On Tree-factor Covered Graphs

Yu Qinglin

Department of Mathematics and Statistics,
Simon Fraser University, Burnaby, Canada,
V5A 1S6

and

Chen Ciping

Department of Mathematics, Shandong University,
Jinan, The People's Republic of China

ABSTRACT

Let T(n) be the set of all trees with at least one and no more than n edges. A T(n)-factor of a graph G is defined to be a spanning subgraph of G each component of which is isomorphic to one of T(n). If every $K_{1,k}$ subgraph of G is contained in a T(n)-factor of G, then G is said to be T(n)-factor k-covered. In this paper, we give a criterion for a graph to be a T(n)-factor k-covered graph.

1. Introduction

Consider a finite connected graph G with the vertex set V(G) and the edge set E(G), which has neither multiple edges nor loops. For any $S \subseteq V(A)$, we denote by G-S the subgraph of G obtained by deleting the vertices of S together with their incident edges, and by $\Delta(G)$ the maximum degree of G. We denote by I(G-S) the set of isolated vertices of G-S, and put $i(G-S)=|I(G-S)|$. The neighbour set of S in G is denoted by $N_G(S)$. In this paper we always suppose that n is an integer and $n \geq 2$.

For a set $\mathcal{F}=\{A,B,...,C\}$ of graphs, an <u>{A,B,. . . ,C}-subgraph</u> of a graph G is a subgraph M of G each component of which is isomorphic to one of the subgraphs in the set {A,B,. . . ,C}. Moreover, if M is a spanning {A,B,. . . ,C}-subgraph, then M is called an {A,B,. . . ,C}-<u>factor</u> of G. An {A,B,. . . ,C}-subgraph M of G is said to be maximum, if G has no {A,B,. . . ,C}-subgraph M' with $|V(M')| > |V(M)|$.

In particular, if $\{A,B,. . . ,C\}=S(n)=\{K_{1,i} : 1\le i\le n\}$, then an {A,B,. . . ,C}-factors of G is also called a star-factor, or an S(n)-factor. If {A,B,. . . ,C}=T(n), the set of all trees with at least one and more than n edges, then an {A,B,. . . ,C}-factor of G is also called a tree-factor, or a T(n)-factor.

A graph G is {A,B,. . . ,C}-factor k-covered, $1\le k\le \Delta(G)$, if for every subgraph $K_{1,k}$ of G there exists an {A,B,. . . ,C}-factor of G containing it.

In [4] Little introduced the concept of an factor-covered graph. This is a graph G with the property that for every edge $e\in E(G)$ there exists a 1-factor containing e. He gave a criterion for classifying 1-factor covered graphs. A defect d-matching in G is a matching covering all but d vertices of G. Little, Grant and Holton in [5] generalized Little's result to defect-d matchings, and showed that a graph G is defect-d covered if and only if it has a defect-d matching and each subset S of V(G) with $|S|+d$ odd components in G-S, is an independent set. In this paper we generalise this idea and consider k-covered graphs. These are graphs with the property that for every subgraph $K_{1,k}$ which they contain there exists a star-factor containing it. We will give a criterion for a graph to be tree-factor k-covered. This is a generalization of the result by Amahashi and Kano in [1]. They showed that a graph has a tree-factor if and only if G-S has at most $n|S|$ isolated vertices for every subset S of V(G).

All notations and definitions not given here can found in [2].

2. Characterization of tree-factor k-covered graphs.

The following two theorems are proved in [1] and [3] respectively.

Theorem A (Amahashi and Kano[1]): Let G be graph. Then G has an $S(n)$-factor if and only if $i(G-S) \leq n|S|$ for every $S \subseteq V(G)$.

Theorem L (Las Vergnas[3]): Let G be a graph. Then G has a $[1,n]$-factor if and only if $i(G-S) \leq n|S|$ for every $S \subseteq V(G)$.

From Theorem A and Theorem L, we can easily derive the following result.

Theorem 1. The graph G has a $T(n)$-factor if and only if $i(G-S) \leq n|S|$ for every $S \subseteq V(G)$.

