A new class of maximal partial line spreads in $PG(3,q)$, q even

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Abstract

In this work we construct many new examples of maximal partial line spreads in $PG(3,q)$, q even. We do this by giving a suitable representation of $PG(3,q)$ in the non-singular quadric $Q(4,q)$ of $PG(4,q)$. We prove the existence of maximal partial line spreads of sizes $q^2-q+1-\bar{t}\bar{z}$, for every pair $(\bar{t},\bar{z}) \in \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the pair sets $\mathcal{P}_1 = \{(t,z) \in \mathbb{Z}\times\mathbb{Z} : \frac{q}{2}-2 \le t \le q-3, 0 \le z \le \frac{q}{2}-2\}$ and
$\mathcal{P}_2 = \{(t,z) \in \mathbb{Z}\times\mathbb{Z} : 0 \le t \le \frac{q}{2}-3, 0 \le z \le q-1\}$, for $q \ge 8$.

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1 Introduction

Let $PG(n,q)$ denote the n-dimensional projective space over the finite field of order q. A *partial line spread* in $PG(3,q)$, is a set of pairwise disjoint lines. A *line spread* in $PG(3,q)$ is a partial line spread in $PG(3,q)$ covering the space. A *maximal partial line spread* in $PG(3,q)$ is a partial line spread in this space which cannot be extended to a larger partial line spread. Many authors have investigated maximal partial line spreads in $PG(3,q)$, but the complete knowledge of them is still far away, especially in the case q even. The aim of this work is to find new examples of maximal partial line spreads in $PG(3,q)$, with q even. To this end we call *regulus* of the non-singular quadric $Q(4,q)$ of $PG(4,q)$ a regulus of a hyperbolic quadric hyperplane section of $Q(4,q)$. Also, for every point V of $Q(4,q)$, we call *lined tangent cone of vertex V of* $Q(4,q)$ the set of all the lines of $Q(4,q)$ through V. As well known, the union of these lines is the tangent cone of vertex V of $Q(4,q)$.
In order to construct our maximal partial line spreads, first we transfer the

whole geometry of $PG(3,q)$ over the non-singular quadric $Q(4,q)$. More precisely we get the following mapping. The points of $PG(3,q)$ are the lines of $Q(4,q)$, and the lines of $PG(3,q)$ are the lined tangent cones and the reguli of $Q(4,q)$. Also, each plane of $PG(3,q)$ is the set of all the lines of $Q(4,q)$ meeting a fixed line of this quadric, and viceversa. Secondly, we consider the non-singular quadric $Q(4,q)$ of $PG(4,q)$, with q even and $q \geq 8$, an elliptic quadric $\mathcal{E}$, hyperplane section of $Q(4,q)$, and a suitable collection of non-singular conics over the quadric $\mathcal{E}$. Through the quadric $\mathcal{E}$ and through the mentioned collection of non-singular conics, we construct a set $\mathcal{F}$ of lined tangent cones and reguli of $Q(4,q)$ such that any two distinct elements of $\mathcal{F}$ have no common line, and such that every lined tangent cone and every regulus of $Q(4,q)$ has a line in common with an element of $\mathcal{F}$. So $\mathcal{F}$ is a maximal partial line spread in $PG(3,q)$, q even and $q \geq 8$, by means of the above mapping of $PG(3,q)$ over $Q(4,q)$. Also we get:

$$|\mathcal{F}| = q^2 - q + 1 - \bar{t}\bar{z},$$

for every pair $(\bar{t},\bar{z}) \in \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the following pair sets:

$$\mathcal{P}_1 = \left\{(t,z) \in \mathbb{Z} \times \mathbb{Z} : \frac{q}{2} - 2 \leq t \leq q-3, 0 \leq z \leq \frac{q}{2} - 2\right\},$$
$$\mathcal{P}_2 = \left\{(t,z) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq t \leq \frac{q}{2} - 3, 0 \leq z \leq q-1\right\}.$$

By this, we get many new values for the sizes of maximal partial line spreads in $PG(3,q)$, with q even, as immediately follows by the state of art presented in section 2.

2 Known results about maximal partial line spreads in $PG(3,q)$, q even

In this section we produce all known results about maximal partial line spreads in $PG(3,q)$, with q even.
In [3] A.A. Bruen proved the following result (also for q odd):

Theorem 2.1. *If $\mathcal{S}$ is a maximal partial line spread in $PG(3,q)$, other than a line spread, we have:*

$$q + \sqrt{q} + 1 \leq |\mathcal{S}| \leq q^2 - \sqrt{q}.$$

The upper bound was given by D.M. Mesner (see [11]) and later by A.A. Bruen,

by using blocking sets theory. Afterwards, in [5] A.A. Bruen and J.A. Thas improved the previous result, ruling out the equal sign on the left.
D.G. Glynn in [7] proved the following result (also for q odd):

Theorem 2.2. *If S is a maximal partial line spread in $PG(3,q)$, then*

$$|S| \geq 2q.$$

In [13] L.H. Soicher gave a complete classification of maximal partial line spreads in $PG(3,4)$, through a computer search. The spectrum of their sizes is the set $\{11,12,13,14,17\}$.
In [3], A.A. Bruen proved the existence of a maximal partial line spread S in $PG(3,q)$ (also for q odd), with

$$|S| = q^2 - q + 1, \quad q > 2.$$

In [5], A.A. Bruen and J.A. Thas constructed a maximal partial line spread S in $PG(3,q)$, with

$$|S| = q^2 - q + 2, \quad q = 2^{2h+1},$$

h integer, $h \geq 1$.
In [6], J.W. Freeman constructed a maximal partial line spread S in $PG(3,q)$, with

$$|S| = q^2 - q + 2, \quad q = 2^{2h},$$

h integer, $h \geq 1$.
In [8], A. Gács and T. Szőnyi constructed a maximal partial line spread S in $PG(3,q)$ (also for q odd), of size

$$|S| = cq + 1,$$

for every integer c satisfying the condition

$$6 \ln q + 1 \leq c \leq q.$$

In [2], A. Beutelspacher showed that in $PG(3,q)$ (also for q odd) there is a maximal partial line spread S of size

$$|S| = q^2 + 1 - nq,$$

for every integer n satisfying the condition

$$0 \leq n \leq \frac{1}{2}q - 1.$$

In [12], S. Rajola and M. Scafati Tallini, showed that in $PG(3,q)$, q even and $q \geq 8$, there is a maximal partial line spread $\mathcal{S}$ of size

