Skew Transversals in Frames

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ABSTRACT

In this paper, we investigate the existence of skew frames with sets of skew transversals. We consider skew frames of type 1^n and skew frames of type $(2^m)^q$ with sets of skew transversals. These frames are equivalent to three-dimensional frames which have complementary 2-dimensional projections with special properties.

1. Introduction

Let V be a set of v elements. Let $G_1,G_2,...,G_m$ be a partition of V into m sets. A $\{G_1,G_2,...,G_m\}$-frame F with block size k, index λ and latinicity μ is a square array of side v which satisfies the properties listed below. We index the rows and columns of F by the elements of V.

(1) Each cell is either empty or contains a k-subset of V.

(2) Let F_i be the subsquare of F indexed by the elements of G_i. F_i is empty for $i=1,2,...,m$. (The subsquares F_i, $1 \leq i \leq m$, are often referred to as the holes of F.)

(3) Let $j \in G_i$. Row j of F contains each element of $V - G_i$ μ times and column j of F contains each element of $V - G_i$ μ times.

(4) The collection of blocks obtained from the nonempty cells of F is a $GDD(v;k;G_1,G_2,...,G_m;0,\lambda)$. (See [13] for the notation for group divisible designs (GDDs).)

If there is a $\{G_1,G_2,...,G_m\}$-frame H with block size k, index λ and latinicity μ such that

(1) $H_i = F_i$ for $i=1,2,...,m$ and

(2) H can be written in the empty cells of $F - \bigcup_{i=1}^{m} F_i$,

then H is called a complement of F and denoted by $\overline{F}$. If a complement of F exists, we call F a complementary $\{G_1,G_2,...,G_m\}$-frame. A complementary $\{G_1,G_2,...,G_m\}$-frame F is said to be skew if at most one of the cells

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(i,j) and (j,i) $(i \neq j)$ is nonempty.

We will use the following notation for frames. If $|G_i| = h$ for $i=1,2,\ldots,m$, we call F a $(\mu,\lambda;k,m,h)$-frame. The type of a $\{G_1,G_2,\ldots,G_m\}$-frame is the multiset $\{|G_1|,|G_2|,\ldots,|G_m|\}$. We will say that a frame has type $t_1^{u_1}t_2^{u_2}\cdots t_k^{u_k}$ if there are u_i G_j's of cardinality t_i, $1 \leq i \leq k$. For notational convenience, if $\mu = \lambda = 1$ and $k = 2$, we will denote a frame simply by its type.

Let $V = \bigcup_{i=1}^{m} V_i$ and let $W = \bigcup_{i=1}^{m} W_i$. Let F be a skew $\{V_1,V_2,\ldots,V_m\}$-frame of type h^m defined on V. Let $\overline{F}$ be the complement of F defined on W. $\overline{F}$ is a skew $\{W_1,W_2,\ldots,W_m\}$-frame of type h^m. Let A be the array of pairs formed by the superposition of F and $\overline{F}$, $A = F \circ \overline{F}$. We say that the skew frame F has property T if there exists a set of hm transversals $\{T_1^i,T_2^i,\ldots,T_h^i \mid i=1,2,\ldots,m\}$ of the array A such that

(i) $T_k^i \cap T_\ell^j = \emptyset$ for $1 \leq k,\ \ell \leq h,\ 1 \leq i,j \leq m$ (excluding the case $k = \ell$ and $i = j$).

(ii) Every element of $(V-V_i) \cup (W-W_i)$ occurs precisely once in T_j^i for $j=1,2,\ldots,h$ $(i=1,2,\ldots,m)$.

(iii) T_j^i contains h empty cells from the hole F_i.

T is called a set of skew transversals if each transversal T_j^i has the property that if cell (x,y) is in the transversal then so is cell (y,x). A set T of skew transversals for a frame F will be listed by the pairs of T in F, $T_F = \{T_j^i \cap F \mid j=1,2,\ldots,h \text{ and } i=1,2,\ldots,m\}$. A skew 3^5 frame from [12] is displayed in Figure 1. This frame has a set of 15 skew transversals which are listed in Table 1.

