Latin rectangles with the constant row-column intersection property

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ABSTRACT

Let L be an $n \times m$ Latin rectangle on a set of v symbols with the property that each symbol occurs in precisely r cells of L. Then L is said to have the row-column intersection property if each row and column of L have precisely r symbols in common. It is shown here that the trivial necessary conditions

(i) $rv = mn$ and

(ii) $r \leq \min\{m,n\}$

are sufficient to guarantee the existence of such a Latin rectangle.

1. Introduction

An $m \times n$ Latin rectangle based on a v-set V is an $m \times n$ array of symbols from V with the property that no symbol appears more than once in each row or column of the array. (It is assumed here that each cell of the array contains precisely one symbol from V). Such a Latin square L will be said to be *r—regular* if each symbol of V occurs in precisely r cells of L. An r-regular Latin rectangle is said to have the row-column intersection property if each row and column have exactly r symbols in common.

It is clear that a necessary condition for the existence of such an r-regular Latin rectangle is the relation

(i) $rv = mn$.

It is also clear that in order for a Latin rectangle to be r-regular the inequality

(ii) $r \leq \min\{m,n\}$

must hold, since each symbol must occur r times, and at most once in each row and column. It is the purpose of this note to show that these conditions are also sufficient.

For convenience, we refer to an r-regular Latin rectangle with the row and column intersection property simply as an *rc—rectangle*. With every rc-rectangle L there is an associated quadruple (v,r,m,n) called the parameter vector of L. The question is then which quadruples of positive integers are parameter vectors of rc-rectangles?

2. Special cases

A $v \times v$ Latin rectangle which is v-regular is called a Latin square of order v. Since in such a Latin square each symbol appears once in each row and column, any Latin square of order v is an rc-rectangle with parameters (v,v,v,v). We extend this result in the following.

Lemma 2.1. Let r and v be positive integers satisfying $1 \leq r \leq v$. Then there exist rc-rectangles with parameters (v,r,r,v) and (v,r,v,r).

Proof. Trivially there exists a Latin square L_v of order v. Take the first r rows L_v. The result is clearly an r-regular r by v Latin rectangle. Since each entry occurs exactly once in each row and at most once in each column, the array is an rc-rectangle with parameters (v,r,r,v). Similarly, by taking the first r columns of L, an rc-rectangle with parameters (v,r,v,r) is obtained. □

Fortunately, smaller rc-rectangles can be composed to produce larger, as is noted below. The construction of the direct product of to Latin rectangles is well known. For the sake of completeness, it is described briefly here.

Let A_1 and A_2 be Latin rectangles defined on the symbol sets V_1 and V_2 respectively. For any symbol x of V_2, let (A_1,x) denote the array obtained from A_1 by replacing each symbol a in A_1 by the ordered pair (a,x). Then the direct product $A_1 \times A_2$ is obtained by replacing each symbol x in A_2 by the array (A_1,x).

Lemma 2.2. Suppose that there exist rc-squares L_1 and L_2 with parameter vectors (v_1,r_1,m_1,n_1) and (v_2,r_2,m_2,n_2) respectively. Then there exists an rc-rectangle with parameters $(v_1v_2,r_1r_2,m_1m_2,n_1n_2)$.

Proof. It is easily verified that the direct product of L_1 and L_2 produces an rc-rectangle with the given parameters. □

It is shown in the next section that lemmas 2.1 and 2.2 are sufficient to solve the problem at hand.

3. The general case

In this section it is shown that the conditions (i) and (ii) of section 1 are sufficient for the existence of rc-rectangles.

Theorem 3.1. Let v,r,m,n be four positive integers satisfying the relations

(i) $vr = mn$ and

(ii) $r \leq \min(m,n)$.

Then there exists an rc-square L whose parameter vector is (v,r,m,n).

Proof. Since r divides mn, we may write $r = ab$ where a and b are positive integers such that a divides m and b divides n. (Either a or b may be 1). Therefore there are positive integers c and d such that $m = ac$ and $n = bd$. Since $vr = mn$, then $v = cd$. Also since $r \leq m$, we have $ab \leq ac$, that is, $b \leq c$, and similarly $a \leq d$. By lemma 2.1, this implies that there exists an rc-rectangle D_1 with parameter vector (d,a,a,d) and an rc-rectangle D_2 with parameter vector (c,b,c,b). Then D_3 $D_1 \times D_2$ is an rc-rectangle with parameter vector (cd,ab,ac,bd), that is (v,r,m,n), as required. □