PRINCIPAL INTERSECTION GRAPH OF COMMUTATIVES RINGS

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ABSTRACT. Let R be a commutative ring. The principal intersection graph of a commutative ring R, noted $G_c(R)$, consist of all proper ideals of R as vertices. Two distinct vertices I and J are adjacent if $I \cap J \neq 0$ and either I or J is a principal (cyclic) ideal. In this paper, we investigate some properties from graph theory of $G_c(R)$ and its algebraic properties where R is a ring.

1. Introduction

The intersection graph of ideals of a ring R is the graph having the set of all ideals as its set of vertices. Two distinct vertices I and J are adjacent if and only if their intersection is non-zero idea and either I or J is a principal (cyclic) ideal. Intersection graph were introduced by Bosak in 1964 [6]. Since, particular intersection graph like small intersection graph, prime intersection graph, semisimple intersection graph are studied respectively in [3, 1, 11, 5, 7]. Recently, several properties of these kinds of graphs were investigated by many authors as Ansari-Toroghy, Nikmehr - Soleymanzadeh and Alwan in 2016; 2017 and 2023 respectively.

In this paper, R is a commutative ring with identity (or eventually a domain). Here, we introduce a particular intersection graph $G_c(R)$ named Principal Intersection Graph, whose set of vertices is the proper ideals of R. We will study the algebraic properties of $G_c(R)$ and also its properties when seen as a graph.

This paper is organized as follow: in the first section, we recall some properties of rings and graph theory. In the second section, we study connectedness, completeness, k-partite and Hamiltonian properties of this intersection graph. We gave a characterization of the connectedness, completeness and Hamiltonian properties of $G_c(R)$ as a principal ideal domain, an Ore domain and a Bezout domain.

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2. Preliminary results

his section, we recall some definitions from ring theory and graph theory.

2.1. Some Properties of Rings.

Definition 2.1.
- *An ideal I of commutative ring R is principal written as $I = aR$ for some $a \in R$, if it is generated by one element.*
- *A ring R is principal if every proper ideal is a principal ideal.*
- *A ring R is an Ore ring if it satisfies the Ore Condition. That is: For all elements a and b in R, $aR \cap bR \neq \{0\}$.*
- *An element a of a ring R is a zero-divisor element if there is $b \neq 0$ such that $ab = 0$.*
- *A ring R is a domain if it has no zero-divisor.*
- *A ring R is an Ore ring if for all elements a and b in R, $aR \cap bR \neq \{0\}$.*
- *A principal ideal domain is a domain such that every ideal of R is a principal ideal.*
- *Bezout ring is a domain in which for any to elements a, $b \in R$, there is $n \geq 0$ such that $Ra^n + Rb^n$ is a principal ideal.*

Example 2.1.
(1) *Every division ring is an Ore ring.*
(2) *The ring of integers $\mathbb{Z}$ is a principal ideal domain.*
(3) *Every principal ideal domain is an Ore domain.*
(4) *Every Ore domain is a Bezout domain.*

2.2. Definitions from Graph Theory.

Definition 2.2.
- *A graph G is pair $(V(G); E(G))$, where $V(G_c(R))$ is the set of vertices of G and $E(G_c(R))$ is the set of edges of $G_c(R))$.*
- *The graph G is empty graph if vertices set $V(G_c(R))$ is empty..*
- *The graph G is null graph if edges set $E(G_c(R))$ is empty..*
- *Let I and J two distinct vertices, $I - J$ means that I and J are adjacent.*
- *The degree of a vertex I of graph $G_c(R))$ which denoted by $deg(I)$ is the number of edges incident on I.*