$$|\mathcal{S}| = q^2 - 2nq + 2n + 1,$$

for every integer n satisfying the condition

$$0 \leq n < \min\left\{\frac{q-1}{4}, \frac{1+\sqrt{2q-1}}{2}\right\}.$$

In [10], D. Jungnickel and L. Storme proved the existence of a maximal partial line spread in $PG(3,q)$, q even and $q \geq 4$, such that

$$|\mathcal{S}| = q^2 - q.$$

In [4], A.A. Bruen and J.W.P. Hirschfeld showed that in $PG(3,q)$, with $(q+1,3) = 1$ (and also for q odd), there is a maximal partial line spread $\mathcal{S}$ such that

$$|\mathcal{S}| = \frac{q^2+q+2}{2}.$$

In [1], J. Bárat, A. Del Fra, S. Innamorati and L. Storme proved that 58 is the largest size for a maximal partial line spread, other than a line spread, in $PG(3,8)$.
Finally, in [9], M. Iurlo and S. Rajola, by a computer search, found new minimums and new density results for the sizes of maximal partial line spreads in $PG(3,q)$, with $q = 8, 16, 32, 64$. More precisely they found the minimum size 30 (previous 41) in $PG(3,8)$, the minimum size 87 (previous 145) in $PG(3,16)$, the minimum size 238 (previous 545) in $PG(3,32)$, the minimum size 623 (previous 1665) in $PG(3,64)$, the density result $31-55$ (previous $56-58$) in $PG(3,8)$, and the density results $88-221, 225-231$ (previous $240-242$) in $PG(3,16)$.

3 On the non-singular quadric $Q(4,q)$ of $PG(4,q)$

Let $Q(4,q)$ be the non-singular quadric of $PG(4,q)$. The quadric $Q(4,q)$ contains no plane. Also, in the case q even, such a quadric admits a nucleus. For every point V of $Q(4,q)$, we denote by Γ_V the tangent cone of vertex V of $Q(4,q)$, i.e. the point set $\Gamma_V = \mathcal{S} \cap Q(4,q)$, where $\mathcal{S}$ is the tangent hyperplane to $Q(4,q)$ at the point V. We denote by Γ'_V the lined tangent cone of vertex V of $Q(4,q)$, that is the set of lines contained in Γ_V and set

$$\Gamma' = \{\Gamma'_V\}_{V \in Q(4,q)} \, .$$

Evidently, if Γ'_V and Γ'_Z are two distinct cones of Γ', then we have either $\Gamma'_V \cap \Gamma'_Z = \emptyset$, or $\Gamma'_V \cap \Gamma'_Z = \{r\}$, with r line of $Q(4,q)$. A hyperplane S_3 of $PG(4,q)$ meets $Q(4,q)$ at an elliptic quadric $\mathcal{E}$, or at a hyperbolic quadric I, or at a tangent cone Γ_V. In particular we have $S_3 \cap Q(4,q) = \Gamma_V$ if and only if S_3 is the tangent hyperplane to $Q(4,q)$ at the point V. Also, for every line r of $Q(4,q)$, the following statements hold:

i) r is tangent to $\mathcal{E}$ (i.e. r has exactly one point in common with $\mathcal{E}$),
ii) either r is tangent to I, or r is contained in I,
iii) either r is tangent to Γ_V, or r is contained in Γ_V.

Furthermore, it is easy to prove the following lemmas.

Lemma 3.1. *The set $\mathcal{E} \cap I$ is a non-singular conic.*

Lemma 3.2. *The set $\Gamma_V \cap \mathcal{E}, V \notin \mathcal{E}$, is a non-singular conic.*

Lemma 3.3. *In the case q even, a hyperplane S_3 of $PG(4,q)$ contains the nucleus of $Q(4,q)$ if and only if S_3 is tangent to $Q(4,q)$.*

In [14] the following result is proved.

Theorem 3.4. *Let α be a plane of $PG(4,q)$, q even, meeting $Q(4,q)$ at a non-singular conic. If the plane α does not contain the nucleus N of $Q(4,q)$, then the hyperplane through α and N is tangent to $Q(4,q)$. Also, there are $q/2$ hyperplanes through α meeting $Q(4,q)$ at elliptic quadrics and $q/2$ at hyperbolic quadrics. If α contains N, then every hyperplane through α is tangent to $Q(4,q)$.*

Now let $\mathcal{I}$ be the set of all hyperbolic quadrics, hyperplane sections of $Q(4,q)$, with q even. For every $I \in \mathcal{I}$, let I_1 and I_2 be the two reguli of I, i.e., for $n \in \{1,2\}$, I_n is a set of lines contained in I which partitions the points of I. Let C be a non-singular conic, plane section of $Q(4,q), C = \pi \cap Q(4,q)$, where π is a plane of $PG(4,q)$ not through the nucleus N of $Q(4,q)$. Also, let $\mathcal{R}(C)$ be the following set:

$$\mathcal{R}(C) = \{I_n | n \in \{1,2\} : C \subset I\} \, . \tag{1}$$

By (1) and by theorem 3.4 we get

$$|\mathcal{R}(C)| = q. \tag{2}$$

Let us prove the following lemma.

Lemma 3.5. *Let $\mathcal{E}$ be an elliptic quadric hyperplane section of $Q(4,q)$, q even, $\mathcal{E} = \mathcal{S}_3 \cap Q(4,q)$, $\mathcal{S}_3$ hyperplane of $PG(4,q)$. Let $\bar{C}$ be a non-singular conic, plane section of $\mathcal{E}$. Then there is one and only one tangent cone $\Gamma_{\bar{V}}$ of $Q(4,q)$, such that $\Gamma_{\bar{V}} \cap \mathcal{E} = \bar{C}$.*

