A CENSUS OF TETRAVALENT GM GRAPHS ON FOURTEEN TO TWENTY VERTICES

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Abstract. Generalized Moore graphs are regular graphs that satisfy an additional distance condition, namely, that there be the maximum number of vertices as close as possible to any particular vertex, when that vertex is considered as root vertex. These graphs form a useful model for the study of various theoretical properties of computer communications networks. In particular, they lend themselves to a discussion of lower bounds for network cost, delay, reliability, and vulnerability. A considerable number of papers have already been published concerning the existence and properties of generalized Moore graphs of valence three, and some initial studies have discussed generalized Moore graphs of valence four, when the number of vertices is less than fourteen. This paper continues the previous studies for those cases when the graph contains a number of vertices that is between fourteen and twenty. In the case of valence three, the graph with a complete second level exists; it is just the Petersen graph. The situation is quite different for valence four; not only does the graph with a complete second level not exist, but the graphs in its immediate "neighbourhood" also fail to exist.

1. Introduction

The topological design of computer communications networks led to the analysis and development of generalized Moore graphs. For a discussion of network cost, delay, and reliability, we refer to [5]; these graphs also satisfy a criterion of minimal average path length, as discussed in [8]. A discussion of the existence or non-existence of the graphs for valence 3 in [4], [6], [7], [9], [10], and [11] provides a fairly complete census of the trivalent generalized Moore graphs for a large number of cases, including all cases in which the number of vertices is relatively small. Initial work on a census of tetravalent generalized Moore graphs was begun in [3], where we considered graphs having 10 or fewer vertices, and in [1] and [2], where we treated the graphs on eleven to thirteen vertices. In the current paper, we discuss the behavior of graphs having 14 to 20 vertices; many of these graphs do not exist.

As in [1] ,[2], and [3], we abbreviate "Generalized Moore Graph" to "GM graph", and we assume that all the graphs under discussion are regular and of valence 4 at every vertex. For completeness, we recall the definition of a generalized Moore graph (of valence 4). If we select a particular node as the root node, then there are 4 nodes at distance 1 from it, 12 nodes at distance 2 from it, 36 nodes at distance 3 from it, 108 nodes at distance 4 from it, etc. Thus, all the other nodes are crowded in as "close" to the root node as is possible. There may, of course, be incomplete levels in the graph; a generalized Moore graph with 61 nodes would have its first three levels full (and containing 1, 4, 12, and 36 nodes); it would have exactly 8 nodes situated on the fourth level, at distance 4 from the root node. The essential feature of a generalized Moore graph is that the distance property (or the "closeness" property) holds, no matter what particular node is selected as the root node. In short, any vertex has the same "distance distribution" as any other vertex, and any node has the maximum possible number of nodes at distances 1,2,3,4, etc.

We also find it useful to refer to the girth of a GM graph; this is the length of the smallest cycle in the graph. If the graph is a complete GM graph (note that this does not have any reference to the complete graph, K_n; a complete GM graph is one in which the m'th level of the graph is full), then the girth of the graph is 2m+1; on the other hand, if the m'th level of the graph is not complete, then there will be edges of the graph that join points in the (m-1)th level to one another; in this case, we note that the girth of the graph is 2m-1.

In general, if we have a GM graph on N nodes which is regular of degree V, then we denote it by the symbol M(N,V). In earlier papers, we discussed the graphs M(N,4) for N less than fourteen; in this paper, we give the results for N = 14 up to N = 20. As before, we often abbreviate M(N,4) to M(N). We also mention that we have again used Kocay's isomorphism testing method, as described in [3], throughout the paper.

2. The Graphs M(14,4).

We employ the notation of [1], [2], and [3]. The number of lines at level 2 is a, the number at level 1 is c, and the number of lines joining vertices at level 2 to vertices at level 1 is b. Then we have:

$$a+b+c = 24,\ 2a+b = 36,\ b+2c = 12;\ b > 8.$$

There are only two solutions: (a,b,c) = (12,12,0) or (13,10,1).

2.1. We consider the (13,10,1) case first. We may arbitrarily join 3 and 4. There are only two subcases; in the first of these, we join 2 and 5 to the same level-2 vertex, as illustrated in Figure 2.1.

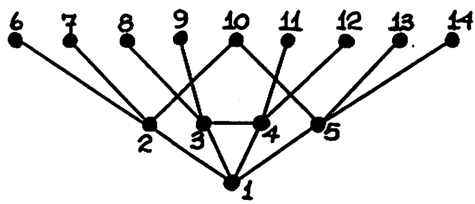


Figure 2.1.

Since we must reach 3 and 4 from 10, we take (10,9) and (10, 11). In order for 10 to reach both 8 and 12, we must take (11,8) and (9,12). Now, 3 must reach 6,7,13, and 14, but there are only 3 edges (two through 8 and one through 9) to use. So this case is impossible.

In the second subcase, 2 and 3 share a common level-2 vertex (Figure 2.2).

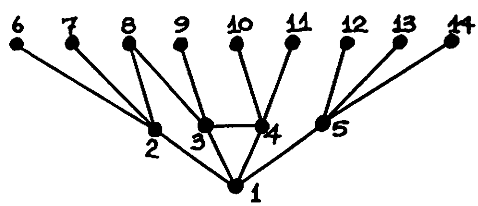


Figure 2.2.

We must join 8 to 4 and 5. Without loss of generality, take (8,10) and (8,12). Again, 3 must reach four vertices at level 2, but is only able to reach three of them. Thus ,there are no graphs M(14,4) which contain triangles.

2.2. We now consider the (12,12,0) case; there are nine subcases.

2.2.1. One level-2 vertex is joined to all level-1 vertices (Figure 2.3).

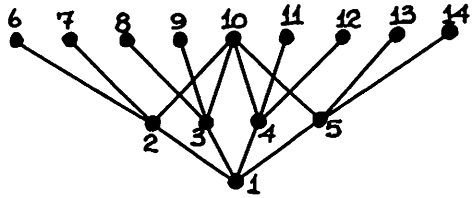


Figure 2.3.

So that 6 may reach 3, 4, and 5, we take (6,8), (6,11), and (6,13). To satisfy the distance property for 2, we must join 7 to 9, 12, and 14. To avoid forming triangles, join 14 to both 8 and 11. Similarly, we must join 13 to both 9 and 12. The graph (Figure 2.4) is completed by (8,12) and (9,11).

