SOME PERFECT ROOM SQUARES

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1. Introduction.

Let H be a finite simple graph. A $1-$*factor* (*perfect matching*) in H is a set of nonadjacent edges which are incident to every vertex of H. A $1-$*factorization* of H is a partitioning of the edge set of H into 1-factors. We will restrict our attention to the graph $H = K_{2n}$ so that each 1-factor contains n edges and each 1-factorization contains $2n-1$ 1-factors. Much research has been done on properties of 1-factorizations of K_{2n} and for an excellent survey the reader is referred to [9]. In this paper we are particularly interested in two properties of 1-factorizations: perfect and orthogonal.

Let F be a 1-factorization of K_{2n}. It is clear that the union of any two 1-factors in F is a 2-factor (the union of cycles). A 1-factorization is called *perfect* if the union of any two 1-factors in F is a Hamiltonian circuit. There has recently been renewed interest in finding perfect 1-factorizations of K_{2n} (see [13],[8]) but there are only two known infinite classes of these objects. If p is a prime then there is a perfect 1-factorization of K_{p+1} called GK_{p+1} and a perfect 1-factorization of K_{2p} called GA_{2p}. In this paper we will construct some perfect 1-factorizations of K_{p+1} which are not isomorphic to GK_{p+1}. For information on perfect 1-factorizations of K_{2n} the reader is referred to [9],[13],[1],[2].

Given 1-factorizations F and G of K_{2n} we say that F and G are *orthogonal* if any two edges which are in the same 1-factor in F are in different 1-factors in G. If F and G are orthogonal 1-factorizations of K_{2n} we can construct a $2n-1$ *by* $2n-1$ array indexed by the 1-factors in F and G where the pair {x,y} is placed in row i column j if {x,y} is an edge in the 1-factors f_i and g_j. This array is called a *Room square* of side 2n - 1. Thus in a Room square, each row contains the edges in a 1-factor and all the rows together comprise a 1-factorization, called the row factorization. The columns do likewise. Noting these facts, we define a *perfect Room Square* to be a Room square in which both the row and column factorizations are perfect. Therefore a perfect Room square is equivalent to a pair of orthogonal perfect 1-factorizations. For further information concerning Room squares and orthogonal 1-factorizations the reader is referred to [9],[4],[10],[7].

Continuing with these ideas, the existence of t mutually orthogonal perfect 1-factorizations of K_{2n} is equivalent to a perfect t-dimensional Room square of side 2n - 1, termed a perfect Room t-cube of side 2n - 1. We will be interested in finding large sets of mutually orthogonal perfect 1-factorizations of K_{2n}. Define P(n) to be the largest t such that there exists t mutually orthogonal perfect 1-factorizations of K_n. Since there are at most n - 2 mutually orthogonal 1-factorizations of K_n [7], we have that P(n) $\leq n-2$ for all even n $\geq$ 4. The existence of perfect 1-factorizations of K_{2p} and K_{p+1} when p is a prime implies that P(2p) $\geq$ 1 and P(p+1) $\geq$ 1. Recently, Alex Rosa [12] discovered constructions which imply that P(12) $\geq$ 2, P(18) $\geq$ 2 and P(20) $\geq$ 2. In this paper we will give further results on P(t) for other values of t $<$ 100.

In the next section we will show how starters can be used to construct mutually orthogonal perfect 1-factorizations. Section 3 introduces a new invariant of 1-factorizations which we will use to determine some isomorphism questions. Finally in Section 4 we discuss P(t) for some small values of t.

2. Constructions Using Starters.

A *starter* of order g in an additive abelian group G, $|G| = g$, is set of pairs $A = \{\{s_i,t_i\}, 1 \leq i \leq \frac{1}{2}(g-1)\}$ satisfying the properties:

(i) $\{s_i\} \cup \{t_i\} = G \setminus \{0\}$,

(ii) $\{\pm(s_i-t_i)\} = G \setminus \{0\}$.

The starter $P = \{\{x,-x\} \mid x \in G\}$ is termed the *patterned* starter. A starter $A = \{\{s_i,t_i\}\}$ is a *strong starter* if $s_i+t_i \neq s_j+t_j$ for all $i \neq j$, and if $s_i+t_i \neq 0$ for all i.

Let $A = \{\{s_i,t_i\}\}$ and $B = \{\{u_i,v_i\}\}$ be two starters in G. We may assume that $t_i-s_i = v_i-u_i$ for all $1 \leq i \leq \frac{1}{2}(g-1)$. A and B are *orthogonal starters* if $u_i-s_i \neq u_j-s_j$ for all $i \neq j$, and if $u_i \neq s_i$ for all i. It is easily shown that if A is a strong starter, then A, $-A = \{\{-s_i,-t_i\}\}$ and P are 3 pairwise orthogonal starters.