Proof. Evidently, we have $\bar{C} = \bar{\gamma} \cap \mathcal{E} = \bar{\gamma} \cap Q(4,q)$, $\bar{\gamma}$ plane of $\mathcal{S}_3$. The hyperplane $\mathcal{S}_3$ does not contain the nucleus N of $Q(4,q)$, since $\mathcal{S}_3 \cap Q(4,q) = \mathcal{E}$ and lemma 3.3. holds. By $N \notin \mathcal{S}_3$ and $\bar{\gamma} \subset \mathcal{S}_3$ it follows that $N \notin \bar{\gamma}$. By $N \notin \bar{\gamma}$, by $\bar{C} = \bar{\gamma} \cap Q(4,q)$, by the fact that $\bar{C}$ is a non-singular conic and by theorem 3.4, it follows that there is one and only one tangent hyperplane to $Q(4,q)$ through $\bar{\gamma}$. This hyperplane meets $Q(4,q)$ at a tangent cone, that we call $\Gamma_{\bar{V}}$. Obviously we have $\Gamma_{\bar{V}} \supset \bar{C}$ and therefore $\bar{V} \notin \mathcal{E}$. By $\bar{V} \notin \mathcal{E}$, $\Gamma_{\bar{V}} \supset \bar{C}$ and by lemma 3.2, we get $\Gamma_{\bar{V}} \cap \mathcal{E} = \bar{C}$. Also, there is no tangent cone of $Q(4,q)$, distinct from $\Gamma_{\bar{V}}$, meeting $\mathcal{E}$ at the conic $\bar{C}$, since there is a unique tangent hyperplane to $Q(4,q)$ through $\bar{\gamma}$. So the lemma is proved. □

Let us prove the following theorem.

Theorem 3.6. *Let $\mathcal{E}$ be an elliptic quadric hyperplane section of $Q(4,q)$, q even. Let C and C' be two non-singular conics, plane sections of $\mathcal{E}$, and let Γ_V and Γ_Z be the tangent cones of $Q(4,q)$ such that $\Gamma_V \cap \mathcal{E} = C$, $\Gamma_Z \cap \mathcal{E} = C'$, according to lemma 3.5. Then we get what follows. If C and C' are distinct conics and Γ'_V and Γ'_Z have a common line r, then we have $C \cap C' = \{A\}$, A common point to r and $\mathcal{E}$. Viceversa, if $C \cap C' = \{A\}$, A point of $\mathcal{E}$, then the line of Γ'_V through A coincides with the line of Γ'_Z through A, and this line is the only common line to Γ'_V and Γ'_Z.*

Proof. Assume $C \neq C'$ and suppose a common line r to Γ'_V and Γ'_Z exists. Also, denote by A the common point to r and $\mathcal{E}$. Evidently, we have $A \in C \cap C'$. Let us prove that $C \cap C' = \{A\}$. To do this, let B be a point of $C \cap C'$ distinct from A. By $C \neq C'$ it follows that $\Gamma_V \neq \Gamma_Z$ and therefore that $V \neq Z$. By $r \in \Gamma'_V$ and $r \in \Gamma'_Z$ we get $V \in r$ and $Z \in r$. So V and Z are two distinct points of $r - \{A\}$. The line u through V and B is a line of $Q(4,q)$, since u is a line of Γ'_V. The line u' through Z and B is a line of $Q(4,q)$, too, since u' is a line of Γ'_Z. Therefore the lines r, u and u' are three lines of $Q(4,q)$ forming a triangle, and the plane through them is contained in $Q(4,q)$. A contradiction, since $Q(4,q)$ contains no plane. The contradiction proves that $C \cap C' = \{A\}$.

Viceversa, suppose $\mathcal{C} \cap \mathcal{C}' = \{A\}$. Let r be the line of Γ'_V through A, and r' the line of Γ'_Z through A. Let γ be the plane of $\mathcal{C}$ and τ_V the tangent hyperplane to $Q(4,q)$ at the point V. Then we have $\Gamma_V = \tau_V \cap Q(4,q)$ and $\gamma \subset \tau_V$. Assume $r \neq r'$. Then we get $r' \not\subset \tau_V$ and therefore $r' \cap \gamma = \{A\}$. Now let $\tilde{S}_3$ be the hyperplane through γ and r'. The hyperplane $\tilde{S}_3$ is distinct from $\tau_V(\tilde{S}_3 \supset r', \tau_V \not\supset r')$ and meets $Q(4,q)$ at a quadric, that we call I. Such a quadric is not an elliptic quadric, since $r' \subset I$. Also, the quadric I is not a tangent cone. To show this, suppose $I = \Gamma_T$, Γ_T tangent cone of $Q(4,q)$ of vertex T. Then $\tilde{S}_3$ is the tangent hyperplane to $Q(4,q)$ at the point T. Furthermore we have $\mathcal{C} \subset \Gamma_T$, $T \notin \mathcal{E}$ and $\mathcal{C} \subset \mathcal{E}$, and therefore $\mathcal{C} \subseteq \Gamma_T \cap \mathcal{E}$, $T \notin \mathcal{E}$. By this and by lemma 3.2 we get $\mathcal{C} = \Gamma_T \cap \mathcal{E}$. Also we have $\Gamma_V \neq \Gamma_T$, since $\tilde{S}_3 \neq \tau_V$. So Γ_V and Γ_T are two distinct tangent cones of $Q(4,q)$ such that $\Gamma_V \cap \mathcal{E} = \Gamma_T \cap \mathcal{E} = \mathcal{C}$: a contradiction, since lemma 3.5 holds. The contradiction proves that I is not a tangent cone. It follows that I is a hyperbolic quadric. We have $\mathcal{C} \subset I$, $\mathcal{C} \subset \mathcal{E}$ and so $\mathcal{C} \subseteq I \cap \mathcal{E}$. By this and by lemma 3.1 we get $\mathcal{C} = I \cap \mathcal{E}$. The line r', which is a line contained in I, is a line of a regulus R of I. Now let v be the line through Z of the regulus of I opposite to R and let E be the common point to v and $\mathcal{E}$. Obviously we have $E \neq A$. By $I \cap \mathcal{E} = \mathcal{C}$ it follows that $E \in \mathcal{C}$. Also, the line v is a line of Γ'_Z, since v is a line of $Q(4,q)$ through Z. By this we get $E \in \mathcal{C}'$. Thus, E is a common point to $\mathcal{C}$ and $\mathcal{C}'$ distinct from A: a contradiction, since $\mathcal{C} \cap \mathcal{C}' = \{A\}$. The contradiction proves that $r = r'$. In addition to this, by $r = r'$ and by the fact that Γ'_V and Γ'_Z are two distinct lined tangent cones of $Q(4,q)$ ($\Gamma'_V \neq \Gamma'_Z$ since $\mathcal{C} \neq \mathcal{C}'$), it follows that the line $r = r'$ is the only common line to Γ'_V and Γ'_Z. So the theorem is proved. □

