Generalized Hilbert Fields, Quaternion Algebra Structure and Block Design Counting

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In [Ka], Kaplansky introduced a class of fields which he called generalized Hilbert fields. These are the fields which have (up to isomorphism) a unique non-split quaternion algebra. Examples of such are real closed fields and the p-adic fields.

For a field F we let $G = \dot{F}/\dot{F}^2$ and for $x \in G$ we set $Q(x) = \left\{\left[\frac{x,y}{F}\right] \mid y \in G\right\}$. Kaplansky showed that F is a generalized Hilbert field if and only if $|Q(x)| \leq 2$ for every $x \in G$.

Generalized Hilbert fields with $|G| < \infty$ satisfying the non-degeneracy condition, $|Q(x)| = 1 \Rightarrow x = 1$ in G, have come to play an important role in the algebraic theory of quadratic forms. It has ben conjectured (the elementary type conjecture) that the Witt rings of generalized Hilbert fields are fundamental building blocks for the generation of all finitely generated Witt rings, a role somewhat analogous to that of the cyclic groups in the theory of finitely generated Abelian groups.

Given two fields F_1 and F_2 with square class groups G_1 and G_2 and quaternion algebra sets Q_1 and Q_2 respectively, we say that F_1 and F_2 have the same quaternion algebra structure if there is a group isomorphism $\alpha: G_1 \to G_2$ satisfying for $a, b, c, d \in G_1$, $\left[\frac{a,b}{F_1}\right] \cong \left[\frac{c,d}{F_1}\right] \Longleftrightarrow \left[\frac{\alpha(a),\alpha(b)}{F_2}\right] \cong \left[\frac{\alpha(c),\alpha(d)}{F_2}\right]$. It turns out (see [MY, Corollary 3.6]) that F_1 and F_2 have the same quaternion algebra structure if and only if they have isomorphic Witt rings thus no mention of Witt rings or quadratic form theory is needed here.

The purpose of this short survey is to report on the recent work concerning the elementary type conjecture in this quaternion algebra setting. Most of this work has been concentrating on (i) the proof of the conjecture in particular cases, (ii) recognizing quaternion algebra structures which can be decomposed and (iii) recognizing basic building blocks such as the quaternion algebra structures arising from generalized Hilbert fields.

If we let B be the subgroup of the Brauer group of F generated by the quaternion algebras then it is not difficult to see, [MY, Section 1], that the mapping $q: G \times G \to B$ via $q(a,b) = \left[\frac{a,b}{F}\right]$ is a symmetric bilinear mapping with G and B groups of exponent two satisfying

(i) $q(a, -a) = 1 \quad \forall\, a \in G$

(ii) $q(a,b) = q(c,d) \Rightarrow \exists\, x \in G$ with $q(a,b) = q(a,x)$ and $q(c,d) = q(c,x)$.

If one begins with arbitrary groups G and B of exponent two, a distinquished element $-1 \in G$ and a bilinear symmetric mapping $q: G \times G \to B$ satisfying (i) and (ii) then it is unknown if there is a field F with $G = \dot{F}/\dot{F}^2$, B the subgroup generated by the quaternion algebras over F and q the quaternion algebra mapping.

There are two important ways to construct new quaternion algebra structures from old. First, if F_1 and F_2 are fields with quaternion algebra structures given by $q_1: G_1 \times G_1 \to B_1$ and $q_2: G_2 \times G_2 \to B_2$ respectively then the natural mapping $q: (G_1 \times G_2) \times (G_1 \times G_2) \to B_1 \times B_2$ is the quaternion algebra structure of some field F. The construction of F is somewhat complicated, see [Ku] for details. We will call this construction the product construction. Second, if F is a field with quaternion algebra structure $q: G \times G \to B$ and Δ is the group $\{1, t\}$ then the mapping $q': G' \times G' \to B'$ where $G' = G \times \Delta$ and $B' \subset B \times G$ defined by

$$q'((a,1),(b,1)) = (q(a,b),1)$$

$$q'((a,1),(b,t)) = (q(a,b),a)$$

$$q'((a,t),(b,1)) = (q(a,b),b)$$

$$q'((a,t),(b,t)) = (q(a,b),-ab)$$

is the quaternion algebra structure of the field of formal Laurent series F((t)) over F, see [BCW]. We call this the power construction.

A list of fundamental examples of fields with their square class size and number of quaternion algebras is given below.