- *If* $|V(G_c(R)))| > 2$, *a path from* I *to* J *is a sequence of adjacent vertices* $I - I_1 - I_2 - \cdots - I_n - J$, *where* $I_i \in V(G_c(R)$.
- *The length a path graph of a graph is the number of edges in this path.*
- *A path using* k *distinct vertices has length* $k - 1$.
- *The distance between two distinct vertices* I *and* J *is denoted by* $d(I; J)$ *is the length of the shortest path connecting* I *and* J.
- *If there is not a path between* I *and* J, $d(I; J) = 0$.
- *The number of vertices of* $G_c(R)$ *is the order of the graph.*
- *The diameter of a graph* $G_c(R))$ *is* $diam(G_c(R))) = sup\{d(I; J)/I; J \in V(G_c(R)))\}$.
- *A graph* $G_c(R))$ *is connected, if for any vertices* I *and* J *of* $G_c(R))$ *there is a path between* I *and* J.
- *If not,* $G_c(R))$ *is disconnected.*
- *A closed path* $I - I_1 - I_2 - \cdots - I_n - I$ *is a cycle.*
- *The girth of* $G_c(R))$ *is the length of the shortest cycle in* $G_c(R))$.
- *It is denoted by* $g(G_c(R)))$. *If* $G_c(R))$ *has no cycle, the girth of* $G_c(R))$ *is infinite.*
- *A clique of graph* $G_c(R))$ *is complete subgraph of* $G_c(R))$.
- *An independent set (or anticlique) of graph* $G_c(R))$ *is null subgraph of* $G_c(R))$.
- *A maximum clique* C *of graph* $G_c(R))$ *is a clique of* $G_c(R))$ *such that for all vertices* x *of* $G_c(R))$, *the graph induced by* $C \cup \{x\}$ *is not one.*
- *The clique number* $w(G_c(R)))$ *of* $G_c(R))$ *ix the number of vertices of a maximum clique of* $G_c(R))$.
- *A Hamiltonian cycle is a cycle that contains every vertex of the graph.*
- *An Hamiltonian graph is graph containing a Hamiltonian cycle.*
- *A graph with no loop or multiple edges is a simple graph.*

3. CONNECTEDNESS, COMPLETENESS AND HAMILTONIAN GRAPH

Definition 3.1. *Let* R *be a ring. The principal intersection graph* $G_c(R)$ *of* R *is the graph with vertices set is the proper ideals of* R *and two distinct vertices* I *and* J *are adjacent if and only if* $I \cap J \neq \{0\}$ *and either* I *or* J *is a principal ideal.*

We shall prove the following important results for the graph $G_c(R)$.

Proposition 3.1. *Let* R *be a ring.* $G_c(R)$ *is an empty graph if and only if* R *is a field.*

Proof:

- $\Longrightarrow$) It is obvious that if $G_c(R)$ is empty graph then $G_c(R)$ has no vertices. Since vertices of $G_c(R)$ are the proper ideals of R, then R has proper ideal, that is R is a field.
- $\Longleftarrow$) Conversely, if R is a field, R has no proper ideal. $\square$

Lemma 3.2. *If the graph $G_c(R)$ is a null graph, then for all $(a;b) \in R\backslash\{1\} \times R\backslash\{1\}$, $ab = 0$.*

Proof: Assume that $G_c(R)$ is a null graph. Let $a \neq 1$ and $b \neq 1$ be two elements of R. Since $G_c(R)$ is a null graph, then $aR \cap bR = \{0\}$. Therefore, we have that

$$ab \in aR \cap bR.$$

Thus,

$$ab = 0$$

$\square$

Proposition 3.3. *Let R be a commutative nonzero ring. The graph $G_c(R)$ is a null graph if and only if $R = \{0; 1\}$.*

Proof:

- $\Longleftarrow$) It is clear that if $R = \{0; 1\}$, then $G_c(R)$ is a null graph.
- $\Longrightarrow$) Let $1 \neq x \in R$ and $G_c(R)$ a null graph. Take $y \in R$ such that $y \neq 1$ and $1 - y \neq 1$, then $xy = 0$. By Lemma 3.2, $x = x - 0 = x - xy = x(1 - y) = 0$. Since R is commutative nonzero ring, then $R = \{0; 1\}$. $\square$

Example 3.1. *If p is prime integer, the graph $G_c(\mathbb{Z}_p)$ is null graph.*

Lemma 3.4. *If R is a domain, then $G_c(R)$ is a connected.*

Proof: Let I and J to vertices of $G_c(R)$. Since I and J are proper ideals of R, there exists $a \neq 0$ and $b \neq 0$ in I and J, respectively, such that $ab \neq 0$. So, we have $ab \in I \cap J$ implies that $I \cap J \neq 0$. If one of the ideals I or J is principal, then I and J are adjacent. Moreover, I and J are not principal ideals. Since $I \cap J \neq 0$, let $0 \neq c \in I \cap J$ and put

$K = cR$. Then $I \cap K \neq 0$ and $K \cap J \neq 0$. Thus $I - K - J$ is a path between I and J. □

Lemma 3.5. *If R is a domain, every connected graph $G_c(R)$ is complete.*

Theorem 3.6. *Let R be a domain. The followings statements are equivalents:*

(1) *$G_c(R)$ is a connected graph;*
(2) *$G_c(R)$ is a complete graph;*
(3) *R is an Ore domain.*

Proof:

(1) $\Longrightarrow$(2) Follows from Lemma 3.5.
(2) $\Longrightarrow$(3) Let a and b to non-zero elements in R. Put on $I = aR$ and $J = bR$. Since $G_p(R$ is connected, I and J principal ideals, then $I \cap J \neq 0$. Hence $aR \cap bR \neq 0$ and $ab \neq 0$. That is R is an Ore domain.
(3) $\Longrightarrow$ (1) follows from Lemma 3.4. □

Proposition 3.7. *Let R be domain. If $G_c(R)$ is a connected graph, then*

$$diam(G_c(R)) \leq 2.$$

Proof: Let I and J be to vertices of $G_c(R)$.

