A GENERALIZATION OF HALL'S THEOREM

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ABSTRACT

A necessary and sufficient condition for a family of finite sets to possess a collection of n compatible systems of distinct representatives (SDR's) is given. A decomposition of finite family of sets into partial SDR's is also studied.

Let $\mathcal{A} = (A_i, i \in I)$ be a system of subsets of a set E indexed by the set I. A system of distinct representatives, or shortly SDR, of $\mathcal{A}$ is a 1-1 function $x: I \to E$ such that $x(i) \in A_i$, $i \in I$. The SDR is also denoted $(x(i), i \in I)$ or $(x_i, i \in I)$.

Since P. Hall [6] showed the criterion for a finite family of sets to possess a SDR, a lot of generalizations of various kinds have been proved (see [8] or for later ones [1],[4],[9] etc.).

Two SDR's $E_j = (x_i^j, i \in A)$, $j = 1,2$, of a family $\mathcal{A} = (A_i, i \in I)$ are called compatible if $x_i^1 \neq x_i^2$, for any $i \in I$. In the first part of this paper we establish a necessary and sufficient condition for a family of finite subsets to possess n pairwise compatible SDR's. This theorem is a generalization of the transfinite form of Hall's theorem due to M. Hall Jr. [5] and at the same time a generalization of a result of A. S. Asratian concerned with finite families [2].

The second part of the paper deals with decompositions of a finite family of sets into partial SDR's.

1. Preliminaries.

Throughout this paper a family $\mathcal{A} = (A_i,\ i \in I)$ will be a family of finite subsets of an underlying set E. If not stated otherwise, the index set I is an arbitrary set of indices.

Let J_i, $i = 1,2,\ldots,n$, be a collection of SDR's of $\mathcal{A}$. Then these SDR's are called compatible if any pair of them is compatible. Let $x \in E$, and $J \subset I$. Then $(x,\mathcal{A},J) = |\{j \in J,\ x \in A_j\}|$. We will often abreviate $(x,\mathcal{A},J)$ to (x,J), whenever convenient. For $J \subset I$, n natural number, we define

$|A(J)|_n = \sum_{x \in E} \min(n,(x,J))$. If $n = 1$ then $|A(J)|_1 = |A(J)| = \bigcup_{j \in J} A_j$ which is standard notation in transversal theory.

We will say that a family $\mathcal{A} = (A_i,\ i \in I)$ satisfies condition $\mathcal{H}_n$ if $|A(J)|_n \geq n|J|$, for any finite subset J of I (which we denote $J \subset\subset I$). If I is also finite, then we replace the condition "for any finite subset" by "for any subset" of I.

For $n = 1$, the statement "A family $\mathcal{A} = (A_i,\ i \in I)$ satisfies condition $\mathcal{H}_n$" becomes: "$\mathcal{A}$ satisfies Hall's condition $\mathcal{H}$, i.e.: For any $J \subset\subset I$, $|A(J)| \geq |J|$.

<u>Remark 1</u>. The property to "satisfy condition $\mathcal{H}_n$" is hereditary in the sense that if a family $\mathcal{A}$ satisfies condition $\mathcal{H}_n$ then $\mathcal{A}$ satisfies also condition $\mathcal{H}_m$ for $m = 1,2,..,n-1$.

Let $J \subset I$. Then a family $\mathcal{A}' = (A_j,\ j \in J)$ is said to be a subfamily of the family $\mathcal{A} = (A_i,\ i \in I)$.

<u>Theorem 1</u>. Let the family $\mathcal{A} = (A_i,\ i \in I)$ satisfy condition $\mathcal{H}_n$. Let $n \geq 2$ be a natural number and J_k, $k = 1,2,\ldots,n$ be subsets of I such that $|A(J_k)|_n = n|J_k|$. Then for $J = \bigcap_{r=1}^{n} J_k$ we also have $|A(J)|_n = n|J|$.

