A Note on the Italian Domination Number and Double Roman Domination Number in Graphs

Maryam Hajibaba and Nader Jafari Rad
Department of Mathematics, Shahrood University of Technology
Shahrood, Iran
E-mail: n.jafarirad@gmail.com

Abstract

An *Italian dominating function* (IDF) on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the property that for every vertex $v \in V$, with $f(v) = 0$, $\sum_{u \in N(v)} f(u) \geq 2$. The weight of an IDF f is the value $w(f) = f(V) = \sum_{u \in V} f(u)$. The minimum weight of an IDF on a graph G is called the *Italian domination number* of G, denoted by $\gamma_I(G)$. For a graph $G = (V, E)$, a double Roman dominating function (or just DRDF) is a function $f : V \longrightarrow \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$ for a vertex v, then v has at least two neighbors assigned 2 under f or one neighbor assigned 3 under f, and if $f(v) = 1$, then v has at least one neighbor with $f(w) \geq 2$. The weight of a DRDF f is the sum $f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a DRDF on G is the double Roman domination number of G, denoted by $\gamma_{dR}(G)$. In this paper we show that $\gamma_{dR}(G)/2 \leq \gamma_I(G) \leq 2\gamma_{dR}(G)/3$, and characterize all trees T with $\gamma_I(T) = 2\gamma_{dR}(T)/3$.

JCMCC **109** (2019), pp. 169-183

Keywords: Italian domination, Double Roman domination.
2010 Mathematical Subject Classification: 05C69.

1 Introduction

We consider finite, undirected, and simple graphs $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$. We denote by $|V(G)| = n(G) = n$ the *order* of G. The *open neighborhood* of a vertex v is the set $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. The *degree* of a vertex v is $\deg(v) = |N(v)|$. We denote the degree of v in G by $\deg_G(v)$ to refer it to G. The maximum and minimum degree among the vertices of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For a subset S of vertices of G, we denote by $G[S]$ the subgraph of G induced by S. The *diameter*, *diam*(G), of G is the maximum distance among all pairs of vertices in G. A diametrical path in G is a shortest path whose length is equal to the diameter of G. A non-trivial graph is a graph of order at least two. A *star* $K_{1,n}$ is a tree with one vertex of degree n and n vertices of degree one. A *double star* is a tree with precisely two vertices that are not leaves, called the centers of the double star. By a *leaf* we mean a vertex of degree one, while a *support vertex* is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex adjacent to at least two leaves. In this paper, we denote the set of all support vertices of T by $S(T)$ and the set of leaves by $L(T)$. A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the parent of v is the neighbor of v on the unique (r, v)-path, while a child of v is any other neighbor of v. A set $S \subseteq V$ in a graph G is called a *dominating set* if $N[S] = V$. The *domination number*, $\gamma(G)$ of G, is the minimum cardinality of a dominating set in G, and a dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. A subset $S \subseteq V$ is a 2-*dominating set* if every vertex of $V - S$ has at least two neighbors in S. The minimum cardinality amongst all 2-dominating sets of G is the 2-*domination number*, $\gamma_2(G)$. For other definitions and notations not given here we refer to [2, 7].

Let $f : V \longrightarrow \{0, 1, 2\}$ be a function having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a neighbor $u \in N(v)$ with $f(u) = 2$. Such a function is called a *Roman dominating function* or just an RDF. The weight of an RDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G, and is denoted by $\gamma_R(G)$, [4].

A generalization of Roman domination called *Italian domination* was introduced by Chellali, Haynes, Hedetniemi and McRae in [3], where it was called Roman $\{2\}$-domination. This parameter was further studied by Klostermeyer and MacGillivray [9], and Henning and Klostermeyer [8]. An *Italian dominating function* (IDF) on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the property that for every vertex $v \in V$, with $f(v) = 0$, $\sum_{u \in N(v)} f(u) \geq 2$. The weight of an IDF f is the value $w(f) = f(V) = \sum_{u \in V} f(u)$. The minimum weight of an IDF on a graph G is called the *Italian domination number* of G, denoted by $\gamma_I(G)$. In [3], what we call $\gamma_I(G)$ is called $\gamma_{\{R2\}}(G)$. A $\gamma_I(G)$-function is an IDF with weight $\gamma_I(G)$. A $\gamma_I(G)$-function f can be represented by a triple $f = (V_0, V_1, V_2)$ (or $f = (V_0^f, V_1^f, V_2^f)$ to refer to f), where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$.