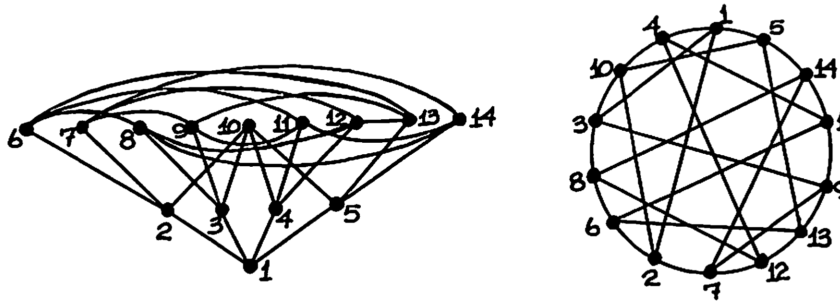


Figure 2.4.

2.2.2 Three level-2 vertices are shared by two level-1 vertices (Figure 2.5).

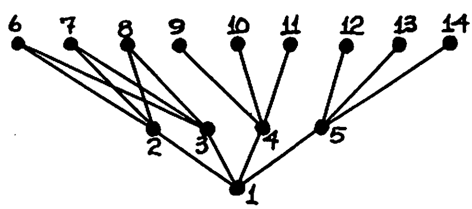


Figure 2.5.

Since 6, 7, and 8 must reach the rest of the level-2 vertices, we may take (6,9), (6,12), (7,10), (7,13), (8,11), and (8,14). To avoid triangles through 9, join 9 to 13 and 14. Similarly, join 12 to 10 and 11. Since 14 can not be joined to 11, we must take (13,11). The graph (Figure 2.6) is completed by (14,10); it is isomorphic to that in Figure 2.4 (rename 2 and 3 as 1 and 10).

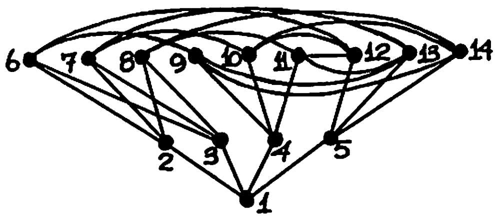


Figure 2.6.

2.2.3. The third subcase is shown in Figure 2.7.

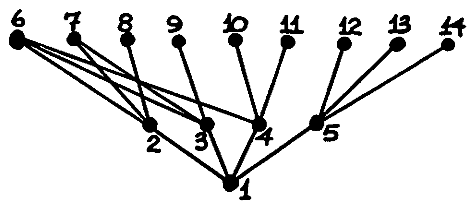


Figure 2.7.

So that 6 may reach 5 in 2 steps, join 6 to 12. So that 6 may reach 13 and 14 in 2 steps, join 12 to 13 and 14; but this forms 2 triangles through 5.

2.2.4. Figure 2.8 depicts the fourth subcase.

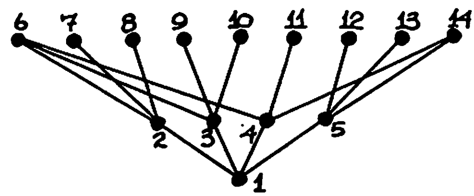


Figure 2.8.

To avoid triangles, we must join 6 to 12 or 13; so take (6,12). So that 6 may reach 13 in 2 steps, we require (12,13); but this forms a triangle through 5.

2.2.5. The fifth subcase is shown in Figure 2.9.

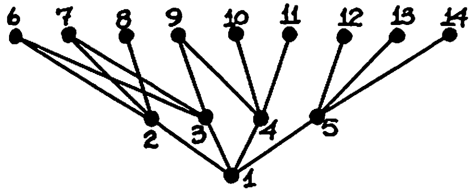


Figure 2.9.

In order for 2 to reach 9 in 2 steps, we must take (8,9); so that 9 may reach 5, take (9,12). To avoid triangles, 9 can only reach 13 and 14 by joining them both to 8. So that 12 may reach 2, and 13 and 14 may reach 3, the entire flower {12,13,14} must be joined to 6 and 7. Since 6 and 7 are equivalent, we may take (7,12) and (6, 14). In order for 14 to reach 4 in 2 steps, we may take (14,10). Now,10 must reach 2 and 3 in 2 steps; it can not be joined to 6 (a triangle results), and so we take (7,10). To avoid triangles, 10 can only be joined to 13. So that 13 may reach 2 in 2 steps, we must take (6,13). Now, the edges through 11 can not be drawn.

2.2.6. Figure 2.10 illustrates the next subcase.

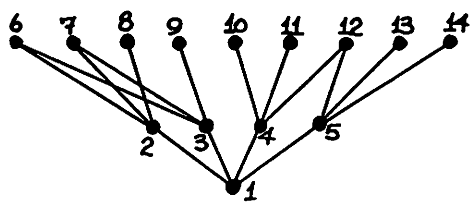


Figure 2.10.

So that 2 may reach 9 in 2 steps, we take (8,9). If 6 were joined to 12, it must reach 10, 11, 13, 14. But it can be joined to only one of these, say 10. Since neither 10 nor 12 may be joined to 11, 13, or 14 without forming triangles, these vertices can not be at maximum distance 2 from 6; thus, the distance property for 6 is not satisfied. The same argument applies to 7. Thus, for 2 and 3 to reach 12, we must join both 8 and 9 to 12; this forms a triangle.

2.2.7. The next case is shown in Figure 2.11.

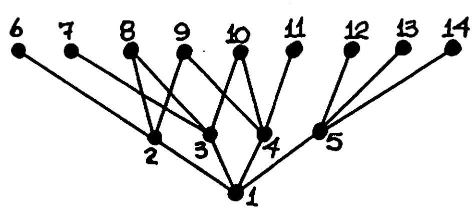


Figure 2.11.

So that 3 may reach 9 without forming triangles, we take (7,9). Since 9 must reach 5 in 2 steps, we take (9,12). So that 4 may reach 8 in 2 steps without forming a triangle, take (8,11). Since 12 may not be joined to 13 or 14 (triangles result), 7 must be joined to 13 and 14 (distance property for 9). So that 10 may reach 9 in 2 steps, we require (10,12). The only join permitting 6 to reach 3 is (6,10). The distance property for 4 requires (11,13) and (11,14). To avoid forming triangles, we must take (8,12); also, 6 must be joined to both 13 and 14. The resulting graph (Figure 2.12) is isomorphic to that in Figure 2.4..

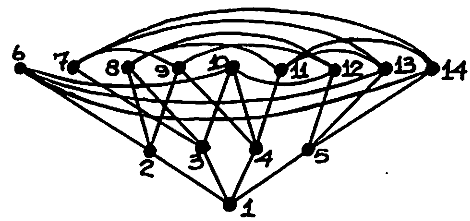


Figure 2.12.