Proof. It is sufficient to prove the statement for n = 2. For n > 2 it can be obtained by induction. Since $\mathcal{A}$ satisfies condition $\mathcal{K}_n$, $|A(J_1 \cap J_2)|_n = n|J_1 \cap J_2| + s$, where s is a nonnegative integer. For m = 1,2 set

$$t_m = \sum_{x \in A(Z_m)} \begin{cases} 0; & (x,J_1 \cap J_2) \geq n \\ n-(x,J_1 \cap J_2); & (x,J_1 \cap J_2) < n \text{ and } (x,J_1 \cap J_2)+(x,Z_m) \geq n. \\ (x,Z_m); & (x,J_1 \cap J_2) + (x,Z_m) < n \end{cases}$$

where $Z_1 = J_1 - J_2$, $Z_2 = J_2 - J_1$.

As $|A(J_m)|_n = n\ |J_m|$ we get

$t_m = n|Z_m| - s$, m = 1,2.

Then
$$\begin{aligned} |A(J_1 \cup J_2)|_n &\leq |A(J_1 \cap J_2)|_n + t_1 + t_2 \\ &= n|J_1 \cap J_2| + s + n|J_1 - J_2| - s + n|J_2 - J_1| - s \\ &= n|J_1 \cup J_2| - s. \end{aligned}$$

However, by assumption, $\mathcal{A}$ satisfies condition $\mathcal{K}_n$, thus s = 0 and the proof is complete.

Let $\mathcal{A}$ be a family of subsets of a set E and let $M \subset E$. The conditions for $\mathcal{A}$ to possess an SDR which contains any element of M are given by several authors (see [8]). (The elements of M are usually referred to as marginal elements). We will make use of the following theorem, which is Theorem 6.6.3 in [8].

Theorem (Hoffman-Kuhn-Rado). Let $\mathcal{A} = (A_i,\ i \in I)$ be a family of finite subsets of E; let $M \subseteq E$; and suppose that no element of M occurs in infinitely many A's. Then $\mathcal{A}$ possesses an SDR which contains M if and only if both of the following conditions are satisfied.

i) $|A(J)| \geq |J|_1$ for all $J \subset\subset I$.

ii) $|\{i \in I;\ A_i \cap N \neq \emptyset\}| \geq |N|$ for all $N \subset\subset M$.

For the reader's convenience, we finish this section by stating Rado's selection principle which will be made use of in what follows.

Theorem (Rado's selection principle, [10]). Let $\mathcal{A} = (A_i, i \in I)$ be a family of finite subsets of a set E. Let $J \subset\subset I$ and let f_J be a choice function of the subfamily $(A_j, j \in J)$. Then there exists a choice function f of $\mathcal{A}$ with the property: for each $J \subset\subset I$, there is a K with $J \subset K \subset\subset I$ and $f|J = f_K|J$.

2. A generalization of the transfinite form of Hall's theorem.

First we will deal with a special case.

Theorem 2. Let $\mathcal{A} = (A_i, i \in I)$ be a family such that $|A_i| = n$, $i \in I$. Then $\mathcal{A}$ possesses n compatible SDR's if and only if $\mathcal{A}$ satisfies condition $\mathcal{K}_n$.

Remark 2. In this case condition $\mathcal{K}_n$ is obviously equivalent to the condition that no element of underlying set E occurs in more than n A's.

Proof. The necessity of condition $\mathcal{K}_n$ is straightforward. To prove the "if" part we will show that $\mathcal{A}$ satisfies the assumptions of Hoffman-Kuhn-Rado theorem for $M = \{x; (x,I) = n\}$. In other words, there exists a SDR of $\mathcal{A}$ which contains all the elements of M.

As it was mentioned above, we have $(x,I) \leq n$, so no element of the underlying set E occurs in infinitely many A's. Directly from Remark 1, $\mathcal{A}$ has property (i). Any element of M occurs in exactly n A's, moreover, $|A_i| = n$, $i \in I$, hence $|\{i \in I: A_i \cap N \neq \emptyset\}| \geq |N|$ for all $N \subset\subset M$. Thus there exists $F_1 = \{x_i^1, i \in I\}$, F_1 containing all the elements of M, F_1 is SDR of $\mathcal{A}$.