Given a starter $A = \{\{s_i,t_i\}, 1 \leq i \leq \frac{1}{2}(g-1)\}$ it is easy to construct a 1-factorization F_A of K_{g+1}. Let $A' = A \cup \{0,\infty\}$, then $F_A = \{A'+g \mid g \in G\}$ where $\infty+g=\infty$. In Figure 1, we give a starter A in the group Z_{11} and the resulting 1-factorization of K_{12}. We define a starter to be a *perfect starter* if the resulting 1-factorization is perfect. The following theorem (first proved by Horton [7] without reference to "perfect") relates orthogonal perfect starters and orthogonal perfect 1-factorizations.

THEOREM 2.1 *If there exist t pairwise orthogonal perfect starters of order n, then there exist t pairwise orthogonal perfect 1-factorizations of K_{n+1} and thus $P(n+1) \geq t$.*

In view of the above theorem, to find lower bounds for P(n) it is convenient to search for large sets of pairwise orthogonal perfect starters of order $n-1$. There are two classes of starters which will give examples of orthogonal perfect starters, these are the so called Mullin-Nemeth starters [11] and the two-quotient starters [5]. The Mullin-Nemeth starters have orders $p=2t+1$ and the two-quotient starters have orders $p=4t+1$ where for both p is a prime power and t is an odd integer.

Let $p=2t+1$ be a prime power with t odd. Let $G=GF(p)^*$ be the multiplicative group of the field $GF(p)$. Let R be the set of all quadratic residues in G and let N be the class of nonresidues. For each a $\in N$ the set $S_a=\{\{x,ax\} \mid x \in R\}$ is called a *Mullin – Nemeth* starter of order p. It is well known that if S_a and S_b are Mullin-Nemeth starters of order p and with a $\neq$ b, then S_a and S_b are orthogonal starters [11]. Also each Mullin-Nemeth starter is a strong starter and $S_{-1}=P$ the patterned starter. We note here that the known class of perfect 1-factorizations GK_{p+1} is generated from the starter S_{-1} when p is a prime.

A 1-factorization is *uniform* [9] (or *semi – regular* [1]) if $F_i \cup F_j = F_k \cup F_m$ for any four 1-factors $F_i, F_j, F_k, F_m, i \neq j, k \neq m$. In [1], Anderson proved that the Mullin-Nemeth starter S_a generates a uniform 1-factorizations provided a is primitive element in $GF(p)$. In fact, it is easy to show that S_a generates a uniform 1- factorization for any a which is a nonresidue in $GF(p)$. We state this in the following theorem (the proof is left for the interested reader).

THEOREM 2.2 *If a is a nonresidue in $GF(p)^*$, then the Mullin-Nemeth starter S_a generates a uniform 1-factorization of K_{p+1}.*

By using this theorem one can easily test whether a given Mullin-Nemeth starter is perfect. One need only consider the graph generated by $S'_a \cup S'_a+1$. If it is a hamiltonian circuit then S_a is perfect, if not then S_a is not perfect. The 1-factorizations given below in Figure 1 are generated by S_7 and S_8 in $GF(11)$. They are both perfect and are orthogonal to each other. Thus $P(12) \geq 2$.

FIGURE 1

Two orthogonal perfect 1-factorizations of K_{12}

Factorization generated by S_7	Factorization generated by S_8
{1,7} {3,10} {4,6} {5,2} {9,8} {0,∞}	{1,8} {3,2} {4,10} {5,7} {9,6} {0,∞}
{2,8} {4,0} {5,7} {6,3} {10,9} {1,∞}	{2,9} {4,3} {5,0} {6,8} {10,7} {1,∞}
{3,9} {5,1} {6,8} {7,4} {0,10} {2,∞}	{3,10} {5,4} {6,1} {7,9} {0,8} {2,∞}
{4,10} {6,2} {7,9} {8,5} {1,0} {3,∞}	{4,0} {6,5} {7,2} {8,10} {1,9} {3,∞}
{5,0} {7,3} {8,10} {9,6} {2,1} {4,∞}	{5,1} {7,6} {8,3} {9,0} {2,10} {4,∞}
{6,1} {8,4} {9,0} {10,7} {3,2} {5,∞}	{6,2} {8,7} {9,4} {10,1} {3,0} {5,∞}
{7,2} {9,5} {10,1} {0,8} {4,3} {6,∞}	{7,3} {9,8} {10,5} {0,2} {4,1} {6,∞}
{8,3} {10,6} {0,2} {1,9} {5,4} {7,∞}	{8,4} {10,9} {0,6} {1,3} {5,2} {7,∞}
{9,4} {0,7} {1,3} {2,10} {6,5} {8,∞}	{9,5} {0,10} {1,7} {2,4} {6,3} {8,∞}
{10,5} {1,8} {2,4} {3,0} {6,5} {9,∞}	{10,6} {1,0} {2,8} {3,5} {7,4} {9,∞}
{0,6} {2,9} {3,5} {4,1} {7,6} {10,∞}	{0,7} {2,1} {3,9} {4,6} {8,5} {10,∞}