2.2.8. The penultimate subcase is shown in Figure 2.13.

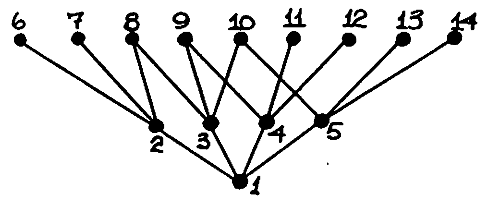


Figure 2.13.

So that 9 may reach 2 and 5 in 2 steps, take (9,7) and (9,13). So that 3 may reach 6 and 14 in two steps without forming triangles, we must take (10,6) and (8,14). Since 11 and 12 are equivalent, take (8,12) and (10,11) in order that 3 may reach 11 and 12 in 2 steps. So that 4 may reach 6 in 2 steps without forming a triangle, take (6,12); then (12,13) is required. So that 13 may reach 2 in 2 steps, 13 must be joined to 6 or 7; both joins result in a triangle.

2.2.9. The final configuration is shown in Figure 2.14.

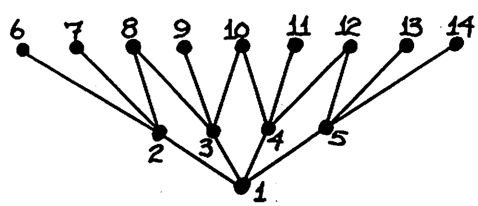


Figure 2.14.

So that 10 may reach 2 and 5 in 2 steps, take (10,7) and (10,13). To allow 3 to reach 6 in 2 steps without forming a triangle, take (9,6); similarly, we must take (11,14) . In order for 10 to reach 6 and 14 in 2 steps, we require the joins (6,13) and (7,14). If (14,8) is a join, the distance property for 14 requires us to join 11 to 6 and 9; but this forms a triangle. Thus, (14,9) and (11,8) must be joins. So that 8 may reach 5 in 2 steps, take (8,13). So that 8 may reach 12, take (11,12). The last 2 joins form a triangle.

We can thus conclude with

Theorem 1. There is only one tetravalent GM graph on 14 vertices.

3. The Graphs M(15,4).

The equations for a,b,c, are

$$a+b+c = 26, \ 2a+b = 40, \ b+2c = 12; \ \ b>9.$$

The only solutions are:

$$(a,b,c) = (15,10,1) \text{ - pattern } \alpha; \text{ and } (14,12,0) \text{ - pattern } \beta.$$

Let us consider (a,b,c) = (15,10,1) first. If all vertices have pattern α, there would be a total of 5 triangles. Furthermore, any particular vertex occurs in exactly one triangle and in no 4-cycles. Hence, the three vertices of any particular triangle must be joined to 6 distinct triangles (to avoid forming 4-cycles, or creating two triangles at a vertex), and this is impossible. Hence there must exist a vertex with pattern β. We now divide the discussion into four cases.

3.1. The two extra lines at level 2 emanate from a common level-2 vertex, as illustrated in Figure 3.1.

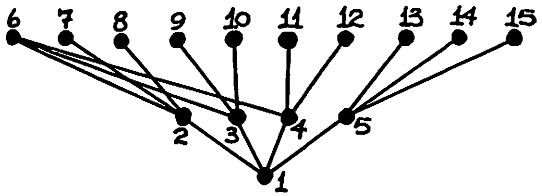


Figure 3.1.

So that 6 may reach 5 in 2 steps, take (6,13). So that 6 may reach 14 and15, we must join 13 to both 14 and 15. But this forms 2 triangles through 5.

3.2. The two extra lines at level 2 come from only 2 different level-1 vertices, as illustrated in Figure 3.2.

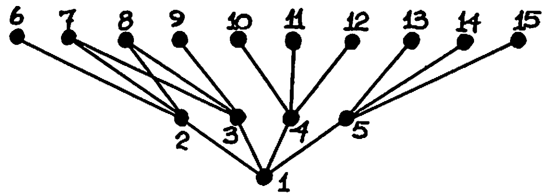


Figure 3.2.

If 9 is joined to either 7 or 8, 3 would be on a triangle and a 4-cycle (impossible). So that 2 may reach 9, we must take (6,9). Now, 6,7, and 8 must all be joined to distinct points in the set {10,11,12} and in the set {13,14,15}; the same holds for 7,8, and 9. Thus, 6 and 9 must be joined to the same element from {10,11,12} and the same element from {13,14,15}. But this forms 2 triangles through 6 and 9.

3.3. Figure 3.3 depicts the next subcase.

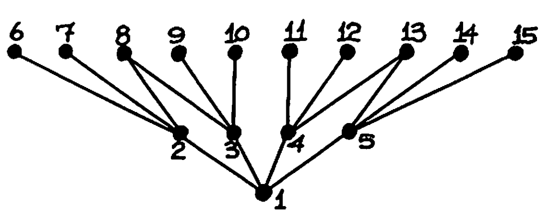


Figure 3.3.

Since the only remaining possibilities are shown in Figures 3.3 and 3.4, we see that there must be two 4-cycles through each point unless the graph contains a triangle; but this would create (15)(2)/4 distinct 4-cycles, which is impossible. Hence the graph must contain at least one triangle. There is no loss of generality in taking the triangle in Figure 3.3 as made up of edges (6,9), (6,11), (9,11). To avoid a 3-cycle and a 4-cycle through 6, we must join 6 to 14 or 15; so we take (6,15). Similarly, we must take (11,14). Then, to avoid forming both a 4-cycle and a 3-cycle through 9, we must take either (9,14) or (9,15); either choices creates 2 triangles through 9.

3.4. Finally, consider the case shown in Figure 3.4.

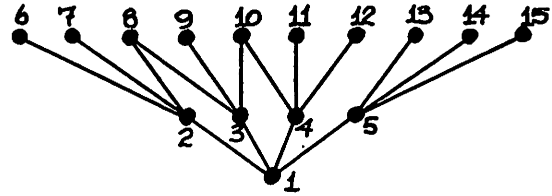


Figure 3.4

There are two 4-cycles through vertex 1, and, therefore, through any β-type vertex. If there are α vertices of type α and β vertices of type β, then $\alpha+\beta = 15$; by counting 3-cycles and 4-cycles, we see that $\alpha \equiv 0 \pmod 3$, $2\beta \equiv 0 \pmod 4$. The only solutions are: $\alpha = 3$, $\beta = 12$; or $\alpha = 9$, $\beta = 6$.