4 On the elliptic quadric $\mathcal{E}$ of $PG(3,q)$, q even

Let $\mathcal{E}$ be an elliptic quadric of $PG(3,q)$, q even, let Ω_1 and Ω_2 be two distinct points of $\mathcal{E}$ and let ω be the line through Ω_1 and Ω_2. Let α and β be the tangent planes to $\mathcal{E}$ at the points Ω_1 and Ω_2 respectively, and l the line $l = \alpha \cap \beta$. Evidently we have $l \cap \mathcal{E} = \emptyset$ and $l \cap \omega = \emptyset$. Let $\pi_1, ..., \pi_{q-1}$ be the planes through l distinct from α and β. Each of such planes meets $\mathcal{E}$ at a non-singular conic. So, for every index $i = 1, ..., q-1$, let $\mathcal{C}_i$ be the non-singular conic $\mathcal{C}_i = \pi_i \cap \mathcal{E}$. Furthermore, let $\pi'_1, ..., \pi'_{q+1}$ be the planes through ω. Each of such planes meets $\mathcal{E}$ at a non-singular conic. So, for every index $j = 1, ..., q+1$, let $\mathcal{C}'_j$ be the non-singular conic $\mathcal{C}'_j = \pi'_j \cap \mathcal{E}$. Let us prove the following theorem.

Theorem 4.1. *For every index $i = 1, ..., q-1$, and for every index $j = 1, ..., q+1$, we have $|\mathcal{C}_i \cap \mathcal{C}'_j| = 1$.*

Proof. Let $\mathcal{C}_i$ and $\mathcal{C}'_j$ be the non-singular conics $\mathcal{C}_i = \pi_i \cap \mathcal{E}$, $\mathcal{C}'_j = \pi'_j \cap \mathcal{E}$, with

$i \in \{1, ..., q-1\}$, $j \in \{1, ..., q+1\}$. Since q is even, the quadric $\mathcal{E}$ determines a null polarity p. Let C_p be the general linear complex determined by p, i.e. the set of isotropic lines of p. As well known, each point of $PG(3,q)$ belongs to its polar plane and C_p consists of the tangent lines to $\mathcal{E}$. For every point X of $PG(3,q)$, we denote by $p(X)$ the polar plane of X with respect to p. Evidently, we have $p(\Omega_1) = \alpha$ and $p(\Omega_2) = \beta$. By this it follows that the line l is the polar line of ω, and viceversa. The pole N_j of π'_j, which is a point of π'_j, is also a point of l, since π'_j is a plane through ω and l is the polar line of ω. So we get $\{N_j\} = \pi'_j \cap l$. Similarly, the pole M_i of π_i which is a point of π_i, is also a point of ω, since π_i is a plane through l and ω is the polar line of l. So we get $\{M_i\} = \pi_i \cap \omega$. Evidently, the distinct points M_i and N_j are conjugated points in p, and therefore the line $r = \pi_i \cap \pi'_j$, which is the line through M_i and N_j, is a line of C_p. So the line r is tangent to $\mathcal{E}$ at a point X, and we get $r \cap \mathcal{E} = \{X\}$. Clearly we have $X \in \pi_i$, $X \in \pi'_j$ and $X \in \mathcal{E}$ and therefore $X \in \mathcal{C}_i \cap \mathcal{C}'_j$. Let us prove that $\mathcal{C}_i \cap \mathcal{C}'_j = \{X\}$. To show this, let Y be a point of $\mathcal{C}_i \cap \mathcal{C}'_j$ distinct from X. By $Y \in \mathcal{C}_i \cap \mathcal{C}'_j$ we get $Y \in \pi_i$, $Y \in \pi'_j$ and $Y \in \mathcal{E}$ and so $Y \in r \cap \mathcal{E}$: a contradiction, since $r \cap \mathcal{E} = \{X\}$ and $Y \neq X$. The contradiction proves that $\mathcal{C} \cap \mathcal{C}' = \{X\}$. So the theorem is proved. □

By the construction of $\mathcal{C}_1, ..., \mathcal{C}_{q-1}$, $\mathcal{C}'_1, ..., \mathcal{C}'_{q+1}$ and by theorem 4.1 we get what follows:

(a)	$\Omega_1 \notin \mathcal{C}_i,\ \Omega_2 \notin \mathcal{C}_i$	$\forall i = 1, ..., q-1,$	
(b)	$\mathcal{C}_{i_1} \cap \mathcal{C}_{i_2} = \emptyset$	$\forall i_1, i_2 = 1, ..., q-1,$	$i_1 \neq i_2,$
(c)	$\mathcal{C}'_{j_1} \cap \mathcal{C}'_{j_2} = \{\Omega_1, \Omega_2\}$	$\forall j_1, j_2 = 1, ..., q+1,$	$j_1 \neq j_2,$
(d)	$\lvert \mathcal{C}_i \cap \mathcal{C}'_j \rvert = 1$	$\forall i = 1, ..., q-1, \forall j = 1, ..., q+1.$	

The non-singular conics $\mathcal{C}_1, ..., \mathcal{C}_{q-1}$, $\mathcal{C}'_1, ..., \mathcal{C}'_{q+1}$, that are univocally determined by $\mathcal{E}$, Ω_1 and Ω_2, will be referred to as *associated conics with* $\mathcal{E}$, Ω_1, Ω_2.

5 The mapping of $PG(3,q)$ over $Q(4,q)$

We remark that the content of this section is also in [12]. Let $Q(5,q)$ be the Klein quadric of $PG(5,q)$. Let $Q(4,q)$ be the non-singular quadric hyperplane section of $Q(5,q)$, $Q(4,q) = Q(5,q) \cap S_4$, S_4 hyperplane of $PG(5,q)$. Also, let $\mathcal{L}$ be the set of lines of $Q(4,q)$. A *linear complex* of $PG(3,q)$ is a set of lines whose Plücker coordinates p_{ij}, $i,j = 0,1,2,3$, $i < j$, satisfy a linear equation, that is a hyperplane section of the Klein quadric. A *general linear* complex is a linear complex with equation

$$\sum_{i<j} a_{ij} p_{ij} = 0, \tag{3}$$

under the condition that $||a_{ij}||$, with $a_{ji} = -a_{ij}$ and $a_{ii} = 0$, is a non-singular matrix. Since $Q(4,q)$ is a hyperplane section of $Q(5,q)$, the points of $Q(4,q)$ represent the lines of a linear complex $\mathcal{B}$ of $PG(3,q)$. Also, $\mathcal{B}$ is a general linear complex with equation (3), since $Q(4,q)$ is a non-singular quadric. Now, let ψ be the Klein mapping, that is the bijection:

$$\psi : r(p_{ij}) \longrightarrow (p_{ij}) \in Q(5,q),\ i < j \ and\ i,j = 0,1,2,3,$$

where $r(p_{ij})$ denotes the line r of $PG(3,q)$ of Plücker coordinates p_{ij}. The general linear complex $\mathcal{B}$ determines the null polarity f of $PG(3,q)$ which associates a point $\overline{x}(\overline{x_0}, \overline{x_1}, \overline{x_2}, \overline{x_3})$ of $PG(3,q)$ with its polar plane $f(\overline{x})$ with equation

$$\sum_{i=0}^{3}\left(\sum_{j=0}^{3} a_{ij}\overline{x_j}\right) x_i = 0.$$

As well known, the point $\overline{x}$ belongs to $f(\overline{x})$ and the lines of $\mathcal{B}$ through $\overline{x}$ constitute the line pencil, of centre $\overline{x}$, lying in $f(\overline{x})$. Now let x be a point of $PG(3,q)$ and F_x the pencil of lines of $\mathcal{B}$ through x; the set $\psi(F_x)$ is a line s of $Q(4,q)$. Let φ be the following bijection:

$$\varphi : x \in PG(3,q) \longrightarrow s \in \mathcal{L}.$$

So φ sends the points of $PG(3,q)$ to the lines of $Q(4,q)$.
Now let us represent the lines of $PG(3,q)$ over $Q(4,q)$. To do this, first we consider the lines of $\mathcal{B}$. Let r be a line of $\mathcal{B}$ and let $x_1, x_2, ..., x_{q+1}$ be the points of r. The lines $\varphi(x_1), \varphi(x_2), ..., \varphi(x_{q+1})$ of $Q(4,q)$ are all distinct and all of them contain the point $\psi(r)$. Therefore $\varphi(r)$ is the lined tangent cone of $Q(4,q)$ of vertex $\psi(r)$. Viceversa, it is easy to prove that, for every lined tangent cone Γ'_V of $Q(4,q)$, there is a line r of $\mathcal{B}$ such that $\varphi(r) = \Gamma'_V$. It follows that φ sends the lines of $\mathcal{B}$ to the lined tangent cones of $Q(4,q)$. Now let r be a line of $PG(3,q)$. The line r', the polar line of r under f, is the axis of the pencil of polar planes of the points of r. In particular it follows that the polar line of a line $r \in \mathcal{B}$, under f, coincides with r. Now let us consider a line r of $PG(3,q)$ not of $\mathcal{B}$ and denote by r' the polar line of r under f. Obviously we get $r \cap r' = \emptyset$. Let $x_1, x_2, ..., x_{q+1}$ be the points of r and let $x'_1, x'_2, ..., x'_{q+1}$ be the points of r'. The lines $\varphi(x_1), \varphi(x_2), ..., \varphi(x_{q+1})$ are mutually disjoint, and the same holds for the lines $\varphi(x'_1), \varphi(x'_2), ..., \varphi(x'_{q+1})$. Every line $\varphi(x'_j)$, $j = 1, ..., q+1$, meets all the lines $\varphi(x_1), \varphi(x_2), ..., \varphi(x_{q+1})$. The hyperplane $\overline{S_3}$ of S_4 through $\varphi(x_1)$ and $\varphi(x_2)$ contains all the lines $\varphi(x'_j)$ and all the lines $\varphi(x_j)$, $j = 1, ..., q+1$. It follows that $\overline{S_3}$ meets $Q(4,q)$ at the hyperbolic quadric the reguli of which are $\varphi(r)$ and $\varphi(r')$. So the line set $R = \varphi(r) = \{\varphi(x_1), \varphi(x_2), ..., \varphi(x_{q+1})\}$ is a regulus of $Q(4,q)$. Viceversa, it is easy to prove that for every regulus R of

$Q(4,q)$ there is a line r of $PG(3,q)$, not of $\mathcal{B}$, such that $\varphi(r) = R$. So φ sends the lines of $PG(3,q)$ not of $\mathcal{B}$ to the reguli of $Q(4,q)$.

Finally, taking into account that φ sends the lines of $\mathcal{B}$ to the lined tangent cones of $Q(4,q)$, it is easy to prove what follows. For every plane π of $PG(3,q)$, the line set $\varphi(\pi)$ consists of all the lines of $Q(4,q)$ meeting a fixed line l of $Q(4,q)$, and the point $\varphi^{-1}(l)$ of $PG(3,q)$ is the pole of π with respect to the null polarity f. Viceversa, for every line l of $Q(4,q)$, there is a plane π of $PG(3,q)$ such that $\varphi(\pi)$ is the set of all the lines of $Q(4,q)$ meeting l. So the following theorem holds.

Theorem 5.1. *The Galois space $PG(3,q)$ is mapped over $Q(4,q)$ as follows:*
The points of $PG(3,q)$ are the lines of $Q(4,q)$.
The lines of $PG(3,q)$ are the lined tangent cones and the reguli of $Q(4,q)$.
A plane π of $PG(3,q)$ is the set of all the lines of $Q(4,q)$ meeting a fixed line l of $Q(4,q)$, and viceversa.

More precisely, the lined tangent cones of $Q(4,q)$ are the lines of a general linear complex $\mathcal{B}$ of $PG(3,q)$, and the other lines of $PG(3,q)$ are the reguli of $Q(4,q)$. Furthermore, in the third statement of theorem 5.1, the line l is the pole of π with respect to the null polarity determined by $\mathcal{B}$. In what follows, we will use only the first two statements of theorem 5.1. The third one has been mentioned for completeness reasons.

6 Construction of maximal partial line spreads in $PG(3,q)$, q even

Let $Q(4,q)$ be a non-singular quadric of $PG(4,q)$, q even, and let $\mathcal{E}$ be an elliptic quadric hyperplane section of $Q(4,q)$. Let Ω_1 and Ω_2 be two distinct points of $\mathcal{E}$ and let $\mathcal{C}_1, ..., \mathcal{C}_{q-1}, \mathcal{C}'_1, ..., \mathcal{C}'_{q+1}$ be the associated conics with $\mathcal{E}, \Omega_1, \Omega_2$. Such conics satisfy the conditions (a), (b), (c) and (d), mentioned in section 4. For every non-singular conic $\mathcal{C}$, plane section of $\mathcal{E}$, there exists one and only one tangent cone Γ_V of $Q(4,q)$ such that $\Gamma_V \cap \mathcal{E} = \mathcal{C}$, as follows by Lemma 3.5. Then, for every index $i = 1, ..., q-1$, let Γ_{V_i} denote the tangent cone of $Q(4,q)$, of vertex V_i, such that $\Gamma_{V_i} \cap \mathcal{E} = \mathcal{C}_i$, and, for every index $j = 1, ..., q+1$, let Γ_{Z_j} denote the tangent cone of $Q(4,q)$, of vertex Z_j, such that $\Gamma_{Z_j} \cap \mathcal{E} = \mathcal{C}'_j$. Obviously we have $V_i \notin \mathcal{E}$ for every $i = 1, ..., q-1$, and $Z_j \notin \mathcal{E}$ for every $j = 1, ..., q+1$.