- If $I \cap J \neq 0$, such that at least of of them is principal, then $I - J$. Thus $d(I, J) = 1$.
- If $I \cap J \neq 0$, I and J both none principal, there are non-zero elements a and b such that $I - cR - J$ with $c = ab$. Thus $d(I, J) = 2$.
- If $I \cap J = 0$ for all nonzero $a \in I$ and $b \in J$, $aR \cap bR = 0$ which contradicts Theorem 3.6.

Hence, $diam(G_c(R)) \leq 2$ □

Proposition 3.8. *Let R be a ring. The following statements are equivalents.*

(1) *$G_c(R)$ is a complete graph;*
(2) *R is essential and R has at most one non-principal ideal.*

Proof:

- (1)$\Longrightarrow$ (2). Assume that $G_c(R)$ is complete and let I be a proper ideal of R. By definition, the vertex I is adjacent to any others vertex, that is I is essential ideal. Then R is essential ring.

  Assume again that R has at least two proper ideals which are not principal. Let I_1 and I_2 be two non- principal ideals of R. The vertices I_1 and I_2 can not be adjacent; that is $G_c(R)$ is not complete. Then R has at most one non-principal ideal.
- (2)$\Longrightarrow$ (1). Let J and K be two vertices of $G_c(R)$. Since R is essential, then $J \cap K \neq 0$. Since R has at most one non-principal ideal, we have two possible cases: either J and K are principal, or exactly one between J and K is principal
  - If J and K are principal, J and K adjacent vertices.
  - If one between J and K is principal, J and K are adjacent vertices. □

  The result follows.

Lemma 3.9. *The graph $G_c(R)$ of a principal ideal domain R is a complete graph.*

Proof: Let I and J two proper ideals of R. Since I and J nonzero ideals, there is non-zero elements a and b in R such that $a \in I$ and $b \in J$. Hence, $0 \neq ab \in aR \cap bR \subseteq I \cap J$ implies that I and J are adjacent. □

Example 3.2. *The graph $G_c(\mathbb{Z})$ is complete because for all n, $m \in \mathbb{Z}$, $n\mathbb{Z} \cap m\mathbb{Z} \neq 0$.*

Corollary 3.10. *If R a field, then $G_c(R[x])$ is a complete graph.*

Lemma 3.11. *The graph $G_c(R)$ of an köthe ring R is a complete graph.*

Proof: Let I and J be two proper ideals of R. Since R is a köthe ring, there is non-zero elements a and b in R such that $a \in I$ and $b \in J$. Hence $aR \cap bR \subset I \cap J$ implies that I and J are adjacent. □

Lemma 3.12. (1) *$G_c(R)$ is a complete graph if and only if R is an essential domain which has at most one non-principal ideal.*
(2) *If R has more than one non-principal ideal, then $G_c(R)$ is a disconnected graph.*

Proof:

(1) If $G_c(R)$ is a complete graph, by proposition 3.8, it has at most one non-principal ideal. For all a and b two non-zero elements in R, $aR \cap bR = abR$. Since $G_c(R)$ is a complete, then $ab \neq 0$. Conversely, if R is an essential domain which has at most one non-principal ideal, then $G_c(R)$ is complete by proposition 3.8.

(2) It is clear that two non-principal ideals of R can not be adjacent vertices of the graph $G_c(R)$. □

Theorem 3.13. *Let R be a Bezout ring. The followings statements are equivalents:*

(1) *$G_c(R)$ is a complete graph;*

(2) *R is a principal ideal domain.*

Proof:

(1) $\Longrightarrow$(2). Since $G_c(R)$ is a complete graph, there is at most one non-principal ideal. Let I this ideal. For all $a_1 \in I$, $I_1 = Ra_1$ is adjacent I, that is $I_1 \cap I \neq 0$. If $I = I_1$, I is principal. Otherwise, there exist $a_2 \in I \backslash I_1$ and let $I_2 = Ra_1 + Ra_2$. If $I \neq I_2 = Ra_1 + Ra_2$, there exists $a_3 \in I \backslash I_2$ and let $I_3 = Ra_1 + a_2 + Ra_3$. Inductively, let $I_n = Ra_1 + \cdots + Ra_n$. If $I \neq I_n$, we choose $a_{n+1} \in I \backslash I_n$. Since $G_c(R)$ is complete, the chain $I_1 - I_2 - \cdots - I_n$ must be finite. Moreover, the ideal $I_n = Ra_1 + \cdots + Ra_n$ is principal because R is Bezout domain. This is a contradiction. Then R is a principal ideal domain.