If the number of type α vertices is 9, then three triangles are formed. There is no loss in generality in taking (5,14,15) as a triangle. Two more 3-cycles must be formed; we may take them as (6,9,11) and (7,12,13). We must still join 5 to 6, 8, 9, 10, 11. Now13 can not be joined to either 6 or 11, or these vertices would be in a 3-cycle and a 4-cycle. Thus,14 and 15 must be joined still join 6 to 10, 12, and 13. Since 14 can not be joined to either 12 or 13, it must be joined to 10. But then neither 9 nor 11 can be joined to either 12 or 13 (they would either be contained in two triangles, or in a triangle and a 4-cycle). Hence we must reject the possibility that there are nine α-type vertices.

Thus, there are only three α-type vertices. They form a triangle, and no vertex is permitted to occur in both a triangle and a 4-cycle. We first consider vertex 3; there are seven edges emanating from vertices 8, 9, and 10, and vertex 3 must reach exactly seven level-2 vertices. Hence, none of vertices 8,9,10, may share any level-2 vertex. We may join 9 to 6,12, and 13, without loss of generality. Now 1,3,8,2, and 1,3,4,10, are the two 4-cycles through vertex 1, and the other β-type vertices must possess a similar configuration. Hence we must take 2,8,7,14, and 4,10,11,15 as 4-cycles (it is easy to trace out the result of the alternative choice 2,8,7,11, and 4,10,7,11, and reach a contradiction). But there are only six 4-cycles in the graph, and the last two are easily found as 7,14,5,13, and 11,15,5,13. We now find that the only possibility for the triangle in the configuration is (9,6,12), and the only possible completion that satisfies the distance property for vertices 2, 4, and 5, is (14,12) and (15,6). The resulting graph is shown in Figure 3.5.

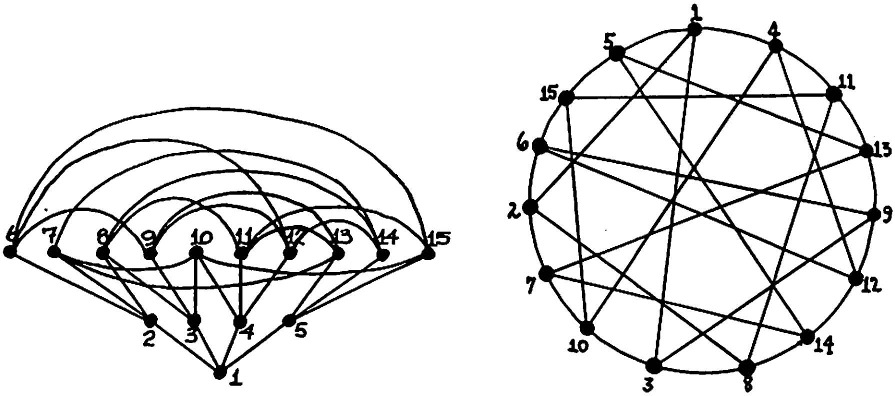


Figure 3.5

Theorem 2. There is exactly one tetravalent generalized Moore graph on fifteen vertices.

4. The Graphs M(16,4), M(17,4), and M(18,4).

A partially complete Moore tree for M(16,4) is shown in Figure 4.1.

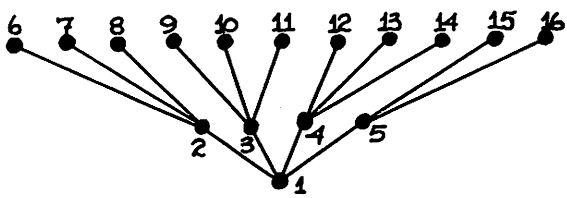


Figure 4.1.

Node 5 must be joined to another level-2 vertex. Without loss of generality, we take (5,14). Now, we count the number of pentagons through node 1. Out of the 16 edges to be joined at level 2, the two that are joined to vertex 14 will each form 2 pentagons. The remaining 14 edges at level 2 produce one pentagon per edge. Hence, the total number of pentagons through vertex 1 is 2(2)+14(1) = 18. Thus, the total number of distinct pentagons is (16×8)/5, which is non-integral. (Alternatively, we might count the number of distinct 4-cycles as 15/4). We may thus conclude with

Theorem 3. There are no tetravalent GM graphs on 16 vertices.

The case of 17 vertices is almost trivial, since the second level is complete; each of the 18 edges at level two (cf. Figure 4.2) produces a pentagon, and so the number of distinct pentagons is 18×17/5. Since this quantity is non-integral, we have

Theorem 4. There are no tetravalent GM graphs on 17 vertices.

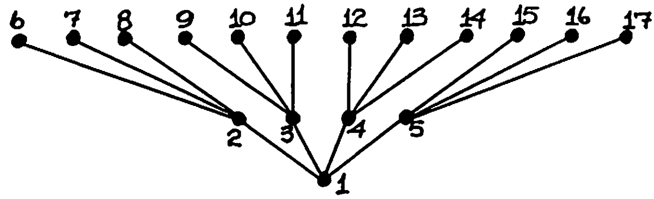


Figure 4.2

The Moore tree for M(18,4) is shown in Figure 4.3; node 18 is joined to exactly one vertex in each of the four flowers at level 2. Since there are 16 edges at level 2, and since each forms a pentagon through vertex 1, we find that the total number of distinct pentagons in the graph is 18×16/5. Since this value is non-integral, we have

Theorem 5. There are no tetravalent GM graphs on 18 vertices.

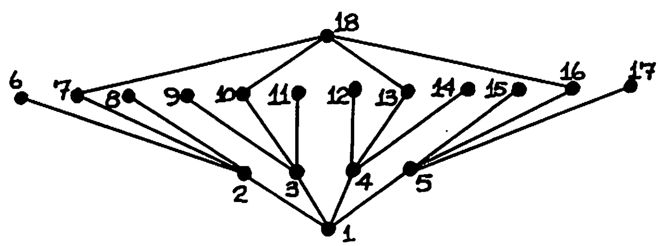


Figure 4.3

5. The Graphs M(19,4).

For M(19,4), there are two possibilities; we first consider the case when vertices 18 and 19 at level 3 are not joined. Because of the girth restriction on the graph, there are no triangles or 4-cycles in the graph. Hence, the 14 edges at level 2 produce 14 pentagons through vertex 1. If all vertices were of this same type, then the number of distinct pentagons in the graph would be 14×19/5, and this is non-integral. Consequently, we conclude that there must be a vertex in the graph for which the two vertices at level 3 are joined.We take this vertex as root node, and consider two subcases.

