A survey of binary factorizations of non-negative integer matrices

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§1. Introduction

Let A be an $m \times n$ matrix with non-negative integer entries. We consider factorizations $A = BC$, where B is $m \times b$ and C is $b \times n$ with entries 0 and 1 only. We say that B and C are binary matrices, and that $A = BC$ is a binary factorization.

Two general problems are

1° To determine the smallest $b = b(A)$ for which there exists a binary factorization of A.

2° To find some structural restrictions on the factors B and C, especially in the extreme case where b is minimal.

We will survey the literature on these problems, making no attempt to be exhaustive. Our emphasis will be on several possible <u>interpretations</u> of $A = BC$, showing the connection with other areas of combinatorics. No proofs are given.

§2. Partitions into rectangles

Given a matrix X, let X_j be the j^{th} column of X. Consider any factorization $A = BC^t$, where A is $m \times n$, B is $m \times r$, C is $n \times r$, and C^t is the transpose of C. Then this factorization may be re-written as

$$A = \sum_{j=1}^{r} B_j C_j^{\,t} \qquad (1)$$

where now each term $B_j C_j^{\,t}$ is an $m \times n$ matrix of very special type: it is either the all-zeroes matrix or has real rank one.

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When B and C are binary, then $B_j C_j^t$ is a "rectangle", that is a binary matrix whose 1's form a rectangular sub-array. When $A = BC^t$, with A, B and C all binary, then (1) has an especially simple and visually appealing interpretation: this is a partition of the set of 1's of A into r rectangles.

This point of view on matrix factorizations is useful, but seems to have been seldom used or even mentioned. (As an aside, I note that Ryser [9, pp. 1-3] points out the connection between certain binary factorizations of the all-ones matrix J and partitions into "connected" rectangles.) In a forthcoming paper [3], David Gregory and myself use this interpretation to derive structural results on binary factorizations of symmetric designs. We recall that a symmetric (v,k,λ)- design is a $v \times v$ binary matrix A such that $AA^t = (k-\lambda)I + \lambda J$. A sample result is that if $k > \lambda^2$ then every (v,k,λ)-design A has only trivial binary factorizations $A = BC$, that is either B or C must be a permutation matrix. Bridges and Ryser [1, p. 442] use the additional hypothesis that $G.C.D.(k,\lambda) = 1$ to derive the same conclusion. Thus, the rectangle interpretation suggests a more general result, with an easy and clear proof.

§3. Bipartite Graphs; the matrix $\bar{I}_n$

There is a standard bijection between bipartite multigraphs and non-negative integer matrices. Given the $m \times n$ matrix A, form the graph $G(A)$ on the vertex-sets $R = \{r_1,\ldots,r_m\}$ and $C = \{c_1,\ldots,c_n\}$ by placing A_{ij} edges between r_i and c_j. Clearly, the bipartite multigraph $G(A)$ completely determines the matrix A. A binary rectangle in A corresponds to a complete bipartite subgraph of $G(A)$ (a "biclique"). Thus, a rectangle partition of A corresponds to a partition of the edge-set of $G(A)$ into bicliques. Orlin [7, p. 418] notes this connection and shows that the computation of $b(A)$ (defined in the introduction) is NP-hard. A recent study of biclique partitions of regular bipartite graphs is by Pullman and Stanford [8].

A lower bound on $b(A)$, which is sometimes tight, is the inequality $b(A) \geq r(A)$, where $r(A)$ is the real rank of A. For example, let $\bar{I}_n$ be the complement of the identity matrix, i.e. $\bar{I}_n$ is the $n \times n$ matrix with zeroes down the main diagonal and ones elsewhere. It is a simple exercise to show that $r(\bar{I}_n) = n$; so $b(\bar{I}_n) \geq n$. Now obviously $b(A) \leq n$ for every $n \times n$ binary matrix A. Thus $b(\bar{I}_n) = n$ for every n. Some methods for estimating $b(A)$ and other exotic "ranks" are discussed in Gregory and Pullman [5].

A somewhat more difficult problem is to classify, in some reasonable fashion, the binary factorizations $\bar{I}_n = BC$, where B and C are $n \times n$; this is a stronger version of problem 2° raised in the introduction. In [2], some progress on this problem is made. For example, it is shown that if $(n-1)$ is prime then the only binary factorizations of $\bar{I}_n$ are trivial, i.e. either B or C must be a permutation matrix. In general, the classification problem seems very difficult; we cannot even classify the factorizations of $\bar{I}_n$ into two circulant binary matrices.

A similar problem, which has a moderately extensive literature, is to classify the circulant binary factorizations of the $n \times n$ all-ones matrix J_n. This is known as the factorization problem for cyclic groups, and has been thoroughly studied by Hajós, de Bruijn, Sands and others. See Hill and Irving [6, especially section 3] for some references to this problem, together with an interesting application to Ramsey numbers.

§4. Directed Graphs

Let A be an $n \times n$ non-negative integer matrix. We may associate with A the directed graph $D(A)$ on the vertex-set $\{v_1,\dots,v_n\}$, where v_i is joined to v_j by A_{ij} directed arcs. A binary rectangle in A corresponds to a complete directed bipartite subgraph of $D(A)$. This relationship is noted by Orlin [7, p. 420] and studied more extensively in [2].

§5. Two applications to symmetric designs

Dinitz and Margolis [4] define a continuous map on an incidence system $(V,\mathcal{B})$ to be a partial mapping $f:V\to V$ such that $f^{-1}(B) \in \mathcal{B} \cup \{\emptyset\}$ for every $B \in \mathcal{B}$. Let us call a continuous map proper if $|f(V)|$ is not $0,1$ or $|V|$. Using results of [4], one can show that a proper continuous map on a symmetric design yields a non-trivial binary factorization of the incidence matrix. We illustrate this by an example.

The matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \qquad (2)$$

which is a $(7,4,2)$-design, has the factorization (i.e. rectangle partition; cf. §2) consisting of a partition of the first three rows of A by three 2×2 rectangles, together with the remaining four row-rectangles of A . A proper continuous map, from which this factorization can be derived, is given by Dinitz and Margolis. (The connection with factorizations is not noted in [4], however.) Using results of [3], some theorems of [4] can be interpreted in a wider context. For example, [4, Cor. 4.9] implies that if k and λ are relatively prime, then a symmetric (v,k,λ)-design has no proper continuous maps. This can be proved easily using [3, Cor. 3.3], which gives a strong structural restriction on partitions into v rectangles of (v,k,λ)-designs with G.C.D.$(k,\lambda)=1$; in particular, the rectangles all have the same dimensions, unlike the factorizations derived from proper continuous maps, as in example (2) above.

Another application to symmetric designs is given in [3], where it is shown that the existence of a geometric line of a certain size in a symmetric design A corresponds to a certain binary factorization of the complement $J - A$. For example, a known result is that the quadratic-residue design $H(q)$, where q is a prime-power congruent to 3 modulo 4 , has no geometric line of size three when $q > 7$; this is equivalent to saying that $J - H(q)$ does not have a binary factorization of a certain sort. This raises an interesting problem. In [3] we conjecture the stronger result that, when $q > 7$, $J - H(q)$ is prime, i.e. has no non-trivial binary factorizations at all. We prove this in [3,Th. 4.5] when $\frac{1}{4}(q+1)$ is an odd prime.

In closing, we recommend Ryser's survey [9], in particular section 5 therein, for further motivation and references in the area of combinatorial matrix factorizations.

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