A function $f : V \rightarrow \{0, 1, 2, 3\}$ is a *double Roman dominating function* on a graph G if the following conditions are met (see [1]).

(i) If $f(v) = 0$, then v has at least two neighbors in V_2 or one neighbor in V_3.

(ii) If $f(v) = 1$, then v has at least one neighbor in $V_2 \cup V_3$.

A double Roman dominating function f can be represented as $f = (V_0, V_1, V_2, V_3)$ (or $f = (V_0^f, V_1^f, V_2^f, V_3^f)$ to refer to f), where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2, 3$. The *double Roman domination number* $\gamma_{dR}(G)$ equals the minimum weight of a double Roman dominating function on G, and a double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR} - function$ of G. Clearly in a double Roman dominating function of weight $\gamma_{dR}(G)$, no vertex needs to be assigned the value 1. Hence we can assume that $V_1 = \emptyset$ for all double Roman dominating functions under consideration.

In this paper we show that for any graph G, $\gamma_{dR}(G)/2 \leq \gamma_I(G) \leq 2\gamma_{dR}(G)/3$. We then give some infinite families of trees achieving equality for the lower bound, and characterize all trees achieving equality for the upper bound. The following are useful.

Proposition 1 (Beeler et al. [1]) *Let G be a graph and $f = (V_0, V_1, V_2)$ a γ_R-function of G. Then $\gamma_{dR}(G) \leq 2|V_1| + 3|V_2|$.*

Corollary 2 (Beeler et al. [1]) *For any nontrivial connected graph G, $\gamma_R(G) < \gamma_{dR}(G) < 2\gamma_R(G)$.*

Proposition 3 (Beeler et al. [1]) *For any graph G, $2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G)$.*

2 Main results

We first state our upper and lower bounds for the Italian domination number in terms of the double Roman domination number.

Theorem 4 *For every graph G, $\gamma_{dR}(G)/2 \le \gamma_I(G) \le 2\gamma_{dR}(G)/3$.*

Proof. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_I(G)$-function. Then $g = (V_0^f, \emptyset, V_1^f, V_2^f)$ is a DRDF for G. Thus, $\gamma_{dR}(G) \le 3|V_2^f| + 2|V_1^f| \le 4|V_2^f| + 2|V_1^f| = 2\gamma_I(G)$, as desired. We next establish the upper bound. Let $g = (V_0^g, \emptyset, V_2^g, V_3^g)$ be a $\gamma_{dR}(G)$-function. Then $h = (V_0^g, V_2^g, V_3^g)$ is an IDF for G. Thus, $\gamma_I(G) \le w(h) = 2|V_3^g| + |V_2^g| = \gamma_{dR}(G) - (|V_3^g| + |V_2^g|)$. On the other hand, since $V_2^g \cup V_3^g$ is a dominating set for G, $\gamma(G) \le |V_2^g| + |V_3^g|$. Therefore, $\gamma_I(G) \le \gamma_{dR}(G) - \gamma(G)$. By Proposition 3, $\gamma_{dR}(G) \le 3\gamma(G)$, and so $\gamma_I(G) \le \gamma_{dR}(G) - \gamma(G) \le \gamma_{dR}(G) - \gamma_{dR}(G)/3 = 2\gamma_{dR}(G)/3$.
■

If $\gamma_{dR}(G)/2 = \gamma_I(G)$ and $f = (V_0^f, V_1^f, V_2^f)$ is a $\gamma_I(G)$-function, then from the proof of Theorem 4, we obtain that $V_2^f = \emptyset$. Thus V_1^f is a 2-dominating set for G and so $\gamma_2(G) \le |V_1^f| = \gamma_I(G)$. Since always $\gamma_I(G) \le \gamma_2(G)$, we obtain that $\gamma_I(G) = \gamma_2(G)$. Thus we obtain the following.

