Validity of Lander's Conjecture for $\lambda = 3$ and $k \leq 500$*

by

K. T. Arasu

ABSTRACT

Lander Conjectured: If D is a (v, k, λ) difference set in an abelian group G with a cyclic sylow p-subgroup, then p does not divide (v, n), where $n = k-\lambda$.

In a previous paper, the above conjecture was verified for $\lambda = 3$ and $k \leq 500$, except for $k = 228$, 282 and 444. These three exceptional values are dealt with in this note, thereby verifying Lander's conjecture completely for $\lambda = 3$ and $k \leq 500$.

1. Introduction

Let G be an abelian group of order v. A (v, k, λ) difference set in G is a subset D of G of size k such that for each element $g \neq 1$ in G, there exist exactly λ ordered pairs $(x, y) \in D \times D$, $x \neq y$, satisfying $g = xy^{-1}$.

An easy counting shows that

$$k(k-1) = \lambda (v-1) \qquad (1)$$

we refer the reader to [3] amd [6] for an excellent treatment on the theory of difference sets and their multipliers.

* Research supported by the Research Incentive Grant #241259 from Wright State University

Lander's conjecture [6] If there exists a (v, k, λ) difference set in an abelian group G whose sylow p-subgroup is cyclic then p does not divide (v, n), where $n = k-\lambda$.

For $\lambda = 1$, the above conjecture is clearly true. For $\lambda = 2$, Dickey and Hughes [4] have checked this for $k \leq 5000$. In fact, they showed more, viz, if there exists an abelian $(v, k, 2)$ difference set, then either $k \leq 9$ or $k \geq 5001$.

In [1], the author studied the above conjecture for $\lambda = 3$ and verified it for $k \leq 500$, except when $k = 228$, 282 and 444. This paper deals with those three exceptional cases.

2. k = 228 Case

We apply the following theorem.

Theorem 1 (Jungnickel and Pott [5]) Let D be a (v, k, λ) difference set in the group G, where $v > k$. Furthermore, let $u \neq 1$ be a divisor of v, let U be a normal subgroup of index u of G, put $H = G/U$ and assume that H is abelian and has exponent u^*. Finally, let p be a prime not dividing u^* and assume $tp^f \equiv -1 \pmod{u^*}$ for some numerical G/U- multiplier t of D and a suitable nonnegative integer f. Then the following hold:

i) p does not divide the square - free part of n, so $p^{2j} \| n$ for some nonnegative integer j,

ii) $p^j \leq v/u$

iii) if $u > k$, then $p^{2j} | v\lambda$ (or equivalently $p^j | k$).

Let D be a hypothetical $(v, 228, 3)$ abelian difference set.

When $k = 228$ and $\lambda = 3$, from (1), we get $v = 3^5 71$. Let $u = 3^5$; $t = 1$, $p = 5$ and U be a subgroup of G of order 71. Theorem 1 applies, since $5^{81} \equiv -1 \pmod{243}$. So, by (iii) of theorem 1, $p | k$ i.e. $5 | 228$, a contradiction. Hence D does not exist.

3. k = 282 Case

Let D be a putative (v, 282, 3) difference set in an abelian group G. Then $v = 3^2 \cdot 5 \cdot 587$. Assume that the sylow 3-subgroup of G is cyclic. Then $G \simeq Z_9 \times Z_5 \times Z_{587}$. By Hall's multiplier theorem, 31 is a multiplier of D. Assume without loss of generality that D is fixed by the multiplier 31. Orbits of Z_9 under $(x \to 31x)$ have sizes 3, 3, 1, 1, 1, of Z_5 have sizes 1, 1, 1, 1, 1 and of Z_{587} have sizes 1, 293, 293. Hence orbits of G under $(x \to 31x)$ have sizes ≥ 293 or ≤ 3. Small size orbits are not enough in number to form a subset of size 282. Hence D cannot exist.

4. k = 444 case

Let D be a hypothetical (v, 444, 3) difference set in an abelian group G. Then $v = 3^2 \cdot 5 \cdot 31.47$. By Hall's theorem, 7 is a multiplier of D. Let H be a subgroup of G of index 5. Let $G/H = \{H_0, H_1, H_2, H_3, H_4\}$ and $s_i = |D \cap H_i|$ for i = 0, 1, 2, 3, 4. It is well-known (for instance, see [2] or [6]),

$$\sum_{i=0}^{4} s_i = k = 444 \text{ and } \sum_{i=0}^{4} s_i^2 = k-\lambda+\lambda|H| = 39780 \quad (2)$$

Since 7 is a G/H multiplier, it follows that (assuming D is fixed by the multiplier 7) $s_1 = s_2 = s_3 = s_4 = b$(say). Let $s_0 = a$. Then (2) becomes

$$\begin{aligned} a + 4b &= 444 \\ a + 4b^2 &= 39780 \end{aligned} \quad (3)$$

Solving for a and b, we obtain a = 72 and b = 93. Orbit sizes under $(x \to 7x)$ of Z_9, Z_5, Z_{31} and Z_{47} respectively are

3, 3, 1, 1, 1

4, 1

15, 15, 1

23, 23, 1

$a = 72$ implies $|D \cap H_0| = 72$, so D picks up

$\{1, 4, 7\} \times \{0\} \times \{0\} \times S$

or $\{2, 5, 8\} \times \{0\} \times \{0\} \times T$, where S and T are among the size 23 orbits of Z_{47}.

In addition D also picks up a size 3 orbit from $D \cap H$, which must be of the form

$\{1, 4, 7\} \times \{0\} \times \{0\} \times \{0\}$

or $\{2, 5, 8\} \times \{0\} \times \{0\} \times \{0\}$

The above orbits yield (3, 0, 0, 0) as a difference of elements of D more than 3 times, contradicting the fact $\lambda = 3$. Hence D does not exist.

References

1. Arasu, K.T., "On Lander's conjecture for the case $\lambda = 3$", J. Comb. Math. and Comb. Computing, 1(1987), 5-11.
2. Arasu, K.T. and Ray-Chaudhuri, D.K., "Multiplier theorem for a difference list", Ars Comb. 22(1986), 119-137.
3. Beth, T., Jungnickel, D. and Lenz, H., Design Theory, Bibliographisches Institut, Mannheim and Cambridge University Press, Cambridge (1985)
4. Hughes, D.R. "On biplanes and semibiplanes", In Combinatorial Mathematics, eds. Holton and Seberry, Lecture Notes in Mathematics, 686, (1978), pp 55-58, New York: Springer-Verlag.
5. Jungnickel, D. and Pott, A., "Two results on difference sets", submitted.
6. Lander, E.S., Symmetric designs: An algebraic approach, London Math Soc. Lecture note Series 74 (1983), Cambridge University Press, Cambridge.

Department of Mathematics & Statistics

Wright State University

Dayton, Ohio 45435