An n-dimensional $\{G_1,G_2,\ldots,G_m\}$-frame of type h^m is an n-dimensional cube of side hm such that each 2-dimensional projection is a $\{G_1,G_2,\ldots,G_m\}$-frame of type h^m ([4]). Let $C_i(F)$ denote the set of pairs which occur in column i of a frame F and let $R_i(F)$ denote the set of pairs which occur in row i of a frame F. The 2-dimensional projections F_1, F_2 and F_3 of a 3-dimensional $\{G_1,G_2,\ldots,G_m\}$-frame of type h^m are said to be in standard form if (i) $R_i(F_1) = R_i(F_2)$, (ii) $C_i(F_2) = C_i(F_3)$, and (iii) $R_i(F_3) = C_i(F_1)$ for all i. The following lemma provides the relationship between skew frames with skew transversals and 3-dimensional frames.

Lemma 1.1 *There exists a 3-dimensional $\{G_1,G_2,\ldots,G_m\}$-frame of type h^m with its 2-dimensional projections F_1, F_2 and F_3 in standard form where*

(i) *F_1 is skew and*

(ii) *F_2 and F_3 are complementary* *if and only if there exists a skew $\{G_1,G_2,...,G_m\}$-frame of type h^m with a set of hm skew transversals.*

Proof. We first use the columns of F_2 where $C_i(F_2) = C_i(F_3)$ for all i to find a set of hm skew transversals of the skew frame F_1. Consider the pairs in $C_i(F_2)$. These pairs will occur in rows $i_1,i_2,...,i_n$ of F_2 and therefore in rows $i_1,i_2,...,i_n$ of F_1 where $n = \frac{h(m-1)}{2}$. These pairs will also occur in rows $j_1,j_2,...,j_n$ of F_3 and therefore in columns $j_1,j_2,...,j_n$ of F_1. Thus, the pairs in $C_i(F_2)$ will occur in cells (i_k,j_k) of F_1 for $k = 1,2,...,n$. Since F_2 and F_3 are complementary, $(\bigcup_{k=1}^{n} i_k) \cap (\bigcup_{k=1}^{n} j_k) = \emptyset$ and $\{(i_k,j_k),(j_k,i_k)\,|k = 1,2,...,n\}$ will be skew transversal of F_1. Thus, we can construct a set of hm skew transversals for F_1.

Conversely, suppose F_1 is a skew frame with a set T of hm skew transversals, $T = \{T_1,T_2,...,T_{hm}\}$. We construct two frames F_2 and F_3 as follows.

(1) If $R_i(F_1) \cap T_j \neq \emptyset$, place $R_i(F_1) \cap T_j$ in cell (i,j) of F_2.

(2) If $C_i(F_1) \cap T_j \neq \emptyset$, place $C_i(F_1) \cap T_j$ in cell (i,j) of F_3.

It is clear from this construction that (i) $R_i(F_1) = R_i(F_2)$, (ii) $C_i(F_2) = C_i(F_3)$ and (iii) $C_i(F_1) = R_i(F_3)$ for all i. Since T_ℓ is a skew transversal, $C_\ell(F_2)$ will have pairs in cells (i_1,ℓ), (i_2,ℓ),...,(i_n,ℓ) of F_2 and $C_\ell(F_3)$ will have pairs in cells (j_1,ℓ), (j_2,ℓ),...,(j_n,ℓ) of F_3 where $(\bigcup_{k=1}^{n} i_k) \cap (\bigcup_{k=1}^{n} j_k) = \emptyset$. Thus, F_2 and F_3 will be complementary frames. This verifies that F_1, F_2 and F_3 are the three 2-dimensional projections of a 3-dimensional frame with the required properties. □

In this paper we investigate the existence of skew frames with sets of skew transversals. In the next section, we describe starter-adder constructions for frames and discuss existence results for skew 1^n frames. We use these results in section 3 to produce infinite classes of skew $(2^n)q$ frames with skew transversals.