Corollary 5 *If $\gamma_{dR}(G)/2 = \gamma_I(G)$, then $\gamma_I(G) = \gamma_2(G)$. Furthermore, $V_2^f = \emptyset$ for every $\gamma_I(G)$-function $f = (V_0^f, V_1^f, V_2^f)$.*

Note that the converse of Corollary 5 does not hold. To see this, note that $\gamma_I(P_3) = \gamma_2(P_3)$, but $\gamma_{dR}(P_3)/2 \ne \gamma_I(P_3)$. We next present necessary and sufficient conditions for a graph achieving equality in the upper bound of Theorem 4.

Corollary 6 *For graph G, $\gamma_I(G) = 2\gamma_{dR}(G)/3$, if and only if $\gamma_{dR}(G) = 3\gamma(G)$, $\gamma_I(G) = 2\gamma(G)$.*

Proof. Assume that $\gamma_I(G) = 2\gamma_{dR}(G)/3$. From the proof of Theorem 4, we obtain that $\gamma_{dR}(G) = 3\gamma(G)$ and $\gamma_I(G) = 2\gamma(G)$. The converse is obvious. ■

Corollary 7 *If for graph G, $\gamma_I(G) = 2\gamma_{dR}(G)/3$, then for any $\gamma_{dR}(G)$-function $f = (V_0^f, \emptyset, V_2^f, V_3^f)$, $V_2^f = \emptyset$.*

Proof. Assume that $\gamma_I(G) = 2\gamma_{dR}(G)/3$. Let $f = (V_0^f, \emptyset, V_2^f, V_3^f)$ be a $\gamma_{dR}(G)$-function. Suppose that $V_2^f \neq \emptyset$. Since $\gamma(G) = |V_2^f| + |V_3^f|$, we have, $\gamma_{dR}(G) = 3\gamma(G) = 3|V_2| + 3|V_3| > 2|V_2| + 3|V_3| = \gamma_{dR}(G)$, a contradiction. ■

2.1 Trees

In this subsection, we first present families of trees achieving equality for the lower bound of Theorem 4, and then characterize all trees achieving equality for the upper bound of Theorem 4. For any positive integer k, let T_k be the class of all trees consisting of the disjoint union of k copies of P_5 plus a path through the central vertices of these copies, as illustrated in Figure 1. Let the i-th copy of P_5 has vertex set $\{v_1^i, ..., v_5^i\}$, where v_j^i is adjacent to v_{j+1}^i for $j = 1, ..., 4$. Clearly for any IDF on T_k, $f(v_1^i)+...+f(v_5^i) \geq 3$, and thus $\gamma_I(T_k) \geq 3k$. On the other hand g defined on $V(T_k)$ by $g(v_2^i) = f(v_4^i) = 0$ for $i = 1, 2, ..., k$, and $g(x) = 1$ otherwise is an IDF for T_k. Thus $\gamma_I(T_k) = 3k$. Similarly, $\gamma_{dR}(T_k) = 6k$. Thus, $\gamma_{dR}(T_k)/2 = \gamma_I(T_k)$.

Lemma 8 *Assume that T' is a tree with $\gamma_{dR}(T')/2 = \gamma_I(T')$, and $w \in V(T')$ is a vertex with $\gamma_{dR}(T'-w) > \gamma_{dR}(T')$. If T is a*

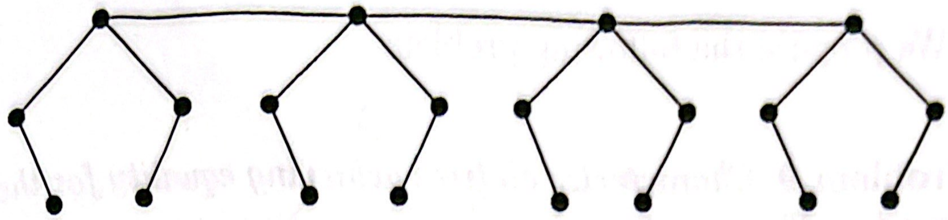

Figure 1: The tree T_4.

tree obtained from T' by adding a path $P_2 = uv$ and adding the edge uw, then $\gamma_{dR}(T)/2 = \gamma_I(T)$.