Define a family $\mathcal{B} = (B_i, i \in I)$ be letting $B_i = A_i - x_i^1$. Any $x \in E$ occurs in at most n-1 B's, moreover $|B_i| = n-1$, $i \in I$, therefore $\mathcal{B}$ also satisfies conditions required for applying the Hoffman-Kuhn-Rado theorem for $M' = \{x, (x, \mathcal{B}, I) = n-1\}$. Thus we obtain a SDR F_2 containing M' which is compatible with F_1. By repeatedly using this procedure, we obtain n compatible SDR's.

Now we can prove a generalization of the transfinite form of Hall's theorem originally stated by M. Hall Jr. [5].

Theorem 3. A family $\mathcal{A} = (A_i,\ i\in I)$ possesses n compatible SDR's if and only if $\mathcal{A}$ satisfies condition $\mathcal{H}_n$.

Proof. Taking into account Theorem 2, all that remains to be shown is to prove that there exists a family $\mathcal{A}' = (A_i',\ i\in I)$ such that $A_i' \subset A_i$, $|A_i'| = n$, $i\in I$.

Consider a partial order "$\leq$" on $\mathcal{B}$, the collection of all families $\mathcal{A}' = (A_i',\ i\in I)$, satisfying $\mathcal{H}_n$ and having property $A_i' \subset A_i$, $i\in I$, given in the following way.

If $\mathcal{A}' = (A_i',\ i\in I)$, $\mathcal{A}'' = (A_i'',\ i\in I)$ are two families from $\mathcal{B}$ then $\mathcal{A}' \leq \mathcal{A}''$ if and only if $A' \subseteq A''$, $i\in I$.

Let $\mathcal{A}_j = (A_i^j,\ i\in I)$, $j\in J$, J an ordered set, be a chain in $(B, \leq)$. Then the family $\bar{\mathcal{A}} = (\bar{A}_i,\ i\in I)$ such that $\bar{A}_i = \bigcap_{j\in J} A_i^j$ has the property $\bar{A}_i \subseteq A_i$, $i\in I$. From the finiteness of all A's follows that $\bar{\mathcal{A}}$ satisfies condition $\mathcal{H}_n$ as well, and, consequently, $\bar{\mathcal{A}} \in \mathcal{B}$. Therefore, $\bar{\mathcal{A}}$ is a lower bound of $(\mathcal{B},\leq)$ and by Zorn's lemma, $(\mathcal{B},\leq)$ has a minimal element, say $\mathcal{A}^* = (\mathcal{A}_i^*,\ i\in I)$.

Suppose that there is an i_o, $i_o\in I$, such that $\mathcal{A}_{i_o}^* = \{x_1,x_2,\ldots,x_m\}$, $m > n$. Moreover, suppose that deleting any x_j from A_{i_o} leads to a violation of $\mathcal{H}_n$, i.e. for any $j = 1,2,\ldots,m$ there exists $J_j \subset\subset I$ such that for $\mathcal{A}_j = (A_k^j,\ k\in J_j)$, $A_k^j = A_k^*,\ k\in J_j)$, $k \neq i_o$, $A_{i_o}^j = A_{i_o}^* - x_j$ we get $|A^j(J_j)|_n < n|J_j|$. This implies that $|A^*(J_j)|_n = n|J_j|$, $j=1,2,\ldots,m$, and, consequently, that $(x_j,\ J_j - \{i_o\}) < n$. According to Theorem 1, for $J = \bigcap_{j=1}^{m} J_j$, we get $|A(J|_n = n|J|$. As $|A(\{i_o\})|_n = m > n.1$, we must have $J \not\ni \{i_o\}$. From $(x_j,\ J_j - \{i_o\}) < n$, $j = 1,2,\ldots,n$, follows directly that $(x_j,J) < n$, $j = 1,2,\ldots,n$, and therefore $|A(J-\{i_o\})|_n =$

$n|J| - m < n\ |J-\{i_o\}| = (n-1)|J|$. We arrived at a contradiction with the assumption that $\mathcal{A}^*$ satisfies condition $\mathcal{H}_n$. Therefore there exists x_j, $1 \leq j \leq m$, such that the family $\mathcal{A}' = (A_i', i\in I)$, $A_i' = A_i^*$, $i\in I - \{i_o\}$, $A_{i_o}' = A_{i_o}^* - \{x_j\}$ satisfies $\mathcal{H}_n$. But this contradicts the assumption that the family $\mathcal{A}^*$ is minimal in $(\mathcal{B},\leq)$. Thus $|A_i^*| = n$, $i\in I$, and the proof is complete.