5.1. Figure 5.1 depicts the subcase when 18 and 19 share two flowers.

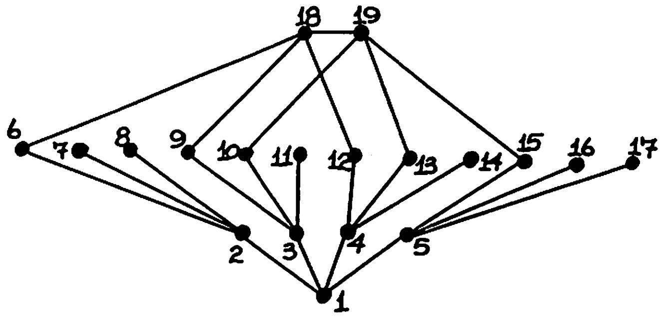


Figure 5.1.

We distinguish two types of vertices: type α vertices are joined to either 18 or 19; type β vertices are joined to neither 18 or 19. The girth restriction ensures that no triangles or 4-cycles may be formed; hence, no two type α vertices may be joined by an edge. Node 15 may be joined to two of 7, 8, 11, and 14, but not to both 7 and 8; hence, we may take (15,14) without loss of generality. Since 16 and 17 must be joined to 12 and 13, and since 16 and 17 are undifferentiated, we may take (16,12) and (17,13).

If 15 is joined to 11, then 16 must be joined to 10, and 17 must be joined to 9. Since 7 and 8 must be joined to all 3 flowers, they must be joined to16 and 17. Since 7 and 8 are undifferentiated, we may take (7,16) and (8,17). Then 6 must be joined to 11 and 14. However, we must join 14 to the {9,10,11} flower, and we can not do this without forming a triangle or a 4-cycle.

If 15 is joined to 7 or 8, we may take (15,8); then we must take (16,7) and (17,6) to avoid forming a 4-cycle. Vertices 16 and 17 must both be joined to the {9,10,11} flower. To avoid forming 4-cycles, 15 and 17 may not be joined to 9 or 10; hence, they must be joined to 11. But this forms a 4-cycle, and so 18 and 19 can not have exactly two flowers in common.

5.2. The case in which 18 and 19 share three flowers is shown in Figure 5.2.

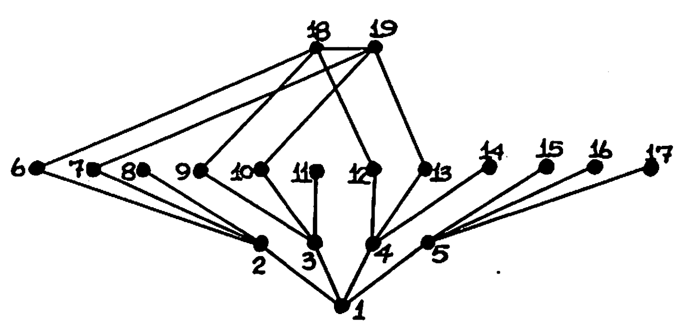


Figure 5.2.

Vertices 15, 16, and 17 must be joined to the 3 other flowers. Without loss of generality, we take (15,14), (16,13), and (17,12). With respect to vertex 11, vertices 16 and 17 are undifferentiable; hence there are only two cases.

We first take (15,11); then we require (16,9) and (17,10). To avoid 4-cycles, vertices 16 and 17 may not be joined to both 6 or 7; hence,they must both be joined to 8. But this creates a 4-cycle.

If we take (16,11), we must take (17,10) and then (15,9). To avoid 4-cycles, 17 must be joined to 8. Now 16 can not be joined to 7; so we must take (16,6) and then (15,7). We must now join 6 and 7 to 14 and 11, respectively. Now, we must take (11,12) so that 11 may reach all 3 flowers without forming triangles or 4-cycles. Vertex 14 can not be joined to 9; so we must take (14,10). To satisfy girth restrictions, we must take (13,8). The last edge is (8,9). By forming an adjacency matrix, it is easily verified that the distance property is satisfied for this graph, which is shown in Figure 5.3.

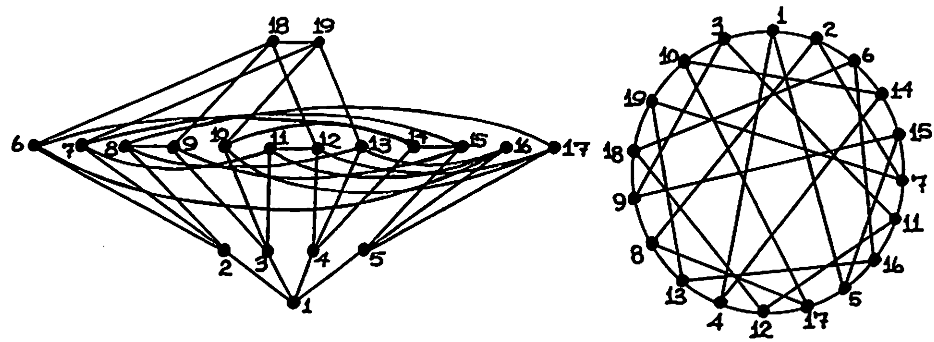


Figure 5.3

We thus have

Theorem 6. There is a unique tetravalent GM graph on 19 vertices.

6. The Graphs M(20,4).

For 20 vertices, the (a,b,c) equations are $a+b+c = 24$, $2a+b = 12$, $b+2c = 36$. There are at most 2 edges at level 3, since 3 edges would form a triangle; thus $a<3$, and we find that (a,b,c) is (0,12,12), (1,10,13), or (2,8,14).

6.1. We begin with (a,b,c) = (2,8,14), and the graph shown in Figure 6.1. The joins (18,19) and (19,20) provide the only way to join the 2 edges at level 3, and we may take (18,6), (18,9), (18,12), (20,11), and (20,14) without loss of generality.