This study was motivated by two recent applications of skew frames with skew transversals. In [7] we describe frame constructions for balanced tournament designs BTDs. These constructions are useful for finding balanced tournament designs with special properties. In particular we are interested in balanced tournament designs with almost orthogonal resolutions which can be used to construct doubly resolvable balanced incomplete block designs [9]. Our main construction for these BTDs requires the existence of skew frames with sets of skew transversals. A similar construction can be used to find odd BTDs with orthogonal resolutions [8].

Table 1

Skew-transversals for the skew 3^6 frame in Figure 1.

10,40	11,41	12,42	31,21	32,22	30,20
31,42	32,40	30,41	22,10	20,11	21,12
20,42	21,40	22,41	11,32	12,30	10,31
20,00	21,01	22,02	41,31	42,32	40,30
41,02	42,00	40,01	32,20	30,21	31,22
30,02	31,00	32,01	21,42	22,40	20,41
30,10	31,11	32,12	01,41	02,42	00,40
01,12	02,10	00,11	42,30	40,31	41,32
40,12	41,10	42,11	31,02	32,00	30,01
40,20	41,21	42,22	11,01	12,02	10,00
11,22	12,20	10,21	02,40	00,41	01,42
00,22	01,20	02,21	41,12	42,10	40,11
00,30	01,31	02,32	21,11	22,12	20,10
21,32	22,30	20,31	12,00	10,01	11,02
10,32	11,30	12,31	01,22	02,20	00,21

Figure 1

A skew frame of type 3^5 with a set of 15 skew transversals

						41 31				40 12	21 42	30 10	32 20	11 22
							42 32		22 40		41 10	12 20	31 11	30 21
								40 30	42 11	20 41		31 22	10 21	32 12
40 20	42 30	21 32							01 41				00 22	31 02
22 30	41 21	40 31								02 42		32 00		01 20
41 32	20 31	42 22									00 40	02 21	30 01	
	10 32	41 12	00 30	02 40	31 42							11 01		
42 10		11 30	32 40	01 31	00 41								12 02	
12 31	40 11		01 42	30 41	02 32									10 00
21 11				20 42	01 22	10 40	12 00	41 02						
	22 12		02 20		21 40	42 00	11 41	10 01						
		20 10	22 41	00 21		11 02	40 01	12 42						
			31 21				30 02	11 32	20 00	22 10	01 12			
				32 22		12 30		31 00	02 10	21 01	20 11			
					30 20	32 01	10 31		21 12	00 11	22 02			

2. Preliminary definitions and results

We will use starter-adder constructions to find skew frames with sets of skew transversals.

Let G be an additive abelian group of order g and let H be a subgroup of G of order h where $g-h$ is even. An $(h,\frac{g}{h})$-frame starter in $G-H$ is a set of pairs $S = \{\{s_i,t_i\} \mid i=1,2,...,\frac{g-h}{2}\}$ satisfying

(1) $\{s_i,t_i \mid i=1,2,...,\frac{g-h}{2}\} = G-H$ and

(2) $\{\pm(s_i-t_i) \mid i=1,2,...,\frac{g-h}{2}\} = G-H$.

An adder A for S is a set of $\frac{g-h}{2}$ distinct elements of $G-H$, $A = \{a_i \mid i=1,2,...,\frac{g-h}{2}\}$, such that $\{s_i+a_i,t_i+a_i \mid i=1,2,...,\frac{g-h}{2}\} = G-H$. An adder A is skew provided $a_i \neq -a_j$ for any i,j.

Lemma 2.1 *[12]. If there exists an $(h,\frac{g}{h})$-frame starter S in $G-H$ and an adder A for S, then there is an $h^{\frac{g}{h}}$ frame. Furthermore, if A is skew, then there is a skew $h^{\frac{g}{h}}$-frame.*

A frame starter $S = \{\{s_i,t_i\} \mid i=1,2,...,\frac{g-h}{2}\}$ in $G-H$ is said to be strong provided $s_i+t_i \notin H$ for $i=1,2,...,\frac{g-h}{2}$ and $s_i+t_i = s_j+t_j$ implies $i = j$. A stronger starter S is called a skew-strong starter if $s_i+t_i \neq -(s_j+t_j)$ for any i,j.