Now let $\mathcal{S}$ and $\mathcal{S}'$ be the following sets of lined tangent cones of $Q(4,q)$:

$$\mathcal{S} = \left\{\Gamma'_{\Omega_1}, \Gamma'_{\Omega_2}, \Gamma'_{V_1}, ..., \Gamma'_{V_{q-1}}\right\},$$

$$\mathcal{S}' = \left\{\Gamma'_{Z_1}, ..., \Gamma'_{Z_{q+1}}\right\}.$$

By means of theorem 5.1 we identify $PG(3,q)$ with defined structure on $Q(4,q)$. So $\mathcal{S}$ and $\mathcal{S}'$ can be considered as sets of lines of $PG(3,q)$, both of them of size $q+1$. The following theorem holds.

Theorem 6.1. *The line sets $\mathcal{S}$ and $\mathcal{S}'$ are the two reguli of a hyperbolic quadric of $PG(3,q)$.*

Proof. By $\Omega_1 \neq \Omega_2$, by (a) and (b) it follows that two distinct lined tangent cones of $\mathcal{S}$ have no common line. So $\mathcal{S}$ is a set of parwise disjoint lines of $PG(3,q)$. We remark that each lined tangent cone of $\mathcal{S}'$ has one line in common with every lined tangent cone of $\mathcal{S}$. To show this, let Γ'_{Z_j} be a cone of $\mathcal{S}'$, $j \in \{1, ..., q+1\}$. Evidently, the cone Γ'_{Z_j} has one line in common with Γ'_{Ω_i}, $i = 1, 2$. Also, for every index $i = 1, ..., q-1$, the conics C_i and C'_j have one common point, since (d) holds. By this and by theorem 3.6 it follows that, for every index $i = 1, ..., q-1$, the cones Γ'_{Z_j} and Γ'_{V_i} have one common line. So the remark is proved. Therefore, in $PG(3,q)$, every line of $\mathcal{S}'$ meets every line of $\mathcal{S}$; this implies that $\mathcal{S}$ and $\mathcal{S}'$ are the two reguli of a hyperbolic quadric of $PG(3,q)$. So the theorem is proved. □

We denote by $\bar{I}$ the hyperbolic quadric of $PG(3,q)$ the reguli of which are $\mathcal{S}$ and $\mathcal{S}'$. Clearly, the points of $\bar{I}$ are the lines of the lined tangent cones of $\mathcal{S}$ (or of $\mathcal{S}'$). We denote by $\overline{F}$ the set of such lines. Evidently, there are $q+1$ lines of $\overline{F}$ through the point Ω_i, $i = 1, 2$, whereas there is one and only one line of $\overline{F}$ through each point of $\mathcal{E} - \{\Omega_1, \Omega_2\}$. So, for every point $X \in \mathcal{E} - \{\Omega_1, \Omega_2\}$, we denote by $r(X, \overline{F})$ the line r of $\overline{F}$ through X. By the definition of $r(X, \overline{F})$ we get:

$$\Gamma'_{V_i} \cap \Gamma'_{Z_j} = r(X, \overline{F}),\ \forall i = 1, ..., q-1,\ \forall j = 1, ..., q+1, \tag{4}$$

where X is the common point to the conics C_i and C'_j.
Now suppose $q \geq 8$ and denote by H_1 and H_2 the following sets:

$$H_1 = \left\{(h,k) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq h \leq \frac{q}{2},\ \ 1 \leq k \leq \frac{q}{2} - 1\right\},$$

$$H_2 = \left\{(h,k) \in \mathbb{Z} \times \mathbb{Z} : \frac{q}{2} + 1 \leq h \leq q-2,\ \ 1 \leq k \leq q\right\}.$$

Let $(\bar{h}, \bar{k})$ be a pair of $H_1 \cup H_2$. The ordered pair $(\bar{h}, \bar{k})$ determines the set $\left\{C_1, ..., C_{\bar{h}}, C'_1, ..., C'_{\bar{k}}\right\}$, which is a proper subset of $\left\{C_1, ..., C_{q-1}, C'_1, ..., C'_{q+1}\right\}$, i.e.

the set of associated conics with $\mathcal{E}, \Omega_1, \Omega_2$. For every index $i = 1, ..., \bar{h}$ the plane π_i, meeting $\mathcal{E}$ (and therefore $Q(4,q)$) at the conic C_i, does not contain the nucleus of $Q(4,q)$. So, let $\mathcal{F}_1$ be the following set of reguli of $Q(4,q)$:

$$\mathcal{F}_1 = \bigcup_{i=1}^{\bar{h}} \mathcal{R}(C_i),$$

where $\mathcal{R}(C_i)$ is the set of reguli (1), for every conic C_i, $i = 1, ..., \bar{h}$. By the definition of $\mathcal{R}(C_i)$ and by lemma 3.1 it follows that every regulus of $\mathcal{R}(C_i)$, $i \in \{1, ..., \bar{h}\}$, is a regulus of a hyperbolic quadric, hyperplane section of $Q(4,q)$, meeting $\mathcal{E}$ at the conic C_i, and viceversa. Let $\mathcal{F}_2$ be the following set of lined tangent cones of $Q(4,q)$:

$$\mathcal{F}_2 = \left\{\Gamma'_{Z_j}\right\}_{j=1,...,\bar{k}}.$$

Obviously, $\mathcal{F}_1$ and $\mathcal{F}_2$ are non-empty sets. Now, let $\mathcal{E}'$ be the point set:

$$\mathcal{E}' = \mathcal{E} - \left[\left(\bigcup_{i=1}^{\bar{h}} C_i\right) \bigcup \left(\bigcup_{j=1}^{\bar{k}} C'_j\right)\right].$$