F	$\|G\|$	$\|Q\|$
$\mathbb{C}$	1	1
F_q	2	1
$\mathbb{R}$	2	2
$\mathbb{Q}_p$	4	2
extension of $\mathbb{Q}_2$ of degree 2k-3	2^{2k-1}	2
extension of $\mathbb{Q}_2$ of degree 2k-2	2^{2k}	2

Notice that all except $\mathbb{C}$ and F_q are generalized Hilbert fields.

The elementary type conjecture can now be stated precisely.

(ETC) The quaternion algebra structure of any field F with $|\dot{F}/\dot{F}^2| < \infty$ can be built up from the quaternion algebra structures of the fundamental examples listed above by using product and power constructions.

It is not too difficult to see that if F does not satisfy the non-degeneracy condition then its quaternion algebra structure is a product of one totally degenerate and one non-degenerate. Totally degenerate structures are easy to classify. Thus from here on out we can and will assume the non-degeneracy condition on our quaternion algebra structures.

<u>Some special cases</u>:

1. If $|G| \leq 32$, the conjecture is true. (see [C1], [KSS], [S] and [M].
2. If F is a pythagorean field (i.e. every sum of squares is a square), the conjecture is true. (see [M])
3. If there are ≤ 4 quaternion algebras over F, the conjecture is true. (see [C2])
4. If $|Q(a)| \leq 4 \ \forall \ a \in G$, the conjecture is true. (see [M])
5. If $|\{Q(a) | a \in G\}| \leq 4$, the conjecture is true. (see [FY3])

Most of the recent work has centered on finding ways of recognizing when a quaternion algebra structure does or does not come from a product or power construction. Notice that sufficiently good recognition theorems would answer the elementary type question.

For $a \in G$ we let $Q_0(a) = \left\{x \in G \,\middle|\, \left[\frac{a, x}{F}\right] = 1\right\}$

Recognition theorems.

1. The quaternion algebra structure of F is a power structure iff there exists $a \in G$ with $|Q_0(a)| = |Q_0(-a)| = 2$. ([BCW]).

2. The quaternion algebra structure of F is a product structure of a power structure and another structure coming from a pythagorean field if and only if there exists $a \in G$ with $|Q_0(a)| = 2$. ([B])

3. The quaternion algebra structure of F is a product of two structures one of which arise from a generalized Hilbert field if and only if there exists $a \in G$ satisfying $|Q_0(a)| = \frac{1}{2}|G|$ and some other technical condition. ([FY1]).

Of course, besides for recognizing product and power structures one must also be able to recognizing when a quaternion algebra structure is the same as one in the list of fundamental examples. With elementary linear algebra one can show that the quaternion algebra structure of a generalized Hilbert field is the same as one in the list. Since the structures arising from C and F_q do not satisfying the non-degeneracy condition it follows that to show a quaternion algebra structure satisfying the non-degeneracy condition arising from a field F is the same as one in the list of fundamental examples it suffices to show that F is a generalized Hilbert field. This is what is done in [FY2] and [FY3] where the main tool used was a generalized block design counting technique.

Recall that to prove $r(k-1) = \lambda(v-1)$ holds for the parameters of any balanced incomplete block design one counts in two different ways the total number of pairs of points (x,y) with x and y appearing in a common block. In a similar way, even with no block design structure present, we fix $a \in G$ and count in two different ways the total number of pairs (x,y) with $x,y \in G\backslash\{1,a\}$ and with $a,\ x \in Q_0(y)$. We obtain

$$\sum_{x\neq 1,a} \frac{1}{|Q(a)\cap Q(ax)|}\cdot\frac{1}{|Q(x)|} = \left[\sum_{y\in Q_0(a)} \frac{1}{|Q(y)|}\right] - \frac{2}{|Q(a)|}$$

This equation becomes somewhat manageable in two cases. First, when all the $Q(x)$ have the same size for $x \neq 1$ notice that both of the sums in the equation will simplify. After a tedious but elementary examination of the simplified equation we obtain

(A) If for every $x \in G\backslash\{1\}$, all $Q(x)$ have the same size $< 2\sqrt{|G|}$ then F is a generalized Hilbert field .

If we now assume that $\{Q(x)\,|\,x \in G\}$ forms a chain under inclusion then it becomes easier to manage the $|Q(a) \cap Q(ax)|$'s in the equation and we obtain

(B) If $\{Q(x)\,|\,x \in G\}$ forms a chain under inclusion the F is a generalized Hilbert field.

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