Proof. Note that by Theorem 4, $2\gamma_I(T' - w) \geq \gamma_{dR}(T' - w) > \gamma_{dR}(T') = 2\gamma_I(T')$, and so $\gamma_I(T' - w) > \gamma_I(T')$. Since by Corollary 5, $V_2^f = \emptyset$ for every $\gamma_I(T')$-function $f = (V_0^f, V_1^f, V_2^f)$, we deduce that $f(w) = 1$ for every $\gamma_I(T')$-function f. Then we can extend a $\gamma_I(T')$-function to an IDF for T by assigning 1 to v and 0 to u, and thus $\gamma_I(T) \leq \gamma_I(T') + 1$. On the other hand, let f_1 be a $\gamma_I(T)$-function. Clearly $f_1(u) + f_1(v) \geq 1$. If $f_1(w) \neq 0$, then $f_1|_{T'}$ is an IDF for T', and so $\gamma_I(T') \leq \gamma_I(T) - 1$. Thus assume that $f_1(w) = 0$. Then $f_1(u) + f_1(v) \geq 2$. Now f_2 defined on T' by $f_2(w) = 1$ and $f_2(x) = f_1(x)$ if $x \neq w$ is an IDF for T', and so $\gamma_I(T') \leq \gamma_I(T) - 1$. Consequently, $\gamma_I(T) = \gamma_I(T') + 1$.

Since $\gamma_{dR}(T')/2 = \gamma_I(T')$, for any $\gamma_I(T')$-function $f = (V_0^f, V_1^f, \emptyset)$, the function $f_1 = (V_0^f, \emptyset, V_1^f, \emptyset)$ is a $\gamma_{dR}(T')$-function with $g'(w) = 2$. Then f_1 can be extended to a DRDF of T by assigning 2 to v and 0 to u, and so $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 2$. Now assume that function $g = (V_0, \emptyset, V_2, V_3)$ be a $\gamma_{dR}(T)$-function. If $g(w) = 0$, then $g(u) + g(v) \geq 3$, and function $g|_{T'-w}$ is a DRDF for $T' - w$. Then $\gamma_{dR}(T) - 2 \leq \gamma_{dR}(T') < \gamma_{dR}(T' - w) \leq w(g|_{T'-w}) \leq \gamma_{dR}(T) - 3$, a contradiction. Hence $g(w) \neq 0$, then $g(u) + g(v) \geq 2$, and function $g|_{T'}$ is a *DRDF* for tree T'. Hence $\gamma_{dR}(T') \leq \gamma_{dR}(T) - 2$ and so $\gamma_{dR}(T) = \gamma_{dR}(T') + 2$. Hence $\gamma_{dR}(T)/2 = (\gamma_{dR}(T') + 2)/2 = \gamma_I(T') + 1 = \gamma_I(T)$. ■

We propose the following problem.

Problem 9 *Characterize all trees achieving equality for the lower bound of Theorem 4.*

We next wish to characterize trees achieving equality for the upper bound of Theorem 4. Let $\mathcal{F}$ be the family of unlabeled trees T that can be obtained from a sequence $T_1, ..., T_j (j \geq 1)$ of trees such that T_1 is a star $K_{1,r}$ for $r \geq 2$, and, if $j \geq 2$, then T_{i+1} can be obtained, recursively, from T_i by one of the following two operations $\mathcal{O}_1$ and $\mathcal{O}_2$, as illustrated in Figures 2 and 3.

Operation $\mathcal{O}_1$: Assume that $\gamma_I(T_i - u) \geq \gamma_I(T_i)$. Then T_{i+1} is obtained from T_i by joining u to the central vertex of a star $K_{1,s}$, for some $s \geq 2$.