Lemma 2.2 *[12]. If there exists a strong frame starter S in $G-H$, then there is an $h^{\frac{g}{h}}$ frame. If S is a skew-strong starter, then there is a skew $h^{\frac{g}{h}}$ frame.*

Let $S = \{\{s_i,t_i\} \mid i=1,2,...,\frac{g-h}{2}\}$ and $S_1 = \{\{u_i,v_i\} \mid i=1,2,...,\frac{g-h}{2}\}$ be two frame starters in $G-H$. S and S_1 are called a pair of orthogonal frame starters if there is an adder A for S so that $S + A = S_1$. A set of frame starters $Q = \{S_1,S_2,...,S_t\}$ is called a set of t orthogonal frame starters if each pair of starters is orthogonal.

A patterned frame starter is a starter $P = \{\{s_i,t_i\} \mid i=1,2,...,\frac{g-h}{2}\}$ where $s_i = -t_i$ for all i. A patterned frame starter can exist in $G-H$ only if $|G|$ is odd. If S is a strong-frame starter in $G-H$ and $|G|$ is odd, then $\{S,-S,P\}$ is a set of 3 orthogonal frame starters and can be used to construct a three-dimensional $h^{\frac{g}{h}}$ frame [4]. An immediate corollary of this result is the following.

Lemma 2.3. *If there is a skew-strong frame starter in $G-H$ and $|G|$ is odd, then there is a frame of type $h^{\frac{g}{h}}$ with a set of g skew transversals.*

We note that this construction actually produces a three-dimensional $h^{\frac{g}{h}}$ frame in which every two-dimensional projection is a skew $h^{\frac{g}{h}}$ frame.

In the remainder of this section, we consider skew frames of type 1^n where $n \equiv 1 (mod\, 2)$, $n \geq 7$. These frames are equivalent to skew Room squares of side n. It is known that skew Room squares of side n exist if and only if $n \equiv 1(mod\, 2)$ and $n \geq 7$ [12]. We can use skew-strong starters for skew Room squares together with Lemma 2.3 to prove the following.

Theorem 2.4. *Let n be an odd prime power, $n \geq 7$, $n \neq 9$.*

(i) There is a skew 1^n frame with a set of n skew transversals.

(ii) There is a skew 1^{5n} frame with a set of $5n$ skew transversals.

Proof. If n is an odd prime power, $n \geq 7$, $n \neq 9$ and $n = 2^k t + 1$ (t odd), skew strong starters for 1^n frames are constructed in [9]. If n is a prime power $n > 9$ and $n = 2^{2^n} + 1$, skew strong starters for 1^n frames can be found in [1]. The quintupling construction due to J. Horton [6] can be used to produce the skew strong starters for (ii). □

The direct product construction for skew Room squares can also be used for skew 1^n frames with sets of skew transversals [12].

Theorem 2.5. *If there exists a skew 1^n frame with a set of n skew transversals and a skew 1^m frame with a set of m skew transversals, then there is a skew 1^{nm} frame with a set of nm skew transversals.*

Theorems 2.4 and 2.5 can be used to construct skew 1^n frames with skew transversals if $n \neq 3m$ and $(m,3) = 1$. In order to complete the spectrum for skew 1^n frames with skew transversals, we need the existence of 1^u frames with sets of skew transversals where $u = 3p$ (p a prime). Unfortunately, the constructions used for Theorem 2.4 cannot be used in this case; skew-strong starters do not exist for 1^u frames when $u = 3p$ [2, 14]. The spectrum of three-dimensional 1^n frames (Room cubes) was determined by using PBD constructions [5]. The following nonexistence result for $n = 9$ rules out using the same constructions.