By $(\bar{h}, \bar{k}) \in H_1 \cup H_2$ it immediately follows that $\mathcal{E}' \neq \emptyset$. Also, let $\mathcal{F}_3$ be the following set of lined tangent cones of $Q(4,q)$:

$$\mathcal{F}_3 = \{\Gamma'_V\}_{V \in \mathcal{E}'}.$$

Finally, let $\mathcal{F}$ be the following set:

$$\mathcal{F} = \mathcal{F}_1 \bigcup \mathcal{F}_2 \bigcup \mathcal{F}_3.$$

Clearly, the set $\mathcal{F}$ is determined by the pair $(\bar{h}, \bar{k})$. The lines of the lined tangent cones and the reguli of $\mathcal{F}$ form a set of lines of $Q(4,q)$, that we call F. The set $\mathcal{F}$ can be considered as a set of lines of $PG(3,q)$. Let us prove that $\mathcal{F}$ is a set of pairwise disjoint lines of $PG(3,q)$. To show this, first we remark that two distinct lined tangent cones $\Gamma'_{Z_{j_1}}$ and $\Gamma'_{Z_{j_2}}$ of $\mathcal{F}_2$ have no common line, since the distinct non-singular conics C'_{j_1} and C'_{j_2} have the common points Ω_1 and Ω_2 and theorem 3.6 holds. Furthermore, two distinct lined tangent cones of $\mathcal{F}_3$ have no common line, since there is no line of $Q(4,q)$ having two distinct points in common with $\mathcal{E}$. Finally, a lined tangent cone of $\mathcal{F}_2$ and a lined tangent cone of $\mathcal{F}_3$ have no common line, since $C'_j \cap \mathcal{E}' = \emptyset$ for every index $j = 1, ..., \bar{k}$. In conclusion, two distinct lined tangent cones of $\mathcal{F}$ have no common line. Also, it is trivial to check that two distinct reguli of $\mathcal{F}_1$ of the same $\mathcal{R}(C_i)$, $i \in \{1, ..., \bar{h}\}$, have no common line. Furthermore, if R_1 and R_2 are two reguli of $\mathcal{F}_1$ such that $R_1 \in \mathcal{R}(C_{i_1})$, $R_2 \in \mathcal{R}(C_{i_2})$, with $i_1, i_2 \in \{1, ..., \bar{h}\}$ and $i_1 \neq i_2$, then such

reguli have no common line, since (b) holds. In conclusion, two distinct reguli of $\mathcal{F}$ have no common line. In addition to this, we remark that every regulus of $\mathcal{F}_1$ has not any lines in common with every lined tangent cone of $\mathcal{F}_3$, since we have $C_i \cap \mathcal{E}' = \emptyset$ for every index $i = 1, ..., \bar{h}$. Now, let R be a regulus of $\mathcal{F}_1$ and Γ'_{Z_j}, $j \in \{1, ..., \bar{k}\}$, a lined tangent cone of $\mathcal{F}_2$. Evidently, we get $R \in \mathcal{R}(C_i)$, $i \in \{1, ..., \bar{h}\}$. Furthermore, the lined tangent cones Γ'_{V_i} and Γ'_{Z_j} meet at the line $r = r(X, \bar{F})$, where $C_i \cap C'_j = \{X\}$, since (4) holds. The regulus R does not contain the line r, since r is a line of Γ'_{V_i} and $R \cap \Gamma'_{V_i} = \emptyset$. By this and by the fact that R cannot contain lines of Γ'_{Z_j} distinct from r, it follows that R and Γ'_{Z_j} have no common line. In conclusion, a regulus and a lined tangent cone of $\mathcal{F}$ have no common line. Thus, any two distinct elements of $\mathcal{F}$ have no common line. This implies that $\mathcal{F}$ is a set of pairwise disjoint lines of $PG(3, q)$, that is a partial line spread in $PG(3, q)$.
Let us prove that $\mathcal{F}$ is maximal. To do this, we have to show that for every $\Gamma'_V \in \Gamma'$ and for every regulus R of $Q(4, q)$ we get $\Gamma'_V \cap F \neq \emptyset$ and $R \cap F \neq \emptyset$. To this end, we remark that every lined tangent cone Γ'_V of $Q(4, q)$, with $V \in \mathcal{E}$, has a line in common with F, since the lines of F cover the points of $\mathcal{E}$. Now, let $\mathcal{T}$ be the following set:

$$\mathcal{T} = \{\Gamma'_V \in \Gamma' : V \notin \mathcal{E}\} \cup \mathcal{R},$$

where $\mathcal{R}$ denotes the set of all the reguli of $Q(4, q)$. It is only to prove that every element of $\mathcal{T}$ has a line in common with F. To show this, let $\bar{T}$ be an element of $\mathcal{T}$ such that $\bar{T} \cap F = \emptyset$, and let $\bar{C}$ be the non-singular conic $\bar{C} = T \cap \mathcal{E}$, where T denotes the union of all the lines of $\bar{T}$ (see lemmas 3.1 and 3.2). Also, let U_1 and U_2 be the following non-empty point sets:

$$U_1 = \bigcup_{i=1}^{\bar{h}} C_i, \quad U_2 = \bigcup_{j=1}^{\bar{k}} C'_j.$$

It is immediate to check that all the $q + 1$ lines of $Q(4, q)$ through a fixed point of $\mathcal{E}' \cup (U_1 \cap U_2)$ are lines of F. By this and by $\bar{T} \cap F = \emptyset$ we get