The above theorem gives a criterion for $T(n)$-factors in a graph. So, if we want to characterize the $T(n)$-factor k-covered graphs, we need only to add more conditions to this. In order to do so, we will require some more definitions and lemmas.

Let G be a graph and $A \subseteq V(G)$. If there exists a $T(n)$-subgraph of G which spans A, then A is called $T(n)$-saturated. Let M be a $T(n)$-subgraph of G, and let $x,y \in V(G)(x \neq y)$. If x and y belong to the same component of M, then we say that x matches with y under M.

For a graph G, $d(G)=\max_{S \subseteq V(G)} \{i(G-S)-n|S|\}$ is called the defect of G. Put $D(G)=\{S: S \subseteq V(G)$ and $i(G-S)-n|S|=d(G)\}$. Clearly, $d(G) \geq 0$ for any graph G, and G has a $T(n)$-factor if and only if $d(G)=0$.

We will need the following three auxiliary lemmas.

Lemma 2([5]). For every maximum $S(n)$-subgraph M of a graph G

$$|V(M)|= |V(G)|-d(G).$$

Lemma 3. For every maximum $T(n)$-subgraph M of a graph G

$$|V(M)|=|V(G)|-d(G).$$

Proof: Given a maximum T(n)-subgraph, we can delete edges to get an S(n)-subgraph on the same number of vertices. The S(n)-subgraph is maximum as if not any larger S(n)-subgraph would also be a larger T(n)-subgraph. □

Lemma 4. Let G be a graph without a T(n)-factor, and $S_0 \in D(G)$. Then there exists a set V_0, $V_0 \subseteq I(G-S_0)$, with $|V_0|=d(G)$, such that $G-V_0$ has a T(n)-factor.

Proof: As $S_0 \in D(G)$, then $i(G-S_0)=n|S_0|+d(G)$. Thus, for any T(n)-subgraph of G, each vertex in S_0 matches with at most n vertices in $I(G-S_0)$. Therefore, for any T(n)-subgraph of G, there exist at least d(G) unsaturated vertices in $I(G-S_0)$.

Let M be a maximum T(n)-subgraph of G, and V_0 be the set of all vertices in $I(G-S_0)$ unsaturated under M. Then $|V_0| \geqslant d(G)$. But by Lemma 3 $d(G)=|V(G)-V(M)| \geqslant |V_0| \geqslant d(G)$ and so $|V_0|=d(G)$. Thus V_0 is the set of all unsaturated vertices of G under the maximum T(n)-subgraph M. So $G-V_0$ has a T(n)-factor M. □

The following theorem is fundamental to the proof of our main theorem.

Theorem 5. Let G be a graph, and K a $K_{1,k}$ subgraph of G, where $1 \leq k \leq \min\{\Delta(G),n\}$.Then has a T(n)-factor containing K if and only if

(1) $i(G-S) \leqslant n|S|$ for every $S \subseteq V(G)$, and

(2) $i(\bar{G}-\bar{S}) \leqslant n|\bar{S}|+(n-k)$ for every $\bar{S} \subseteq V(\bar{G})$, where $\bar{G}=G-V(K)$.

Proof: We first prove the necessity of the conditions.Suppose Let M be a T(n)-factor in G which contains K. Denote by C the component containing K in M. Let $A=V(C)-V(K)$. Since $C \in T(n)$, we have $|V(C)| \leqslant n+1$. Moreover $|V(K)|=k+1$, thus $|A| \leqslant n-k$. Set $\bar{G}=G-V(K)$. Because $G-V(C)=\bar{G}-A$ has a T(n)-factor,then by Theorem 1 we have $i(\bar{G}-A-S) \leq n|S|$ for every $S \subseteq V(\bar{G})-A$. Let $\bar{S} \subseteq V(\bar{G})$. Then $\bar{S}-A \subseteq V(\bar{G})-A$ and $i(\bar{G}-A-(\bar{S}-A)) \leqslant n|\bar{S}-A|$. Therefore

$$i(\bar{G}-\bar{S}) \leq i(\bar{G}-A-(\bar{S}-A))+|A| \leq n|\bar{S}-A|+|A| \leq n|\bar{S}|+(n-k).$$

Consequently, condition (2) holds.