Lemma 2.6. *There does not exist a skew 1^9 frame with a set of 9 skew transversals.*

Proof. There are 267 three-dimensional frames of type 1^9. None of these frames has a skew two-dimensional projection, [3]. □

Theorem 2.4 and the direct product (Theorem 2.5) are used to prove the following.

Theorem 2.8. *Let $n \equiv 1(mod\,2)$, $n \geq 7$ and $n \neq 3m$ where $m \geq 5$ and $(m,3) = 1$. There exists a skew frame of type 1^n with a set of n skew transversals.*

It should be noted that using product constructions and some small designs of side n where $n \equiv 0(mod\,3)$ we can construct infinitely many of these designs whose side is divisible by 3.

3. Skew $(2^n)^q$ frames with skew transversals

In this section, we use skew-strong frame starters in $(GF(q)\times Z_2^n) - (\{0\}\times Z_2^n)$ where $q \equiv 1(mod\,4)$ is a prime power to construct skew $(2^n)^q$ frames with sets of skew transversals.

To describe the skew-strong frame starters, we need the idea of a doubling-scheme in the additive group Z_2^n [4]. This scheme uses the canonical identification between elements in Z_2^n and the non-negative integers less than 2^n written in base 2. All of the arithmetic is still in Z_2^n. A doubling-scheme in Z_2^n, $\mathbf{D} = (C,D)$, consists of two lists C and D each containing 2^{n+1} elements in Z_2^n. One list $C = (c_i \mid 0 \leq i \leq 2^{n+1}-1)$ is defined by

$$c_{2i} = i \quad \text{(written base 2)} \quad 0 \leq i \leq 2^{n-1}-1$$

$$c_{2i+1} = c_{2i} \quad 0 \leq i \leq 2^{n-1}-1$$

$$c_i = i-2^n \quad \text{(written base 2)} \quad 2^n \leq i \leq 2^{n+1}-1.$$

The other list $D = (d_i \mid 0 \leq i \leq 2^{n+1}-1)$ is defined by

$$d_i = i \quad \text{(written base 2)} \quad 0 \leq i \leq 2^n-1$$

$$d_{2i} = i \quad \text{(written base 2)} \quad 2^{n-1} \leq i \leq 2^n-1$$

$$d_{2i+1} = d_{2i} \quad 2^{n-1} \leq i \leq 2^n-1.$$

As an example of this definition, we list a doubling scheme in Z_2^2.

$$C = (00,00,01,01,00,01,10,11)$$
$$D = (00,01,10,11,10,10,11,11)$$

The following property of doubling-schemes can be proved by induction [4].

Lemma 3.1 [4]. *If* $\mathbf{D} = (C,D)$ *is a doubling scheme in* Z_2^n, *then* $\{d_i - c_i \mid 0 \le i \le 2^n - 1\} = \{d_i - c_i \mid 2^n \le i \le 2^{n+1} - 1\} = Z_2^n$.

This implies $d_i - c_i \ne d_j - c_j$ if $0 \le i,j \le 2^n - 1$ or if $2^n \le i,j \le 2^{n+1} - 1$. We also note that $d_i - c_i = d_i + c_i$ since the group is Z_2^n. Doubling-schemes have another important property which we will need.

Lemma 3.2. *If* $\mathbf{D} = (C,D)$ *is a doubling-scheme in* Z_2^n, *then*

$$c_{2i} = 2^n - 1 - d_{2^{n+1} - 2i - 1} \text{ (written base 2) } 0 \le i \le 2^n - 1$$

and

$$d_{2i} = 2^n - 1 - c_{2^{n+1} - 2i - 1} \text{ (written base 2) } 0 \le i \le 2^n - 1.$$

Proof. This can be verified directly from the definitions of C and D listed above. □

We note that this tells us that

$$c_{2i} - d_{2i} = -d_{2^{n+1} - 2i - 1} + c_{2^{n+1} - 2i - 1} = c_{2^{n+1} - 2i - 1} - d_{2^{n+1} - 2i - 1}$$

for $0 \le i \le 2^n - 1$.