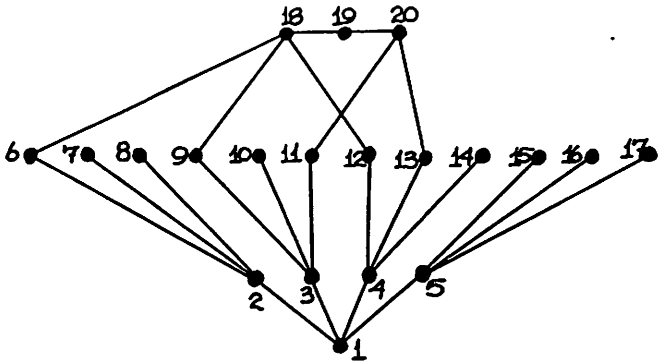
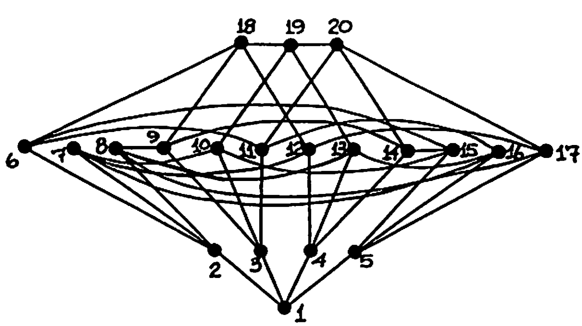


Figure 6.1.

We first consider the join (20,8). Now 19 must be joined to a vertex in each of two flowers; call these vertices A and B. These two vertices must not be joined to any vertex that is joined to 18, 19, or 20 (triangles or 4-cycles would result). As well, each vertex must be joined to two different flowers. One of the flowers to which A and B must be joined has only one available vertex. Since there are only two flowers, and A and B may not be joined to the same vertex, the graph can not be completed. Thus, vertex 20 can not be joined to 8 (or, by symmetry, to 7).

Thus, without loss of generality, we may take the join (20,17). We designate a vertex joined to 20 as "a 20-vertex", and we designate a vertex not joined to any of 18, 19, or 20 as "a spare vertex". Now, suppose that 19 is joined to 7, and to a spare vertex A. Vertex 7 must be joined to two spare vertices so that vertex A may be joined to 8 and to another spare vertex without forming a 4-cycle. Since16 may not be joined to a vertex joined to either 18 or 19, it must be joined to a 20-vertex and to a spare vertex. There is only one spare vertex to which 6 may be joined, namely, the vertex joined to A (this can not be 8). Also, the 20-vertex in the same flower as A must be joined, without forming a 4-cycle, to the same spare vertex as 7. To avoid triangles or 4-cycles, 8 must be joined to an 18-vertex and a 20-vertex. This leaves only one way for vertex 6 to join a 20-vertex, and there is only one possible join for the spare vertices not in the 7 flower or the A flower. There is also only one join for the 20-vertex in the same flower as the join from A. Now, the 18-vertex joined to 8 can not be joined to any vertex without forming a triangle or a 4-cycle. Hence, 19 may not be joined to 7 (or to 16).

Thus, we try the joins (19,10) and (19,13). Both 10 and 13 must be joined to spares; so we take (10,7) and (10,15), along with (13,8) and (13,16), without loss of generality. Now, 6 must be joined to a spare; since 7 and 8 are symmetric as regards 6, we take (6,15). Vertex 15 can not be joined to 12, and so we must take the edge (15,14); also, we require (14,9), since any other join would violate the girth restriction. Since 7 must be joined to the {12,13,14} flower, we must take (7,12); also, 6 can not be joined to16 or 17, and so it must be joined to a 20-vertex, namely, 11. Since 8 must be joined to the three other flowers, the girth restriction requires (8,9) and (8,17). The last three joins are (7,16), (16,11), and (17,12). It is easy to verify that the distance property is satisfied, and the graph is shown in Figure 6.2.



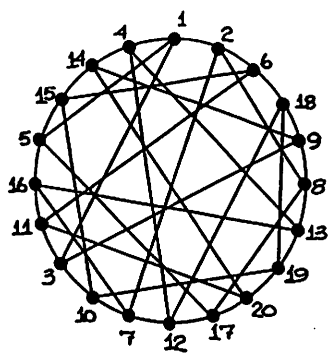


Figure 6.2

6.2. We next examine the case (a,b,c) = (1,10,13); we take (18,19) as the join at level 3, and discuss 3 subcases.

6.2.1. We first consider (Figure 6.3) when 18 and 20, as well as 19 and 20, share a vertex. Without loss of generality, we join 20 to 6, 9, 14, and 17; then we may take (18,6), (18,10), (18,12), as well as (19,17) and (19,13).

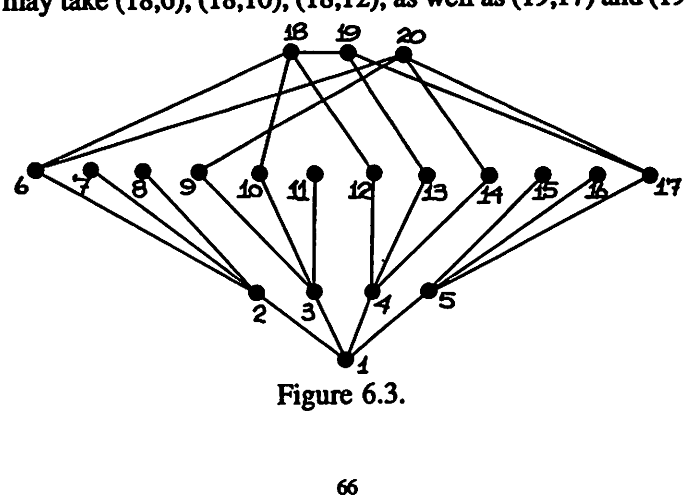


Figure 6.3.

If 19 is joined to 11, we must join 6 to 15 or 16; similarly, we must join 17 to 7 or 8. By symmetry, we may take (6,15) and (17,8). Now, 15 may not be joined to an 18-vertex or to a 20-vertex; so we must join 15 to both 11 and 13, and this violates the girth restriction.

If we join 19 to 7, then 6 may not be joined to11 (if it were,11 would have to be joined to both 13 and 15; then 8 would have to be joined to 10 and 12, and this would form a 4-cycle). If we join 6 to15, the girth restriction forces us to join 15 to both 11 and 13. Thus 17 must be joined to 8 (any other join would form either a triangle or a 4-cycle). Also, the girth restriction forces the following joins: (13,9), (9,16), (16,7), and (16,12). Now, 8 must be joined to all three other flowers. Therefore, 8 must be joined to 12; since (8,10) would form a 4-cycle, we must take (8,11). The only available join for 7 is (7,14). But then we must take (10,11), and this forms a triangle.