Using these ideas we can test all the Mullin-Nemeth starters in $GF(p)$ to see which are perfect. Remembering that orthogonality of these starters is quaranteed, we get the following lower bounds for P(n), the number of pairwise orthogonal perfect 1-factorizations of K_n.

THEOREM 2.3. P(12) ≥ 3, P(20) ≥ 5, P(24) ≥ 9, P(44) ≥ 3, P(48) ≥ 5, P(60) ≥ 5, P(68) ≥ 7, P(72) ≥ 7, P(80) ≥ 9, and P(84) ≥ 17.

Proof. In the following table we list the prime p and the Mullin-Nemeth starters which are perfect and thus generate perfect 1-factorizations of K_{p+1}.

Prime	Orthogonal perfect starters
11	S_7, S_8, S_{10}
19	$S_2, S_3, S_{10}, S_{13}, S_{18}$
23	$S_5, S_7, S_{10}, S_{11}, S_{14}, S_{15}, S_{20}, S_{21}, S_{22}$
43	S_8, S_{27}, S_{42}
47	$S_{10}, S_{31}, S_{33}, S_{44}, S_{45}$
59	$S_{13}, S_{39}, S_{50}, S_{56}, S_{58}$
67	$S_5, S_{11}, S_{12}, S_{27}, S_{28}, S_{61}, S_{66}$
71	$S_7, S_{14}, S_{47}, S_{61}, S_{66}, S_{68}, S_{70}$
79	$S_3, S_{15}, S_{30}, S_{53}, S_{58}, S_{63}, S_{74}, S_{77}, S_{78}$
83	$S_8, S_{13}, S_{18}, S_{32}, S_{39}, S_{43}, S_{46}, S_{47}, S_{52}, S_{53}, S_{56}, S_{58}, S_{60}, S_{66}, S_{73}, S_{74}, S_{82}$

Another class of starters called two-quotient starters was introduced by Dinitz in [5]. Let p = 4t+1 be a prime power with t an odd integer. Let $C_0 \subset GF(q)^* = G$ be the unique subgroup of order t (index 4) and let g be a primitive element in G. Let C_0, C_1, C_2, C_3 be the multiplicative cosets of C_0, where $C_i = g^i C_0$. Define the set $S(a_0,a_1)$ $=\{\{x,a_0x,\},\{y,a_1y\} \mid x \in C_0^{a_0}, y \in C_1^{a_1}\}$ where $C_i^{a_i} = (1/(a_i - 1))C_i$ for $i = 0,1$. Certain conditions (see [5]) on a_0 and a_1 allow $S(a_0,a_1)$ to be a starter. In this case $S(a_0,a_1)$ is termed a *two − quotient starter*. The following conditions assuring orthogonality are proven in [5].

LEMMA 2.4 *Let* $S(a_0,a_1)$ and $S(b_0,b_1)$ *be two-quotient starters. Then* $S(a_0,a_1)$ and $S(b_0,b_1)$ *are orthogonal starters if and only if*

$$\frac{b_0-a_0}{b_1-a_1}\cdot\frac{a_1-1}{a_0-1}\cdot\frac{b_1-1}{b_0-1} \notin C_1, \quad a_0 \neq b_0 \text{ and } a_1 \neq b_1.$$

As was the case with Mullin-Nemeth starters we can now generate all of the two-quotient starters on the computer and test to find which ones are perfect. Then with the list of perfect two-quotient starters we can check them pairwise for orthogonality and search for a maximal set of these. It can be checked that these two-quotient starters do not necessarily generate uniform 1-factorizations, thus the test for perfect is a bit longer than it was for the Mullin-Nemeth starters. We have the following theorem.

THEOREM 2.5 P(30) ≥ 5, P(38) ≥ 7, P(54) ≥ 8.

Proof. In the following table we list the prime p and the a maximal set of two-quotient starters which are perfect and pairwise orthogonal. Thus they generate perfect 1-factorizations of K_{p+1} which are pairwise orthogonal.

Prime	Orthogonal perfect starters
29	S(8,27), S(11,14), S(14,8), S(27,11), S(28,28).
37	S(2,18), S(6,22), S(17,23), S(19,35), S(24,29), S(31,32), S(36,36).
53	S(3,51), S(6,52), S(8,30), S(11,29), S(18,26), S(29,11), S(33,20), S(52,6).