Theorem 3.3. *If* $q \equiv 1 (mod\, 4)$ *is a prime power, then there is a skew frame of type* $(2^n)^q$ *which has a set of* $2^n q$ *skew transversals.*

Proof. A skew frame of type $(2^n)^q$ is constructed in [4]. We show that this frame has a set of $2^n q$ skew transversals.

Let w be a primitive element in $GF(q)$ and let $q = 4t+1$. Define $Q = \{w^{2i} \mid 0 \le i \le t-1\}$. Let $\mathbf{D} = (C,D)$ be a doubling scheme in Z_2^n. S is a skew-strong starter in $(GF(q) \times Z_2^n) - (\{0\} \times Z_2^n)$ for a skew $(2^n)^q$ frame F.

$$S = \{\{(x,c_i),(wx,d_i)\},\ 0 \le i \le 2^n - 1,\ i \equiv 0 (mod\, 2)$$
$$\{(-x,c_i),(-wx,d_i)\},\ 0 \le i \le 2^n - 1,\ i \equiv 1 (mod\, 2)$$

$$\{(-wx,c_i),(-w^2x,d_i)\},\ 2^n \leq i \leq 2^{n+1}-1,\ i \equiv 0(mod\ 2)$$
$$\{(wx,c_i),(w^2x,d_i)\},\ 2^n \leq i \leq 2^{n+1}-1,\ i \equiv 1(mod\ 2)\} \mid x \in Q\}$$

The corresponding adder for S is A where

$$A = \{(-x(w+1),-(c_i+d_i)),\ 0 \leq i \leq 2^n-1,\ i \equiv 0(mod\ 2)$$
$$(x(w+1),-(c_i+d_i)),\ 0 \leq i \leq 2^n-1,\ i \equiv 1(mod\ 2)$$
$$(wx(w+1),-(c_i+d_i)),\ 2^n \leq i \leq 2^{n+1}-1,\ i \equiv 0(mod\ 2)$$
$$(-wx(w+1),-(c_i+d_i)),\ 2^n \leq i \leq 2^{n+1}-1,\ i \equiv 1(mod\ 2) \mid x \in Q\}$$

We construct $2^n q$ skew transversals as follows. Let

$$S_i = \{\{(x,c_i),(wx,d_i)\},\{(wx,c_j),(w^2x,d_j)\} \mid |c_i-d_i| = |c_j-d_j| \text{ and } x \in Q\}$$

for $i \equiv 0(mod\ 2)$, $0 \leq i \leq 2^n-1$.
Let

$$S_i = \{\{(-x,c_i),(-wx,d_i)\},\{(-wx,c_j),(-w^2x,d_j)\} \mid |c_i-d_i| = |c_j-d_j| \text{ and } x \in Q\}$$

for $i \equiv 1(mod\ 2)$, $0 \leq i \leq 2^n-1$. Then $S = \bigcup_{i=0}^{2^n-1} S_i$. (We have used Lemma 3.2 to construct this partitioning of S.) We define

$$A_i = \{(-\frac{x}{2}(w+1),0),(-\frac{wx}{2}(w+1),0) \mid x \in Q\}$$

for $i \equiv 0(mod\ 2)$, $0 \leq i \leq 2^n-1$ and

$$A_i = \{(\frac{x}{2}(w+1),0),(\frac{wx}{2}(w+1),0) \mid x \in Q\}$$

for $i \equiv 1(mod\ 2)$, $0 \leq i \leq 2^n-1$.
Let $A_{ij} = A_i + (0,j)$ where j is written base 2, $0 \leq j \leq 2^n-1$. Then $\bigcup_{j=0}^{2^n-1} (S_i+A_{ij})$ generates a set of 2^n skew transversals of F.

