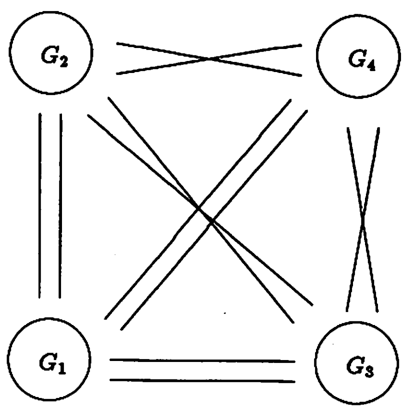

Fig. 1

Lemma 3. The universal graph V_{d+2} of dimension $d+2$ is (isomorphic to) the s-power of V_d for all even $d \geq 0$.

Proof. Let V_d be the subspace of V_{d+2} generated by the symplectic basis elements except $u_{d/2+1}, v_{d/2+1}$. Every element of V_{d+2} lies in one of the four cosets V_d, $u_{d/2+1} + V_d$, $v_{d/2+1} + V_d$ and $u_{d/2+1} + v_{d/2+1} + V_d$ of V_d. It is easily checked that, by bilinearity of $[\cdot,\cdot]$, each of these cosets is a graph isomorphic to V_d and that its adjacencies are as required in def. 6. For instance, for all $x,y \in V_d$,

$$[u_{d/2+1}+x, u_{d/2+1}+y] = [u_{d/2+1}, y]+[x, u_{d/2+1}]+[x,y] = [x,y] = [u_{d/2+1}+x, y],$$

and

$$[u_{d/2+1}+x, v_{d/2+1}+y] = 1+[x,y] = [u_{d/2+1}+x, u_{d/2+1}+v_{d/2+1}+y]. \square$$

Theorem 3. Let V be a regular symplectic space. The symplectic dimension of a graph G of order n given as a subset $X \subseteq V$ as well as an optimal embedding of G can be found in time $O(n^3)$. Therefore, the symplectic dimension of a graph given by its adjacency matrix can be computed in cubic time.

Proof. Let $X \subseteq V_d$ be a symplectic embedding of G of dimension $\leq d$ (see lemma 1). If $n = 1$ then $dim(X) = 0$ and $X' = \{0\}$ is an optimal embedding of X so assume $n > 1$. First assume, in view of the proof of lemma 2, that G has no isolated vertices and that G is linearly irreducible, i.e., is its own linear core . Let v be any nonzero vertex in X. By lemma 2, G must have dimension $|G|$. Since V_d is a regular symplectic space, there exists a symplectic basis element not orthogonal to v. Swapping symplectic pairs if necessary, assume without loss of generality that $[v, u_{d/2}] \neq 0$. Now replace every vertex $x \in X$ by $\tau(x) := x + [v + u_{d/2}, x](v + u_{d/2})$. It is easy to check that τ is an isometry of V_d, hence $\tau(X)$ is another embedding of G in which $v = u_{d/2}$. Thus assume with no loss of generality that $u_{d/2} \in X$. Let

$$X = A \cup (u_{d/2}+B) \cup (v_{d/2}+C) \cup (u_{d/2}+v_{d/2}+D)$$

be the decomposition of X induced by the s-power decomposition of V_d of lemma 3, where $A,B,C,D \subseteq V_{d-2}$ are the projections of X into the four cosets of the subspace V_{d-2} of two less dimensions. Suppose we have constructed an optimal symplectic embedding $X' \subseteq V_{d'}$ of the set $Y := A \cup B \cup C \cup D$, i.e., an isomorphism $f: Y \to X'$. The mapping f naturally induces an embedding $\phi : X \to V_{d'+2}$ of X given by

$$\phi(X) = f(A) \cup (u_{d'/2+1} + f(B)) \cup (v_{d'/2+1} + f(C)) \cup (u_{d'/2+1} + v_{d'/2+1} + f(D)).$$

By lemma 3, $\phi(X)$ is an optimal symplectic embedding of X of dimension d. Clearly all the operations involved (including those described before def. 5) produce a linear core of a given embedding in cubic time.

Now, suppose G is not its own linear core, i.e., that another embedding of G can be obtained by replacing some vertex $v \in X$ by a linear combination $\sum_{x \neq v} \lambda_x x$ of the other elements in X. If $Y \subseteq V_{d'}$ is an optimal embedding of $X - \{v\}$ of dimension d', then $Y \cup \{z\}$, where $z := \sum_{x \neq v} \lambda_x y_x$ and y_x is the vertex corresponding to $x \in X$, is an embedding of G of dimension d', whenever z is distinct from the remaining vertices in Y. If, on the contrary, all such linear combinations produce another element of Y, G cannot have dimension d'. For in that case, an embedding $Y' \subseteq V_{d'}$ of G of dimension d' with a core Y_0' would extend to an isometry $\phi : V_{d'} \to X$ because Y_0' must be a basis of $V_{d'}$ by lemma 2; therefore $\phi^{-1}(v)$ would be a linear combination of elements of Y distinct from other elements in Y, so the corresponding linear combination of elements of X would be distinct from all other elements of X and could replace v. Therefore G has dimension $d' + 2$ since v can be replaced by $z + u_{d'/2+1}$ to obtain an embedding of dimension $d' + 2$.