6.2.2. In our second subcase, only 18 and 20 share a vertex. Without loss of generality, the following nine joins may be made: join 20 to 6, 11, 14, and 17; take (18,6), (18,9), and (18,15); join 19 to 10 and 13. If 19 is joined to 7, then 6 may be joined to 15 or 16; by symmetry, take (6,15). Thus 15 can not be joined to an 18-vertex or to a 20 vertex (4-cycles are formed). Hence, 15 must be joined to 10 and 13, creating a 4-cycle . But if we take (19,16), then we require (6,15), (15,10), and (15,13), as above. So this case is impossible.

Finally, we consider the case when no level-3 vertices share a level-2 vertex. Again, without loss of generality, we make the following nine joins: join 18 to 6, 9, and 12; join 20 to 8, 11, 14, and 17; take (19,10) and (19,13).

If 19 is joined to 7, 6 may not be joined to an 18-vertex or a 19-vertex. Thus, we may take (6,11) and (6, 15). Since 15 can not be joined to an 18-vertex, it must be joined to both 10 and 14; then 10 can only be joined to 8. To avoid 4-cycles, 7 must be joined to both 14 and 16. Then 9 must be joined to both 16 and 17; but this forms a 4-cycle. So we can not join 19 to 7, and must try joining it to 15; then 7 may not be joined to 16 (if it were, the other two joins from 6 can not be made without forming either a triangle or a 4-cycle). Since (7,16) is impossible, we must take (16,6). Without loss of generality, we may also take (6,11). We must still join 16 to a 19-vertex and to a 20-vertex, and the only possible joins are (14,16) and (10,16). To avoid triangles and 4-cycles through 9, we take (9,7). Now we must take (12,7), since 12 must be joined to exactly one 20-vertex; but this forms a 4-cycle.

6.3. We divide the case (a,b,c) = (0,12,12) into five subcases.

6.3.1 If we join 18, 19, and 20 to different level-2 vertices, we may take the following twelve joins: (18,6), (18,9), (18,12), (18,15), (19,7), (19,10), (19,13), (19,16), (20,8), (20,11), (20,14), (20,17), as in Figure 6.4.

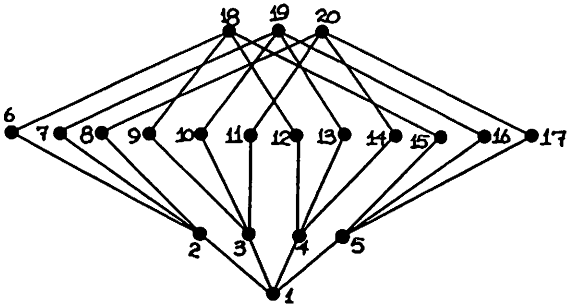


Figure 6.4.

We must join 6 to one 19-vertex and to one 20-vertex; so we may take (6,10) and (6,14). This forces the joins (10,17) and (14,16). Vertex 17 must be joined to an 18-vertex, and the only valid choice is 12. Similarly, 16 must be joined to an 18-vertex, and we may take (16,9). This forces us to join 9 to 8. The remaining joins (8,13), (13,15), (15,11), (11,7), and (7,12) are now all forced. The graph is shown in Figure 6.5.

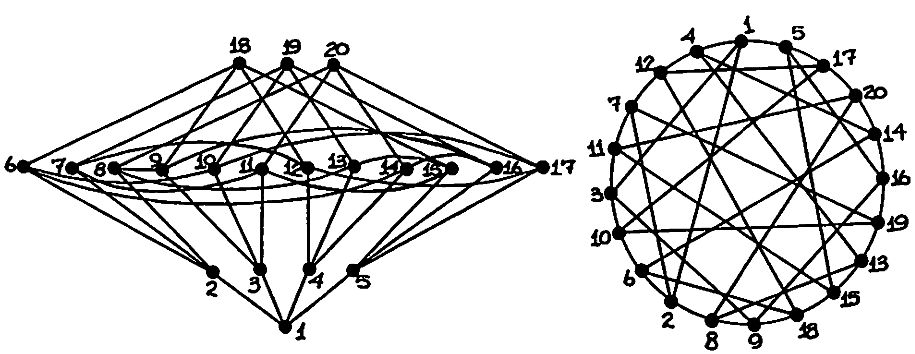


Figure 6.5.

6.3.2. We next suppose that 18 and 19 share a vertex, but that no other level-2 vertices are shared. Without loss of generality, we may take joins (18,6), (19,6), (18,9), (18,12), (18,15), (19,10), (19,13), (19,16), (20,8), (20,11), (20,14), (20,17). The situation is shown in Figure 6.6.

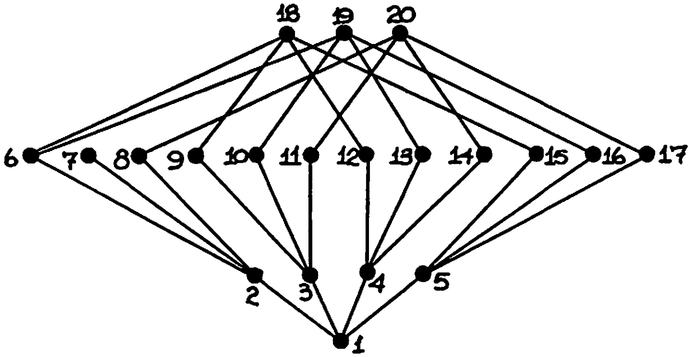


Figure 6.6.

Without loss of generality, we may take (7,9), (7,13), and (7,17), as well as (6,11). Similarly, we take (8,12) and (8,16). Since 12 can not be joined to 16, we must take (12,10). The girth restriction then forces (16,9). Now, 10 can not be joined to any other vertex without forming either a triangle or a 4-cycle. Hence, the graph cannot be completed.

6.3.3. In this case, 18 and 19 share a vertex, and 19 and 20 share a vertex. We arbitrarily join 18 and 19 to 6, and join 19 and 20 to 7. Without loss of generality, we may join 18 to 9, 12, and 15; also, we may join 19 to 10 and 13, and may join 20 to 8, 11, and 14. The situation is shown in Figure 6.7.

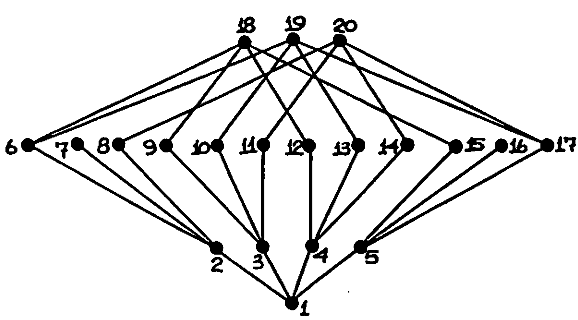


Figure 6.7.