The following two corollaries are now immediate.

Corollary 1 [5]. A family $\mathcal{A} = (A_i, i\in I)$ possesses a SDR if and only if $\mathcal{A}$ satisfies condition $\mathcal{H}$, i.e. Hall's condition: $\forall J \subset\subset I$: $|A(J)| \geq |J|$.

Corollary 2 [2]. Let $\mathcal{A} = (A_i, i = 1,2,\dots,m)$ be a finite family of subsets of E. Then $\mathcal{A}$ possesses n compatible SDR's if and only if $\mathcal{A}$ satisfies condition $\mathcal{H}_n$.

Remark 3. In [2] the author uses a slightly different notation. In addition, he exhibits an algorithm for constructing, if possible, n compatible SDR's of $\mathcal{A}$ in $O(n.m^3 + n^2.m.|E|)$ "elementary" steps. In [3], the lower bound for the number of different collections of compatible SDR's is given.

The proof of Theorem 3 is "direct" in that it does not utilize the fact that for finite families the criterion has already been shown. If we take this into account, using Rado's selection principle we can obtain a shorter proof.

Theorem 4. A family $\mathcal{A} = (A_i, i\in I)$ possesses n compatible SDR's if and only if any finite subfamily does.

Proof. Again, the "only if" part is obvious. To prove sufficiency, associate with the family $\mathcal{A}$ a family $\mathcal{A}' = (A_i', i\in I)$ such that A_i' contains exactly all ordered n-tuples of elements from A_i.

Formally, $A_i' = \{(x_1,\ldots,x_n);\ x_j \in A_i,\ j = 1,2,\ldots,n\}$, $i \in I$. Let $J \subset\subset I$ and let $F_k = (x_j^k,\ j \in J)$, $k = 1,2,\ldots,n$, be n compatible SDR's of $(A_i, i \in J)$. Define a choice function $f_J\colon J \to E^n$ (E^n is the set of all ordered n-tuples of elements from the underlying set E) by $f_J(j) = (x_j^1, x_j^2,\ldots,x_j^n)$, $j \in J$. According to Rado's selection principle we get a choice function $f\colon I \to E^n$, such that for any $J \subset\subset I$ there exists K, $J \subset K \subset\subset I$, with the property $f|J = f_K|J$. This implies immediately that if $f(i) = (x_i^1, x_i^2,\ldots,x_i^n)$, $f(j) = (x_j^1,x_j^2,\ldots,x_j^n)$ then $x_i^m \neq x_i^k$, $1 \leq m \neq k \leq n$, and $x_i^k \neq x_j^k$, $k = 1,2,\ldots,n$. Therefore, setting $F_k = (x_i^k,\ i \in I)$, $k = 1,2,\ldots,n$, we get n compatible SDR's of $\mathcal{A}$.

3. Decomposition of a family of sets into maximal SDR's.

Theorem 2 and Lemma 1 of [2], which is a version of Theorem 2 for finite families of sets, can be thought of as a decomposition of a family of sets into SDR's. Clearly, this is possible only in the case when the cardinalities of all sets of the family are the same. A family which contains sets with different cardinalities can be decomposed only into partial SDR's. One theorem of this type is König's theorem [7] which says that the edges of any bipartite graph with maximum degree k can be decomposed into k matchings. After translation into terms of transversal theory, it gives a decomposition of a family of sets into partial SDR's. (As a matter of fact, the above mentioned Lemma 1 of [2] is an immediate consequence of König's theorem.) However, we do not know anything about the size of these partial SDR's. For instance, one could require these SDR's to have the maximal possible size.