Finally, if G has isolated vertices, let X_1 be the nonempty set of isolated vertices of X. Find as before an optimal embedding Y of $(X - X_1)$ and solve a system of linear equations to find a subset $Z \subseteq (X - X_1)^{\perp}$ in $V_{d'}$ of cardinality $|X_1|$, where $d' = 2\lceil log|X_1|\rceil$. Clearly $Y \cup Z$ is an optimal embedding of G. □

We conclude this section with some examples of symplectic embeddings and dimensions of various graphs. Some of them are easy consecuences of general facts about the symplectic spaces V_d (All logarithms are taken to base 2).

Examples.

1. The universal d−dimensional graph has order 2^d and has largest degree $\Delta(V) = 2^{d-1}$. Of course,

$$dim(V) = log|V|.$$

2. Thus, if G has maximum degree $\Delta(G)$, $dim(G) \geq 1 + log\,\Delta\,(G)$.

3. The structure of the graph V_d determines many properties of the graphs of dimension d. For instance, since $Sp(V_d)$ acts transitively on $V_d - \{0\}$ [1, p. 138], these vertex deleted graphs are regular of valence 2^{d-1}. It follows that for any graph G, $log|V(G)| \leq dim(G)$. In fact,

$$log(|V(G)|) \leq dim(G) \leq |V(G)| + c$$

are optimal bounds on the dimensions of any graph G. The second inequality is a consequence of the fact that there is an absolute constant c such that any $n \times n$ adjacency matrix is a submatrix of an invertible matrix modulo 2 with $n + c$ columns.

4. The largest totally isotropic subspace of V_d (i.e., satisfying the identity $[x, y] = 0$) has dimension $d/2$ and contains $2^{d/2}$ elements. Therefore

$$dim(K'_n) = 2\lceil log\, n \rceil.$$

5. The s-power decomposition of lemma 2 implies by induction that

$$dim(K_n) = 2\lfloor n/2 \rfloor.$$

This allows an estimate on the size of the largest clique of a graph G in terms of its dimension (see theorem 4 in section 4).

4. Applications

Symplectic embeddings offer the possibilitiy of new algorithms for graph-theoretic problems. For example, by the results of section 3, all graphs can be given as subsets of a fixed symplectic space V of dimension d (see theorem 3). Obviously the symplectic dimension is an isomorphism invariant. Moreover,

Proposition 1. Two linear cores $G, H \subseteq V$ are isomorphic if and only if there is an isometry $\sigma \in Sp(V)$ such that $\sigma(G) = H$, i.e., iff they belong to the same orbit under the induced action of $Sp(V)$ on the power set of V. □

Based on this observation, one can obtain a new algorithm for graph isomorphism based on a decomposition theorem of Dieudonné [11, theorem 2.18] for isometries of a symplectic space. The detailed description and a full analysis of its complexity and experimental performance will appear in [6].

From the constructions and remarks in the previous section one obtains the following curious result.

Proposition 2. Any graph of order 2^d and symplectic dimension d is isomorphic to V_d and its automorphism group is (isomorphic to) the full symplectic group $Sp(V_d)$. □

Theorem 4. $\omega(G) \leq dim(G)$, i.e., a (largest) clique in a graph G contains at most $dim(G) + 1$ vertices. This bound is optimal. □

This is an important estimate since deciding whether an arbitrary graph has a clique of size at least a given integer m is NP−complete [4, problem GT19]. Since this upper bound can be computed in P−time, it follows that problem GT19 remains NP−complete even if m is bounded by $dim(G) + 1$.

There is a number of other features that make this approach to graph theoretic problems suitable for practical implementation. On a practical level, the operations involved −essentially bit manipulation− are very suitable for automatic execution. On a theoretical level, symplectic spaces and symplectic groups $Sp_d(V)$ give rise to one of the classical families of finite simple groups of Lie type. The virtue of their action lies in the fact that it is the same group acting *uniformly* on *all* graphs of dimension d, and, moreover, on objects with a *linear and geometric* structure, which yields global information on a graph from its local properties such as orthogonality and restriction to a basis. This action on the set of subsets of V does not seem to have been considered at all in the mathematical literature and seems to be a mathematically interesting problem in itself. Moreover, the above construction can be done over any field F of char(F) > 2, where the symplectic groups are generated by only 2 elements.

5. Conclusion

Other implicit representations of graphs have been suggested [3], [7],[14],[2], [10], [9]. They generally apply to restricted classes of graphs or are not uniform for all graphs, and the labeling does not seem to relate to isomorphism, if it does at all, by known geometric structures of the type presented here.

The basis for a general, algorithmic analytic graph theory has been developed. This paper provides a polynomial time algorithm to find a most efficient labeling of the vertices of *arbitrary* graphs into an equal number of strings (over the binary alphabet) of smallest possible length d (the coordinates of a set of vectors in a symplectic space V_d with respect to a standard symplectic basis). One of the advantages of this vertex labeling is that adjacency is *implicitely* represented by the values of a single bilinear map defined on V_d which takes on a particularly simple form that can be evaluated easily. Furthermore, the isometries of the host space naturally correspond to graph isomorphism in the same way that the motions of ordinary euclidean space correspond to the congruences of elementary geometric figures. This approach makes explicit a geometric aspect of graph theory that allows the calculation of graph invariants useful in estimates of computationally hard graph-theoretic parameters (clique size). Further work on this approach is likely to provide not only efficient algorithms for graph theoretic problems, but also novel approaches to other problems in graph theory (see [6]).

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