Note that $S_i + A_{ij}$ contains as a first component of each ordered pair the patterned starter in $GF(q)$. ($S_i + A_{i0} = \{\{(\frac{x}{2}(1-w),c_i), (-\frac{x}{2}(1-w),d_i)\}, \{(\frac{wx}{2}(1-w),c_j), (-\frac{wx}{2}(1-w),d_j)\} \mid |c_i-d_i| = |c_j-d_j|$ and $x \in Q\}$ for $i \equiv 0(mod\ 2)$ and $0 \leq i \leq 2^n-1$). Thus $\bigcup_{j=0}^{2^n-1} (S_i+A_{ij})$ contains every element in $(GF(q)\times Z_2^n) - (\{0\}\times Z_2^n)$ precisely once.

Let $A(S_i)$ be the set of adder elements of A which correspond to the starter elements in S_i. Then $A(S_i) - A_i = \{(-\frac{x}{2}(w+1),-(c_i+d_i)), (-\frac{wx}{2}(1+w),-(c_i+d_i)) \mid x \in Q\}$ for $i \equiv 0 (mod\, 2)$, $0 \le i \le 2^n-1$ and $A(S_i) - A_i = \{(\frac{x}{2}(w+1),-(c_i+d_i)), (\frac{wx}{2}(1+w),-(c_i+d_i)) \mid x \in Q\}$ for $i \equiv 1 (mod\, 2)$, $0 \le i \le 2^n-1$. Since the elements of $A(S_i) - A_i$ are distinct and $(A(S_i)-A_i) \cap (A(S_j)-A_j) = \emptyset$ for $0 \le i,j \le 2^n-1$ $(i \ne j)$, $\bigcup_{j=0}^{2^n-1} (S_i+A_{ij})$ is a transversal of F. It is skew since the frame F is skew.

This verifies that $\{\bigcup_{j=0}^{2^n-1} (S_i+A_{ij}) \mid i=0,1,\ldots,2^n-1\}$ is a set of 2^n skew transversals. Since each of these transversals can be used to generate a set of q skew transversals, F has a set of $2^n q$ skew transversals. □

As an example of this construction, we list in Table 2 the starters and adders for a skew 2^9 frame with a set of 18 skew transversals. The frame is displayed in Figure 2.

Table 2

Starters and adders for a skew 2^9 frame with 18 skew transversals

S	10,0		01,0	20,0		02,1	02,0		21,1	01,1		12,1	
A		22,0			11,1			10,1				20,0	
	12,0		22,0	21,0		11,1	11,0		10,1		22,1		20,1
		02,0			01,1			12,1				21,0	
S_0	10,0		01,0	01,1		12,1	12,0		22,0		22,1		20,1
A_{00}		11,0			10,0			01,0				12,0	
A_{11}		11,1			10,1			01,1				12,1	
S_1	20,0		02,1	02,0		21,1	21,0		11,0		11,0		10,1
A_{10}		22,0			20,0			02,0				21,0	
A_{11}		22,1			20,1			02,1				21,1	