In what follows we will deal with this kind of decomposition, and we will confine ourselves to finite families.

Let $F' = (x_i', i \in J_1)$, $F'' = (x_i'', i \in J_2)$ be SDR of $\mathcal{A}' = (A_i, i \in J_n)$, $\mathcal{A}'' = (A_i, i \in J_2)$, respectively, where $\mathcal{A}'$, $\mathcal{A}''$ are subfamilies of the family $\mathcal{A}$. We will say that F' and F'' are compatible if $x_i' \neq x_i''$ for any $i \in J_1 \cap J_2$.

Definition. Let $0 = a_o < a_1 < a_2 < \dots < a_n$ be natural numbers. Let $\mathcal{A} = (A_i, i \in I)$ be a family and let $I = J_1 \cup J_2 \cup \dots \cup J_n$ be a decomposition of I such that if $i \in J_k$ then $|A_i| = a_k$. Then we will say that $\mathcal{A}$ is decomposable into maximal SDR's if there exist $a_i - a_{i-1}$ SDR's of $\mathcal{A}_i = (A_j, j \in I - \bigcup_{j=1}^{i-1} J_j)$, $i = 1,2,\dots,n$, such that all these a_n (partial) SDR's are pairwise compatible.

To illustrate the difference between the decomposition given by König's theorem and the decomposition into maximal SDR's, consider the family $\mathcal{A} = (A_1, A_2, A_3)$, where $A_1 = A_2 = \{a,b\}$, $A_3 = \{a,b,c\}$. Then $\mathcal{A}$ has a decomposition into three partial SDR's but cannot be decomposed into maximal SDR's because it does not possess two compatible SDR's (of length 3).

The following theorem gives a criterion of decomposability into maximal SDR's for a family such that the cardinality of its sets attains one of two possible values.

Theorem 5. Let a,b be natural numbers, $a > b$. Let $\mathcal{A} = (A_i, i \in I)$ be a family of subsets of a set E such that the cardinality of any A_i is either a or b. Then $\mathcal{A}$ is decomposable into maximal SDR's if and only if

i) $\mathcal{A}$ satisfies condition $\mathcal{K}_b$

ii) no $x \in E$ occurs in more than a A's.

Proof. For the sake of simplicity, we may take $I = \{1,2,\dots,n\}$ and $|A_i| = a$, $i = 1,2,\dots,m$, $|A_i| = b$, $i = m+1,\dots,n$.

Let $\mathcal{A}$ be decomposable into maximal SDR's, i.e. there exist b compatible SDR's of $\mathcal{A}$. By Theorem 3, $\mathcal{A}$ satisfies the condition $\mathcal{H}_b$. Moreover, there exist a-b compatible SDR's of $(A_i,\ i = 1,2,\ldots,m)$ which are also compatible with SDR's of $\mathcal{A}$, i.e. alltogether any element of E can occur in at most a A's.

Now we show that i) and ii) are sufficient for $\mathcal{A}$ to be decomposable into maximal SDR's. From i) and Theorem 3 we get that $\mathcal{A}$ possesses b compatible SDR's, say $F_k = (x_i^k,\ i = 1,2,\ldots,n)$, $k = 1,2,\ldots b$. We will construct a-b compatible SDR's of $(A_i,\ i = 1,2,\ldots,m)$, denoted $G_t = (y_i^t,\ i = 1,2,\ldots,m)$, $t = 1,2,\ldots,a\text{-}b$, which are compatible with F_k. The construction will proceed by induction. But $y_1^t = v_t$, $t = 1,2,\ldots,a\text{-}b$, where $\{v_1,\ldots,v_{a-b}\} = A_1 - \{x_1^k,\ k = 1,2,\ldots,b\}$. Suppose now that the elements y_i^t, $i = 1,2,\ldots,s$, $s < m$, $t = 1,2,\ldots, a\text{-}b$, have already been chosen, and we want to choose elements y_{s+1}^t, $t = 1,2,\ldots, a\text{-}b$. Let $z_1,\ldots,z_{a-b}$ be the elements of the set A_{s+1} - $\{x_{s+1}^k,\ k = 1,2,\ldots,b\}$. We will make use of the following procedure. Set $y_{s+1}^1 = z_1$. If $y_j^1 \neq z_1$, $j = 1,2,\ldots,s$, then $(y_j^1,\ j = 1,2,\ldots,s+1)$ is an SDR of $(A_i,\ i = 1,2,\ldots,s+1)$. Let $y_c^1 = z_1$, for some c, $1 \leq c \leq s$.