It now remains to prove the sufficiency of the condition. We will give an augmenting path procedure to construct a T(n)-factor containing K. Suppose that condition (1) and (2) hold. It follows from condition (1) that G has a T(n)-factor.

If $\bar{G}$ has a T(n)-factor $\bar{F}$, then $\bar{F} \cup K$ is a T(n)-factor of G containing K. Suppose then that $\bar{G}$ has no T(n)-factor. From the conditions we have $n-k \geq d(\bar{G}) > 0$. Let $S_0 \in D(\bar{G})$. Then $i(\bar{G}-S_0)-n|S_0|=d(\bar{G})$. It follows from Lemma 4 that there exists a set V_0, $V_0 \subseteq I(G-S_0)$, and $|V_0|=d(\bar{G})$, such that $\bar{G}-V_0$ has a T(n)-factor M_0; that is, M_0 is a T(n)-factor in $G-V(K)-V_0$.

Suppose that $N_G(\{v\}) \cap N(K) \neq \phi$ for every $v \in V_0$. For each $v \in V_0$ choose a vertex $v' \in N_G(\{v\}) \cap V(K)$, and let E_0 be the set of edges (v,v'). So $|E_0|=d(G)$ and $T_0=G[E_0 \cup E(K)]$ is a spanning tree containing K in $G[V_0 \cup V(K)]$. Since

$$|E(T_0)|=d(\bar{G})+|E(K)| \leq (n-k)+k=n,$$

T_0 is a T(n)-factor containing K in $G[V(K) \cup V_0]$, and $M_0 \cup T_0$ is a T(n)-factor of G containing K .

Otherwise, there exists a vertex v, $v \in V_0$, such that $N_G(v) \cap V(K)=\phi$. For every $A \subseteq S_0$, put $N_I(A)=N_G(A) \cap I(\bar{G}-S_0)$, and let $N_{M_0}(A)$ denote the set of vertices which A matches under M_0.

Set $S_1=\{x:\ x \in S_0,\ \text{and}\ (x,v) \in E(G)\}$

$S_2=\{x:\ x \in S_0 \setminus S_1,\ \text{and}\ N_I(\{x\}) \cap N_{M_0}(S_1) \neq \phi\}$

$S_3=\{x:\ x \in S_0 \setminus (S_1 \cup S_2),\ \text{and}\ N_I(\{x\}) \cap N_{M_0}(S_2) \neq \phi\}$

...

$S_m=\{x:\ x \in S_0 \setminus (\bigcup_{j=1}^{m-1} S_j),\ \text{and}\ N_I(\{x\}) \cap N_{M_0}(S_{m-1}) \neq \phi\}$.

Observe that $S_1 \neq \phi$.

Since $S_0 \in D(\bar{G})$, the subset $\bigcup_{j=1}^{m} S_j$ of S_0 satisfies $|N_{M_0}(\bigcup_{j=1}^{m} S_j)| = n|\bigcup_{j=1}^{m} S_j|$. Thus, if $N_{M_0}(\bigcup_{j=1}^{m} S_j) \cap N_I(V(K))=\phi$, then $i(G-\bigcup_{j=1}^{m} S_j) \geq$

$n|\bigcup_{j=1}^{l} S_j|+|\{v\}|$, which is contrary to (1). Therefore there exists a vertex y_l, $y_l \in (N_{M_0}(S_l)-S_l)\cap N_I(V(K))$.

Let x_i be the vertex matching y_i in S_i, and y_{i-1} be a vertex which is adjacent to x_i in $N_I(S_{i-1})$ $(i=2,\ldots,l)$. Let x_1 be a vertex matching y_1 in S_1. Set

$M_1=M_0-\{(x_1,y_1),\ldots,(x_l,y_l)\}\cup\{(x_1,v),\ (x_2,y_1),\ldots,(x_l,y_{l-1})\}$,

and $V_1=(V_0-\{v\})\cup\{y_l\}$. Then, from the construction of S_j, M_1 is a $T(n)$-factor in $\bar{G}-V_1$.