$$\bar{C} \subset U_1 \Delta U_2, \tag{5}$$

where the symbol Δ denotes the symmetric difference operation. Also, taking into account that $(\bar{h}, \bar{k}) \in H_1 \cup H_2$, it is easy to verify that each associated conic with $\mathcal{E}, \Omega_1, \Omega_2$, contains a point not of $U_1 \Delta U_2$. So, each of such conics is not contained in $U_1 \Delta U_2$. By this and by (5) it follows that $\bar{C}$ is not a conic of the set $\{C_1, ..., C_{q-1}, C'_1, ..., C'_{q+1}\}$. We remark that $\bar{C}$ has at most two distinct points in common with $U_1 - U_2$. To show this, let X, Y and Z be three distinct points of $\bar{C} \cap (U_1 - U_2)$. By the definition of F and by $\bar{T} \cap F = \emptyset$, it follows that $\overline{T}$ contains the three distinct lines $r_1(X,\overline{F})$, $r_2(Y,\overline{F})$ and $r_3(Z,\overline{F})$ of $\overline{F}$, which are three distinct points of the hyperbolic quadric $\overline{I}$ of $PG(3,q)$. So $\overline{T}$ is a line of $PG(3,q)$ having three distinct points in common with $\overline{I}$, and therefore $\overline{T}$ is a line contained in $\overline{I}$, that is a line of a regulus of $\overline{I}$. Thus, $\overline{T}$ is a lined tangent cone of $\mathcal{S}\cup\mathcal{S}'$. By this and by $\overline{T}\in\mathcal{T}$, it follows that $\overline{T}$ is a lined tangent cone of $\mathcal{S}\cup\mathcal{S}'$ distinct from Γ'_{Ω_1} and Γ'_{Ω_2}, that is a lined tangent cone of the set $\left\{\Gamma'_{V_1},...,\Gamma'_{V_{q-1}},\Gamma'_{Z_1},...,\Gamma'_{Z_{q+1}}\right\}$, and therefore that $\overline{C}$ is a conic of the set $\{C_1,...,C_{q-1},C'_1,...,C'_{q+1}\}$: a contradiction, since $\overline{C}$ is not a conic of this set, as already noticed. The contradiction proves the remark, that is that $\overline{C}$ has at most two distinct points in common with U_1-U_2.

Now, we give a lower bound for the integer $\overline{k}$. In order to do this, first we remark that

$$U_2-U_1=\bigcup_{j=1}^{\overline{k}}(C'_j-U_1). \tag{6}$$

Also, by the fact that $\overline{C}$ is not a conic of the set $\{C'_1,...,C'_{q+1}\}$, it follows that $\overline{C}$ has at most two distinct points in common with C'_j, for every index $j=1,...,q+1$. So $\overline{C}$ has at most two distinct points in common with the point set C'_j-U_1, for every index $j=1,...,\overline{k}$. By this and by (6) we get

$$|\overline{C}\cap(U_2-U_1)|\leq 2\overline{k}. \tag{7}$$

By (7), by the fact that $\overline{C}$ has at most two distinct points in common with U_1-U_2 and by (5), we have:

$$q+1\leq 2\overline{k}+2,$$

and therefore

$$\overline{k}\geq\frac{q}{2}. \tag{8}$$

Now, we give an upper bound for the integer $\overline{h}$. In order to do this, first we remark that

$$U_2-U_1=\left[\bigcup_{i=\overline{h}+1}^{q-1}(C_i\cap U_2)\right]\bigcup\{\Omega_1,\Omega_2\}. \tag{9}$$

Also, since $\overline{C}$ is not a conic of the set $\{C_1,...,C_{q-1}\}$, it follows that $\overline{C}$ has at most two distinct points in common with C_i, for every index $i=1,...,q-1$. So $\overline{C}$ has at most two distinct points in common with the point set $C_i\cap U_2$, for every index $i=\overline{h}+1,...,q-1$. Then the conic $\overline{C}$ contains at most $2(q-1-\overline{h})$ distinct points of the set

$$\bigcup_{i=\overline{h}+1}^{q-1}(C_i\cap U_2).$$

Also, the conic $\bar{C}$ contains at most one of the points Ω_1 and Ω_2, since $\bar{C}$ is not a conic of the set $\{C'_1, ..., C'_{q+1}\}$. Thus, taking into account the equality (9), we get

$$|\bar{C} \cap (U_2 - U_1)| \leq 2(q - 1 - \bar{h}) + 1. \tag{10}$$

By (10), by the fact that $\bar{C}$ contains at most two distinct points of $U_1 - U_2$ and by (5), we have:

$$q + 1 \leq 2(q - 1 - \bar{h}) + 3,$$

and therefore

$$\bar{h} \leq \frac{q}{2}. \tag{11}$$

Thus, the pair $(\bar{h}, \bar{k})$ satisfies both of conditions (8) and (11): a contradiction, since $(\bar{h}, \bar{k}) \in H_1 \cup H_2$ and there is no pair of $H_1 \cup H_2$ satisfying both of (8) and (11). The contradiction proves that every element of $\mathcal{T}$ has a line in common with F. It follows that the partial line spread $\mathcal{F}$ of $PG(3, q)$ is maximal.
Also, by the definition of $\mathcal{F}$ and by (2) we get

$$|\mathcal{F}| = q^2 - \bar{k}q + 2\bar{k} + \bar{h}\bar{k} - \bar{h} - 1. \tag{12}$$

Now, let $\bar{t}$ and $\bar{z}$ be the following integers:

$$\begin{aligned} \bar{t} &= q - 2 - \bar{h}, \\ \bar{z} &= \bar{k} - 1. \end{aligned} \tag{13}$$

By (12) and (13) we get:

$$|\mathcal{F}| = q^2 - q + 1 - \bar{t}\bar{z}. \tag{14}$$

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the following pair sets:

$$\mathcal{P}_1 = \left\{(t, z) \in \mathbb{Z} \times \mathbb{Z} : \frac{q}{2} - 2 \leq t \leq q - 3,\ 0 \leq z \leq \frac{q}{2} - 2\right\},$$
$$\mathcal{P}_2 = \left\{(t, z) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq t \leq \frac{q}{2} - 3,\ 0 \leq z \leq q - 1\right\}.$$

It is easy to verify that:

$$\begin{aligned} (\bar{h}, \bar{k}) \in H_1 &\iff (\bar{t}, \bar{z}) \in \mathcal{P}_1, \\ (\bar{h}, \bar{k}) \in H_2 &\iff (\bar{t}, \bar{z}) \in \mathcal{P}_2. \end{aligned} \tag{15}$$

By (15) it follows that

$$(\bar{h}, \bar{k}) \in H_1 \cup H_2 \iff (\bar{t}, \bar{z}) \in \mathcal{P}_1 \cup \mathcal{P}_2.$$

Thus, in $PG(3, q)$, with q even and $q \geq 8$, there exists a maximal partial line spread $\mathcal{F}$ of size (14), for every pair $(\bar{t}, \bar{z}) \in \mathcal{P}_1 \cup \mathcal{P}_2$.

For $q \geq 16$, q even, formula (14) allows us to find many new cardinalities. In particular, for $q = 16$, we find the size 223 and all the sizes from 232 to 239; for $q = 32$ we find 192 new sizes. Moreover, for $q = 8, 16$, we find many of the results already obtained in [9] by a computer search. The number of new sizes increases for larger value of q.

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