Figure 2

A skew 2^9 frame with a set of 18 skew transversals

		01,1 12,1			02,0 21,1	12,0 02,0			21,0 11,1	10,0 01,0			11,0 10,1	22,1 20,1			20,0 02,1
			01,0 12,0	02,1 21,0			12,1 22,1	21,1 11,0			10,1 01,1	11,1 10,0			22,0 20,0	20,1 02,0	
	12,0 01,1			11,1 22,1		21,0 20,1			00,0 12,1	22,0 02,0		20,0 11,0			01,0 21,1	02,1 00,1	
12,1 01,0					11,0 22,0		21,1 20,0	00,1 12,0			22,1 02,1		20,1 11,1	01,1 21,0			02,0 00,0
21,1 02,1			22,0 11,1			00,0 21,0		12,1 10,1			01,0 00,1	02,0 12,0			10,0 22,1		11,0 01,1
	21,0 02,0	22,1 11,0					00,1 21,1		12,0 10,0	01,1 00,0			02,1 12,1	10,1 22,0		11,1 01,0	
	22,0 12,1	20,1 21,1			21,0 00,1			10,0 20,0		12,1 10,1			00,0 22,1	11,0 02,0			12,0 11,1
22,1 12,0			20,0 21,0	21,1 00,0					10,1 20,1		02,0 10,0	00,1 22,0			11,1 02,1	12,1 11,0	
11,0 21,0		12,0 00,0		10,0 12,1		20,0 10,1				21,1 22,1		22,0 01,1		00,1 11,1			01,0 20,1
	11,1 21,1		12,1 00,1		10,1 12,0		20,1 10,0				21,0 22,0		22,1 01,0		00,0 11,0	01,1 20,0	
	01,0 10,1		02,0 22,1	00,1 01,1			10,0 02,1		22,0 21,1			12,1 20,1		20,0 00,0		21,0 12,0	
01,1 10,0		02,1 22,0			00,0 01,0	10,1 02,0		22,1 21,0					12,0 20,0		20,1 00,1		21,1 12,1
10,1 11,1			11,0 20,1		12,0 02,1	22,1 00,1		01,0 22,0			20,0 12,1				02,0 01,1	00,0 10,0	
	10,0 11,0	11,1 20,0		12,1 02,0			22,0 00,0		01,1 22,1	20,1 12,0				02,1 01,0			00,1 10,1
	20,0 22,1	21,0 01,0		22,0 10,0			02,0 11,1		11,0 00,1		00,0 20,1	01,1 02,1				10,1 21,1	
20,1 22,0			21,1 01,0		22,1 10,1	02,1 11,0		11,1 00,0		00,1 20,0			01,0 02,0				10,0 21,0
02,0 20,0			00,0 02,1	01,0 11,0		11,1 12,1		20,1 01,1			12,0 21,1		10,0 00,1		21,0 10,1		
	02,1 20,1	00,1 02,0			01,1 11,1		11,0 12,0		20,0 01,0	12,1 21,0		10,1 00,0		21,1 10,0			

We note that this construction differs from the constructions for n-dimensional frames in [4]. It does not construct sets of mutually orthogonal frame starters.

We can use PBD-closure to construct $(2^n)^u$ frames for $u \equiv 1\, modulo\, 4$. Definitions and results on PBD-closure can be found in [4]. Let n be a positive integer. We define $F_n = \{u \mid$ there exists a $(2^n)^u$ frame with a set of $2^n u$ skew transversals $\}$. The next result follows immediately from Theorem 4.1 in [4].

Theorem 3.4. *Let n be a positive integer. F_n is PBD-closed.*

We use this result to prove the following.

Theorem 3.5. *Let n be a positive integer. If $u \equiv 1(mod\,4)$ and $u \neq 33,57,93$ or 133, then there exists a skew frame of type $(2^n)^u$ which has a set of $2^n u$ skew transversals.*

Proof. There exist PBD(u,{5,9,13,17}) for all $u \equiv 1(mod\,4)$ and $u \neq 29,33,49,57,93,129$ or 133. Since there exist skew frames of type $(2^n)^q$ with sets of skew transversals where $q \equiv 1(mod\,4)$ is a prime power (Theorem 3.3), there exist skew frames of type $(2^n)^u$ for $u \equiv 1(mod\,4)$ and $u \neq 33,57,93,129$ or 133. The PBD(129,{5,29}) constructed in [15] is used to take care of the case $u = 129$. □

4. Conclusions

Completing the spectra for skew 1^n frames and skew $(2^n)^q$ frames with skew transversals appear to be difficult problems. As we noted earlier, the nonexistence of a skew 1^9 frame with 9 skew transversals causes problems with the usual PBD constructions for these frames. New constructions appear to be needed to take care of the remaining cases for skew 1^n frames with skew transversals. The spectrum of skew frames of type 2^q has not been completely determined [12]. The existence of several frames of small order could be used with PBD-closure to determine the spectrum for skew frames of type $(2^n)^q$ and skew frames of type $(2^n)^q$ with skew transversals, for $n \geq 1$.

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