If $N_C(\{u\})\cap V(K)\neq\phi$ for every $u\in V_1$, then the proof is finished by the earlier argument. Otherwise, for any vertex in V_1 which is not adjacent to $V(K)$, we again apply the argument just given. Eventually, we reach a vertex set V_p and a $T(n)$-factor M_p in $\bar{G}-V_p$, so that every vertex in V_p is adjacent to $V(K)$, or $N_C(\{v\})\cap V(K)\neq\phi$ for all $v\in V_p$. This complete the proof. □

<u>Corollary 6</u>. Let G be a graph and $1\leq k\leq n$. If $i(G-S)\leq n|S|-(n+1)k$ for every $S\subseteq V(G)$, then G is $T(n)$-factor k-covered.

<u>Proof</u>: It is obvious that condition (1) of Theorem 5 is satisfied. For any given $K_{1,k}$ subgraph K of G, set $\bar{G}=G-V(K)$. For every $\bar{S}\subseteq V(\bar{G})$, we have

$$i(\bar{G}-\bar{S})=i(G-V(K)-\bar{S})\leq n|V(K)\cup\bar{S}|-(n+1)k$$
$$=n|\bar{S}|+n(k+1)-(n+1)k=n|\bar{S}|+(n-k).$$

Thus, condition (2) of Theorem 5 is also satisfied. So G has a $T(n)$-factor containing K and G is $T(n)$-factor k-covered. □

Now we are ready to prove our main theorem.

Theorem 7. Let G be a graph, and $1 \leq k \leq n$. Then G is T(n)-factor k-covered if and only if:

(1) $i(G-S) \leq n|S|$ for every $S \subseteq V(G)$ and

(2) $i(G-S) > n|S|-(n+1)k$ implies that $\Delta(G[S]) < k$.

Proof: Suppose that G is T(n)-factor k-covered. It is obvious that condition (1) holds. Suppose there exists a subset of vertices $S_0, S_0 \subseteq V(G)$, such that $n|S_0| \geq i(G-S_0) > n|S_0|-(n+1)k$, and $\Delta(G[S_0]) \geq k$. Since $\Delta(G[S_0]) \geq k$, then $G[S_0]$ contains a $K_{1,k}$ subgraph K. Set $\bar{G}=G-V(K)$ and $\bar{S}=S_0-V(K)$. Then

$$i(\bar{G}-\bar{S})=i(G-V(K)-(S_0-V(K)))=i(G-S_0) \geq n|S_0|-(n+1)k+1$$
$$=n|\bar{S}|+(n-k)+1$$

and so by Theorem 5 there exists no T(n)-factor containing K in G, a contradiction.

We next prove the sufficiency of the theorem. Suppose that there were a $K_{1,k}$ subgraph K of G such that G had no T(n)-factor containing K. Set $\bar{G}=G-V(K)$. By Theorem 5 there exists a set $\bar{S}$, $\bar{S} \subseteq V(\bar{G})$, such thst $i(\bar{G}-\bar{S}) > n|\bar{S}|+(n-k)$. Set $S_0=\bar{S} \cup V(K)$. Then

$$i(G-S_0)=i(\bar{G}-\bar{S}) > n|\bar{S}|+n-k=n|S_0|-k(n+1).$$

But $\Delta(G[S_0]) \geq \Delta(K)=k$, which is contrary to (2). □

Note that from Theorem A and Theorem 1, it follows that the star-factor and tree-factor problem are equivalent. But for the covering problem they are not. For example the path of lenth 3, P_3, is T(3)-factor 1-covered, but not S(3)-factor 1-covered. Since $S(n) \subseteq T(n)$, the S(n)-factor covered graphs are also T(n)-factor covered, but the converse is not true. Thus we leave the following open problem, characterize the star-factor covered graphs.

Acknowledgement

The authors wish to express their thanks to Professor K.Heinrich for her kind help.

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