If 7 is joined to 17, it can only further be joined to 9. Since there are 3 edges from 7, the graph can not be completed. If 7 is joined to 15, it may also be joined to 10 and 14; then we require joins (12,10) and (6,11). Then 16 must be joined to 8, 9, and 13. Now, we can not insert enough edges through 8.

If we suppose that 7 is joined to 14, then, since 10 may not be joined to13, 13 must be joined to both 8 and 15. Now 15 can not be joined to 8 or 10; hence, the graph can not be completed.

If 7 is joined to 16, then we may take the join (6,11); this forces the join (11,16). If (16,12) is a join, then 17 must be joined to 9; this forces the join (9,7). Now 7 can not be joined to 13 (if it were, girth constraints force (13,15) and then (15,8)). Since 2 edges emanate from 10 and 14, and only one edge emanates from each of 8 and 12, we require the join (10,14). Now, either (8,10) or (8,14) violates the girth conditions.

If, instead, 16 is joined to 13, then 17 must be joined to 9 or 12. If (17,9) is a join, then we require (9,7), to avoid triangles or 4-cycles. Similarly, 7 must be joined to 14, and this forces the join (13,8). Now, 12 can not be joined to 8, 14, or 15; thus, the graph can not be completed. If (17,12) is a join, then we must join 12 to 7; this forces the join (7,10). Also, 8 must be joined to both an 18-vertex and a 19-vertex; since no 19-vertex provides a valid join to 8, the graph can not be completed.

6.3.4. In this case, 18 and 19 share a common level-2 vertex, as do the pairs (18,20) and (19,20). Figure 6.8 illustrates this situation using the joins (18,6), (18,10), (18,12), (18,15), (19,6), (19,9), (19,13), (19,16), (20,7), (20,9), (20,12), (20,17).

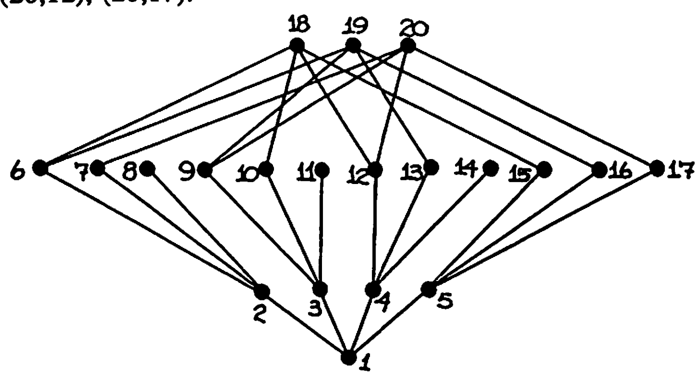


Figure 6.8.

If 6 is joined to11, we must join 11 to both 14 and 17. We may not join 8 to 12 (for then 8 could not be joined to the {9,10,11} flower). Hence, 12 must be joined to 16. This forces the join (16,8), and then 8 must be joined to both 10 and 14. The girth restrictions now force (14,15). Now the joins (7,10) and (7,13) produce a 4-cycle.

If 6 is joined to 17, then 17 may be joined to 14, without loss of generality. Since 14 can not be joined to both 10 and 11, (14,8) must be a join. If we join 11 and 12, we are forced to have joins (11,16), (11,8), (8,15), and (14,10). Also, 13 can only be joined to 7 and 15; but now 9 can not be joined to any vertex without forming either a triangle or a 4-cycle. Consequently, let us use the join (12,16). This forces the join (16,11). Since 11 may not be joined to both 7 and 8, it must be joined to 14; then we must choose (7,11). Since 8 must be joined to the other 3 flowers, we take (8,9). Now 10 can not be joined to 7 or 15, and hence the graph can not be completed.

6.3.5. Finally, if 18, 19 and 20 are joined to the same vertex, say 6, then the following joins can then be made: (18,9), (18,12), (18,15), (19,10), (19,13), (19,16), (20,11), (20,14), (20,17). See Figure 6.9.

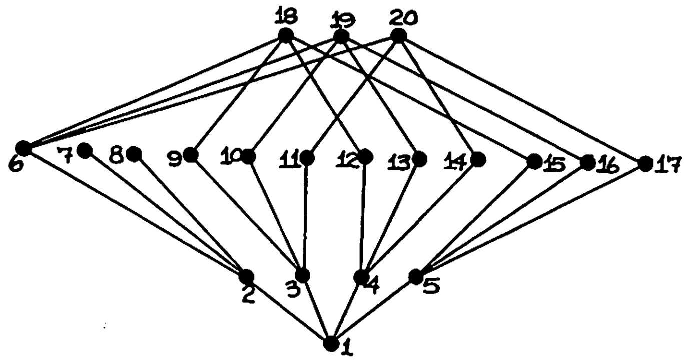


Figure 6.9.

Both 7 and 8 must be joined to the other three flowers without forming a 4-cycle. Without loss of generality, we may join 7 to 9, 13, and 17; then we must join 8 to 10, 14, and 15. If we consider vertex 11, we see that it can join two pairs of points: (12,16) and (13,15). If 11 is joined to both 12 and 16, joins (12,17) and (9,14) are forced; now, any edge from 10 will violate the girth restriction. On the other hand, if 11 is joined to both 13 and 15, then 16 must be joined to one of 9, 12, and 14. Then 12 must be joined to 17 and then to 10. The remaining join is (9,16). The graph (Figure 6.10) is isomorphic to that in Figure 6.2.

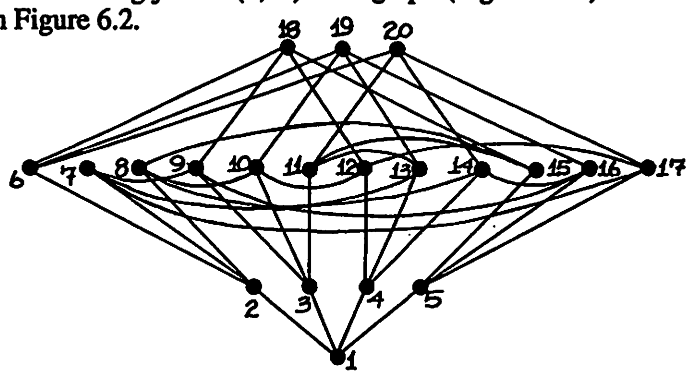


Figure 6.10.

We may thus conclude with

Theorem 7. There are two non-isomorphic GM graphs on 20 vertices.

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