(2) $\Longrightarrow$(1) follows from Lemma 3.9 □

Corollary 3.14. *Let R be a Bezout domain. The followings statements are equivalents:*

(1) *$G_c(R)$ is a complete graph;*

(2) *R is a principal ideal domain.*

(3) *For two ideals I and J, $I \cap J = 0$ implies $I = 0$ or $J = 0$.*

(4) *For all $(a; b) \in R^2$, $aR \cap bR = 0$ implies $a = 0$ or $b = 0$.*

(5) *Every non-zero ideal of R is indecomposable.*

Remark 3.1. (1) *If R is Bezout domain, for all vertices $I_1, I_2, \cdots, I_n \in G_c(R)$, $I_1 - I_2 - \cdots - I_n - I_1$ is a cycle.*

(2) *$girth(G_c(R)) = 3$*

Proposition 3.15. *If R is Bezout domain, N and K two vertices of $G_c(R)$ such that $K \subset N$, then $deg(K) \leq deg(N)$.*

Proof: Let N and K two vertices of $G_c(R)$ such that $K \subset N$. If J is another vertex of $G_c(R)$ then $J \cap K \neq 0$. Since R is Bezout principal ideal domain and $J \cap K \subset J \cap J \cap N$, then $J \cap N \neq 0$. □

Theorem 3.16. *The followings statements are equivalents in a Bezout domain R.*

(1) *$G_c(R)$ is a complete;*
(2) *R is an integral domain and has at most one non-principal ideal ;*
(3) *R is a principal ideal domain.*

Proof:

- (1)$\Longleftrightarrow$ (2) follows from Lemma 3.12 (1)
- (3)$\Longleftrightarrow$ (1) follows from Theorem 3.13. □

Corollary 3.17. *The graph $G_c(R)$ of a Bezout domain is a regular graph.*

Proof: Since R is a Bezout domain, $G_c(R)$ is complete in view of Theorem 3.16; Then $G_c(R)$ is regular graph. □

Proposition 3.18. *If R is an Ore domain with k proper principal ideals, then the clique number*

$$w(G_c(R)) = k$$

.

Proof: Let I_1, I_2, ..., I_k the k proper principal ideals, $I_{k+1}, I_{k+2}, \ldots, I_n$ the $n-k$ proper non-principal ideals of R. Since R is an Ore domain, for every vertex I_i for $i \in \{1, 2, \ldots, k\}$ $I_i \cap I_j \neq 0$ for $j > k$. That is I_i for $i \in \{1, 2, \ldots, k\}$ adjacent to each other vertex in the graph. Then the graph induced by the path $\{I_{k+1}, I_{k+2}, \ldots, I_n\}$ is complete. Thus $w(G_c(R)) = k$. □

Here we recall a result from [5].

Lemma 3.19. *(Ore Theorem) If G is a simple graph with order $n \geq 3$ and $deg(v) + deg(w) \geq n$ for each pair of non-adjacent vertices v and w, then G is a Hamiltonian graph.*

Theorem 3.20. *Let R a domain with k proper non-principal ideals and k' proper principal ideals of R. The followings statements are equivalents.*

(1) $G_c(R)$ *is a simple graph and* $k' \geq k$*;*

(2) $G_c(R)$ *is Hamiltonian graph.*

Proof:

- (1) $\Longrightarrow$ (2). The order of $G_c(R)$ is $n = k' + k$. Let L_k the set of non-principal ideals, L'_k the set of principal ideals, (I, J) a pair of non-adjacent vertices of $G_c(R)$. Three cases are possibles:
  - case 1: If I and J are non - principal ideals, then $deg(I) + deg(J) = 2k' \geq k' + k = n$.
  - case 2: If I and J are principal ideals, then $deg(I) + deg(J) = 2(n-1) \geq k' + k = n$.
  - case 3: If exactly one ideal between I or J is principal, $deg(I) + deg(J) = k' + 2(n-1) \geq n$.

  By the previous Lemma 3.19, $G_c(R)$ is a Hamiltonian graph.
- (2) $\Longrightarrow$ (1). Assume that $G_c(R)$ is not a simple graph or $k' < k$;
  - If $G_c(R)$ is not a simple graph clearly $G_c(R)$ is not Hamiltonian graph.
  - If $k' < k$, since $n = k + k'$ there is no cycle containing every vertex of $G_c(R)$. That is, there is no Hamiltonian cycle. Then $G_c(R)$ is not Hamiltonian. □

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