3. Invariants of 1-factorizations.

The obvious question concerning all of the perfect 1-factorizations generated thus far is whether or not they are isomorphic. The usual tests for isomorphism of 1-factorizations utilize the cycle structure of the unions on the 1-factors (see [9]). Obviously, since all of these 1-factorizations are perfect this method will not be useful to us. We propose a new invariant called the train of the 1-factorization.

Trains were first described for Steiner triple systems by White [16]. Recently, Colbourn *et al* [3] and Stinson [14] have also discussed trains as invariants of triple systems. Our definition of trains in 1-factorizations is very similar to that of trains in triple systems. Let $F=\{f_1,f_2,\ldots,f_n\}$ be a 1-factorization of K_{n+1}. For any two points x,y, define which(x,y) $= f_i$ if {x,y} is an edge in the 1-factor f_i. Also, for any point x and any 1-factor f_i, define other(x,f_i) $=$ y if {x,y} is an edge in the 1-factor f_i. A *train* is a directed graph T whose vertices are the $\binom{n+1}{2}\times n$ subsets of the form {x,y,f} where x,y are points in K_{n+1} and f is a 1-factor in F. T is regular of outdegree 1; the edge leaving {x,y,f} is directed to {other(x,f), other(y,f), which(x,y)}. It is obvious that if two 1-factorizations are isomorphic, then so are their associated trains.

As was done by Stinson, we shorten this invariant by using only the indegree sequence of the train. That is, to each 1-factorization F we will associate a sequence $t_0,t_1,t_2,\ldots$ where t_i equals the number of vertices in the train of F with indegree i. We now can use this invariant to determine whether some of the perfect 1-factorizations discussed earlier are isomorphic.

In K_{12} we have found 3 orthogonal perfect 1-factorizations, these are generated by S_7, $S_8 = -S_7$ and S_{10}. S_7 and S_8 have indegree sequence 330, 176, 165, 0, 55 while S_{10} has sequence 110, 506, 110. Thus we see that S_{10} is not isomorphic to S_7. Displayed below we give the 1-factorizations and the indegree sequences of their associated trains for the 1-factorizations of K_{20} and K_{24} given in Theorem 2.3. Again we can see that none of these 1-factorizations are isomorphic.

K_{20}

1-factorizations	indegree sequence of train
S_2, $S_{10} = -S_2$	648, 2242, 684
S_3, $S_{13} = -S_3$	1026, 1900, 513, 171
S_{18}	342, 2926, 342

K_{24}

1-factorizations	indegree sequence of train
S_5, $S_{14} = -S_5$	1265, 3818, 1265
S_7, $S_{10} = -S_7$	2277, 2806, 506, 506, 253
S_{11}, $S_{21} = -S_{11}$	2530, 2300, 1012, 253, 253
S_{15}, $S_{20} = -S_{15}$	1771, 3312, 759, 506
S_{22}	506, 5336, 506

It is our hope that the indegree lists of trains will provide a useful tool for solving isomorphism questions concerning 1-factorizations of K_n.

4. Low orders.

We would just like to take a quick look at the small orders 4, 6, 8 and 10. The only 1-factorization of K_4 is GK_4 and it is obviously perfect. It has no orthogonal mates, thus P(4) = 1. GK_6 is perfect in K_6 and since there is no pair of orthogonal 1-factorizations of K_6 we have that P(6) = 1.

For K_8 it was shown by Wallis [15] that there is a unique perfect 1-factorization up to isomorphism (again GK_8). He also showed that there are no pair of orthogonal perfect 1-factorizations of K_8 and so P(8) = 1. In 1973 Gelling [6] listed all the 1-factorizations of K_{10}. He found 396 nonisomorphic ones and only one of these was perfect. In the Gelling listing the perfect one is #396, but it is also clearly GA_{10}. We have run a computer test and have shown that there are not two copies of GA_{10} which are orthogonal. Thus we have that P(10) = 1.

From these results we see that the example given in Section 2 of three pairwise orthogonal perfect 1-factorizations of K_{12} is the smallest possible case where P(n) > 1.

5. Summary and conclusion.

In this paper we have used starters to construct pairwise orthogonal perfect 1-factorizations of K_n. We have also introduced a new invariant of 1-factorizations called the indegree list of a train which can discern nonisomorphic perfect 1-factorizations.

We believe that there may be a new infinite class of perfect 1-factorizations which arise from either the Mullin-Nemeth starters or the 2-quotient starters. It is our hope that one can find some set of conditions on a and p which will insure that S_a generates a perfect 1-factorization in K_{p+1}.

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