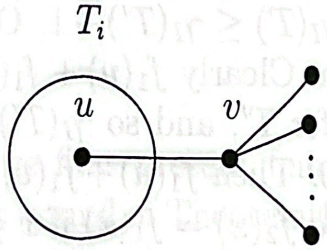

Figure 2: *Operation* $\mathcal{O}_1$.

Operation $\mathcal{O}_2$: Assume $u \in V(T_i)$. Then T_{i+1} is obtained from T_i by joining u to the central vertex of a star $K_{1,s}$, for some $s \geq 1$, and then subdividing the new edge.

Next we show that every tree T in the family $\mathcal{F}$ satisfies in $\gamma_I(T) = 2\gamma_{dR}(T)/3$.

Lemma 10 *If* $\gamma_I(T_i) = 2\gamma_{dR}(T_i)/3$ *and* T_{i+1} *is obtained from* T_i *by Operation* $\mathcal{O}_1$*, then* $\gamma_I(T_{i+1}) = 2\gamma_{dR}(T_{i+1})/3$.

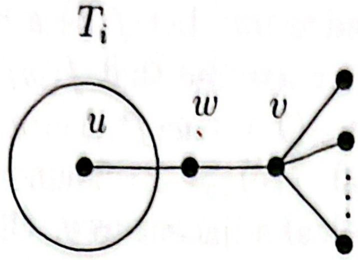

Figure 3: *Operation $\mathcal{O}_2$.*

Proof. Assume that $u \in V(T_i)$ and $\gamma_I(T_i - u) \geq \gamma_I(T_i)$. Since $\frac{2}{3}\gamma_{dR}(T_{i-u}) \geq \gamma_I(T_i - u) \geq \gamma_I(T_i) = \frac{2}{3}\gamma_{dR}(T_i)$, we obtain that $\gamma_{dR}(T_i - u) \geq \gamma_{dR}(T_i)$. Let v be the central vertex of a star $K_{1,s}$ according to the Operation $\mathcal{O}_1$. Let f be a $\gamma_{dR}(T_{i+1})$-function. Then clearly $f(v) = 3$, since v is a strong support vertex of T_{i+1}. If $f(u) \neq 0$, then $f|_{T_i}$ is a DRDF for tree T_i and so $\gamma_{dR}(T_i) \leq w(f|_{T_i}) = w(f) - 3 = \gamma_{dR}(T_{i+1}) - 3$, and otherwise $f|_{T_i - u}$ is a *DRDF* for forest $T_i - u$ and so by applying Theorem 4, $\gamma_{dR}(T_i) = 3\gamma_I(T_i)/2 \leq 3\gamma_I(T_i - u)/2 \leq \gamma_{dR}(T_i - u) \leq w(f|_{T_i - u}) \leq w(f) - 3 = \gamma_{dR}(T_{i+1}) - 3$. On other hand any $\gamma_{dR}(T_i)$-function can be extended to a DRDF for T_{i+1} by assigning 3 to v and 0 to any leaf adjacent to v, and so $\gamma_{dR}(T_{i+1}) \leq \gamma_{dR}(T_i) + 3$. Thus $\gamma_{dR}(T_{i+1}) = \gamma_{dR}(T_i) + 3$. Now assume that D is a $\gamma(T_i)$-set, then $D \cup \{v\}$ is a dominating set for tree T_{i+1}, and so $\gamma(T_{i+1}) \leq \gamma(T_i) + 1$, and so by applying Corollary 6 and Proposition 3 on T_i, we obtain that $\gamma_{dR}(T_{i+1}) \leq 3\gamma(T_{i+1}) \leq 3\gamma(T_i) + 3 = \gamma_{dR}(T_i) + 3 = \gamma_{dR}(T_{i+1})$. Thus $\gamma_{dR}(T_{i+1}) = 3\gamma(T_{i+1})$. Similarly, we can see that $\gamma_I(T_{i+1}) = 2\gamma(T_{i+1})$ and so again by Corollary 6, $\gamma_I(T_{i+1}) = 2\gamma_{dR}(T_{i+1})/3$. ■

