# A NOTE ON A PARTICULAR FAMILY OF EXACT COVERINGS

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## 1. INTRODUCTION.

Suppose that we have a set of v elements, and that we wish to have an exact covering of all the pairs. Furthermore, this covering is to have all its blocks of the same length, namely, k+1, with the sole exception of one long block that contains $k^2$-k+1 elements. We naturally restrict ourselves to the case when k>2 in order to ensure that the long block actually is longer than the other blocks.

We can immediately obtain two necessary conditions. First, it is clear that all the pairs that do not occur in the long block must occur in the short blocks; hence, we find that

$$v(v-1) - (k^2-k+1)(k^2-k) \equiv 0 \pmod{k(k+1)}.$$

Also, suppose that x is an element in the long block; then the elements not in the long block must occur with x in the short blocks, in sets of k. Hence, we have

$$v - (k^2-k+1) \equiv 0 \pmod{k}.$$

These two necessary conditions are easily discussed. From the second congruence, we immediately have $v \equiv 1 \pmod{k}$. So we write

$$v = 1+ka;$$

substitute this value of v in the first congruence to yield

$$\{ka-(k^2-k)\}\{ak+1+(k^2-k)\} \equiv 0 \pmod{k(k+1)}.$$

Hence, we find

$$(a+2)(a-3) \equiv 0 \pmod{(k+1)}.$$

When k+1 is composite, there will, in general, exist several solutions for this congruence. However, there are always solutions given by setting $a \equiv 3$ or $a \equiv -2$, modulo (k+1).

Thus, we may write

$$v = 1+k\{m(k+1)+3\} \text{ or } v = 1 +k\{m(k+1)-2\}, \text{ that is,}$$

$$v \equiv 3k+1 \quad\text{or}\quad v \equiv 1-2k \equiv k^2-k+1 \pmod{k(k+1)}.$$

So we have these two solutions for all values of k (and there will, in general be others when k+1 is composite).

For $k = 3$, we get the well known result (cf. [1]) that, provided that v is at least 22, we can cover all $v(v-1)/2$ pairs by a 7-block and a set of $(v-7)(v+6)/2$ quadruples whenever v is congruent to 7 or 10 (mod 12). There is a similar exclusion of small values of v for other values of k; this question is discussed in more detail in the next section.

The case $k = 4$ is particularly interesting, in that it is the only case when there is a single congruence class, since $3k+1 \equiv k^2-k+1$, modulo $k(k+1)$, when $k = 4$. Consequently, we see that we may be able to cover all the pairs by a 13-block and a set of quintuples if v is congruent to 13 mod 20.

The next few possibilities are a 21-set and sextuples when v is congruent to 1,6,16, or 21 (mod 30); a 31-set and septuples when v is congruent to 19 or 31 (mod 42); a 43-set and octuples when v is congruent to 22 or 43 (mod 56); a 57-set and nonuples when v is congruent to 25 or 57 (mod 72); a 73-set and decuples when v is congruent to 28 or 73 (mod 90).

## 2.EXCLUDED SMALL VALUES.

The case when $v = mk(k+1)+3k+1$ is especially interesting when we look at the excluded small values. We note that the number of blocks of length $(k+1)$ has to be

$$\{v(v-1) - (k^2-k+1)(k^2-k)\}/k(k+1)$$

$$=m^2k(k+1) + 6mk + m + 6 +(1-k)(k-2).$$

However, it is necessary that each element from the long block occur in

$$\{mk(k+1)-k^2+4k\}/k = m(k+1)-k+4$$

of the blocks of length $k+1$, and we thus have the condition

$$(k^2-k+1)(m(k+1)-k+4) \leq m^2k(k+1)+6mk+m+6-k^2+3k-2.$$

This condition reduces to

$$0 \leq \{(k+1)m-(k-4)\}\{m-(k-2)\};$$

hence, we see that, if m is assigned the value $(k-2)$, all of the short blocks contain exactly one element from the long block. For values of m greater than $k-2$, there are short blocks that are disjoint from the long block. Of course, for values of m greater than zero but less than $k-2$, the covering becomes impossible.

Let us now take the critical value $k-2$ as the value of m, and let us delete all elements that occur in the long block. The result is a set of blocks, all of length k; these blocks form a resolvable balanced incomplete block design which possesses parameters

$$v = (k-2)(k+1)k+3k+1-k^2+k-1 = k(k^2-2k+2), \; k, \; \lambda = 1.$$

In the particular case that $k = q+1$, where q is a prime power, this design has parameters given by

$$v = (q+1)(q^2+1), \; b = (q^2+1)(q^2+q+1), r = q^2+q+1, \; k = q+1, \lambda = 1.$$

But these are just the parameters of the set of all lines in projective 3-space over GF(q). It is well known (cf.[3]) that the projective designs (15,35,7,3,1) is resolvable; indeed, this is just the Kirkman Schoolgirl Problem. For (40,130,13,4,1), and (85,357,21,5,1), we refer to [4]; indeed, Denniston [2] has shown that the general projective design PG(3,q) is resolvable. Hence, we see that these initial members of the set of covering designs with v congruent to 3k+1 modulo k(k+1) do indeed exist whenever q is a prime power.

## 3. QUESTIONS AND CONJECTURES.

If the projective design does not exist, can one still find a resolvable balanced incomplete block design with the same parameters (that is, in the case when q is not a prime power)? In view of the eighty designs with parameters (15,35,7,5,1)[3] and the enormous number of designs with the parameters (40,130,13,4,1)[4], it would seem reasonable that some resolvable design should exist for each value of k.

Suppose that m is greater than k-2; deletion of the elements from the long block then leaves a pairwise balanced design with some blocks of length k and others of length k+1. Can this design be constructed for all m values greater than k-2; one would expect the answer to be affirmative, although a general demonstration would be very difficult.

In a more modest vein, could one prove the existence of all the covering designs comprising a 13-set and set of quintuples? This case has the nice feature that the possible values of v all lie in one congruence class, modulo 20. The design does exist trivially when v = 13; it does not exist for v = 33; it does exist for v = 53 (all of the quintuples have an element in common with the 13-set). Recurrence methods should be able to settle the question in this case, provided that solutions for some of the smaller values of v could be found. We do not even know whether the design can be constructed when v = 73 (one 13-set, 255 blocks of length 5; 195 of the quintuples intersect the long block in a single element, and the other 60 quintuples are disjoint from the long block).

The case of a 21-set, together with sextuples, is also worthy of special attention, since this is the first case when more than two congruence classes, modulo 30, are involved (we recall that v can be congruent to 1,6,16, or 21 (mod 30)).

It is perhaps superfluous to mention that the utility of this particular type of covering derives from the fact that, whenever k is a prime power, there exists a symmetric balanced incomplete block design on $k^2$-k+1 points. This fact allows one to replace the long block by a set of blocks of length k,and these blocks cover the same pairs that were covered by the long block; the end result is thus a covering which is a pairwise balanced design having $k^2$-k+1 blocks of length k and all other blocks of length k+1. Of course, any other covering of the pairs in the long block can be used to give different over-all results.

## REFERENCES


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[3] R.A. Mathon, K.T. Phelps, and A. Rosa, *Small Steiner Triple Systems and their Properties*, Ars Combinatoria **15** (1983), 3-110.

[4] R.A. Mathon and A. Rosa, *Tables of Parameters of BIBDs with $r \leq 41$, including Existence, Enumeration, and Resolvability Results*, Annals of Discrete Math **26** (1985), 275-308.