Then, from ii), there exists either k, $1 \leq k \leq b$ such that $z_1 \neq x_i^k$, $i = 1,2,\ldots,n$, or t, $2 \leq t \leq a\text{-}b$ such that $z_1 \neq y_j^t$, $j = 1,2,\ldots s$. We will illustrate this step of the procedure for the case $z_1 \neq x_i^k$, where $1 \leq k \leq b$; the other case would be treated in the same manner.

In order not to violate the condition that $(y_j^1,\ j = 1,2,\ldots s+1)$ be an SDR of $(A_i,\ i = 1,2,\ldots,s+1)$ we will interchange some elements of F_k and $(y_j^1,\ j = 1,\ldots,s+1)$ which have the same subscript, i.e. the elements, which belong to the same A_i. Define the sequences $\{\bar{x}_{\alpha_i}\}$, $\{\bar{y}_{\alpha_i}\}$, $i = 1,2,\ldots,\ell$, as follows:

$\bar{x}_{\alpha_1} = y_c^1 = z_1$, $\bar{y}_{\alpha_1} = x_{\alpha_1}^k$, where $\alpha_1 = c$

Suppose $\{\bar{x}_{\alpha_i}\}$, $\{\bar{y}_{\alpha_i}\}$, $i = 1,2,\ldots,p$, have already been constructed. If $\bar{y}_{\alpha_p} \neq y_j^1$, $j = 1,2,\ldots,s$, then we put $\ell = p$. If $\bar{y}_{\alpha_p} = y_d^1$ then $\alpha_{p+1} = d$ and

$$\bar{x}_{\alpha_{p+1}} = y_d^1,\ \bar{y}_{\alpha_{p+1}} = x^k_{\alpha_{p+1}}.$$

Clearly, after at most $\ell \leq s$ steps we must get $\bar{y}_{\alpha_\ell} \neq y_j^1$, $j = 1,2,\ldots,s$.

Now we interchange the elements of F_k and $(y_j^1,\ j = 1,2,\ldots,s)$ with subscripts $\alpha_1,\ldots,\alpha_\ell$ and get $\bar{F}_k = (\bar{x}_i^k,\ i = 1,2,\ldots,n)$, where $\bar{x}_i^k = x_i^k$, $i \notin \{\alpha_1\ldots,\alpha_\ell\}$, $\bar{x}_i^k = \bar{x}_i$, $i \in \{\alpha_1,\ldots,\alpha_\ell\}$. Similarly, we get $(\bar{y}_j^1,\ j = 1,2,\ldots,s+1)$.

Because $z_1 \neq x_i^k$, $i = 1,2,\ldots,n$, $\bar{F}_k$ is an SDR of $\mathcal{A}$ and $(\bar{y}_j^1,\ j = 1,2,\ldots,s+1)$ is an SDR of $(A_i,\ i = 1,2,\ldots,s+1)$.

Obviously, by the above described process of interchange, the condition of compatibility could not be violated. We can repeat the above procedure for elements $z_2,\ldots,z_{a-b}$ and construct the element y^t_{s+1}, $t = 2,\ldots,a-b$, such that $H_t = (y_j^t,\ j = 1,2,\ldots,s+1)$ is an SDR of $(A_i,\ i = 1,2,\ldots,s+1)$, $t = 1,2,\ldots,a-b$, H_t are mutually compatible, and at the same time compatible with F_k, $k = 1,2,\ldots,b$. The proof is complete.

Acknowledgement

This paper was written while the author was visiting the Department of Mathematics and Statistics, McMaster University. He would like to thank the department for its hospitality.

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RW/AR1/AR2/FNL/EP
July 20, 1987