Lemma 11 *If $\gamma_I(T_i) = 2\gamma_{dR}(T_i)/3$ and T_{i+1} is obtained from T_i by Operation $\mathcal{O}_2$, then $\gamma_I(T_{i+1}) = 2\gamma_{dR}(T_{i+1})/3$.*

Proof. Assume that $u \in V(T_i)$ and v is the central vertex of a star $K_{1,s}$, where $s \geq 1$, and w is the new vertex obtained by subdividing the edge uv. Let f be a $\gamma_{dR}(T_{i+1})$-function. If $f(v) = 3$, then we may assume that $f(w) = 0$, and so $f|_{T_i}$ is a DRDF for T_i. Thus $\gamma_{dR}(T_i) \leq w(f|_{T_i}) = w(f) - 3 = \gamma_{dR}(T_{i+1}) - 3$. Thus assume that $f(v) \neq 3$. Then, v is a weak support vertex. Let v_1 be the leaf adjacent to v. Clearly we may assume that $f(v) \neq 2$. Thus $f(v) = 0$, and so $f(v_1) = f(w) = 2$. Then g defined on $V(T_i)$ by $g(u) = \max\{1, f(u)\}$, and $g(x) = f(x)$ if $x \neq u$ is a DRDF for T_i. Thus $\gamma_{dR}(T_i) \leq w(g) \leq \gamma_{dR}(T_{i+1}) - 3$. On other hand any $\gamma_{dR}(T_i)$-function can be extended to a *DRDF* of T_{i+1} by assigning weight 3 to v and weight 0 to its neighbors, and so $\gamma_{dR}(T_{i+1}) \leq \gamma_{dR}(T_i) + 3$. Thus $\gamma_{dR}(T_{i+1}) = \gamma_{dR}(T_i) + 3$. Now assume that D is a $\gamma(T_i)$-set, then $D \cup \{v\}$ is a dominating set for T_{i+1}. Then $\gamma(T_{i+1}) \leq \gamma(T_i) + 1$, and so by applying Corollary 6 and Proposition 3, $\gamma_{dR}(T_{i+1}) \leq 3\gamma(T_{i+1}) \leq 3\gamma(T_i)+3 = \gamma_{dR}(T_i)+3 = \gamma_{dR}(T_{i+1})$. Thus $\gamma_{dR}(T_{i+1}) = 3\gamma(T_{i+1})$. Similarly we show that $\gamma_I(T_{i+1}) = 2\gamma(T_{i+1})$ and so by Corollary 6, $\gamma_I(T_{i+1}) = 2\gamma_{dR}(T_{i+1})/3$. ■

We are now ready to provide a constructive characterization for all trees T with $\gamma_I(T) = 2\gamma_{dR}(T)/3$.

Theorem 12 *For a tree T of order $n \geq 3$, $\gamma_I(T) = 2\gamma_{dR}(T)/3$, if and only if $T \in \mathcal{F}$.*

Proof. The proof of sufficiency follows by an induction on the number of operations performed to construct a tree T and applying the Lemmas 10 and 11. We now prove the necessity part. Let $T \in \mathcal{F}$ be a tree of order $n \geq 3$. We proceed by an induction on the order n of T with $\gamma_I(T) = 2\gamma_{dR}(T)/3$ to show that $T \in \mathcal{F}$. Clearly, $diam(T) \geq 2$. If $diam(T) = 2$, then T is a star and so $T \in \mathcal{F}$. Hence we may assume that $diam(T) \geq 3$. Suppose that $diam(T) = 3$. Therefore, T is a double star. Let a and b be the central vertices of T. If $\deg(a) = 2$, then assigning 2 to b, 1 to the leaf adjacent to a, and 0 to any other leaf of T yields an IDF for T, and thus $\gamma_I(T) \leq 3$. Furthermore, it is obvious that $\gamma_I(T) \neq 2$. Thus $\gamma_I(T) = 3$. Similarly, it can be easily seen that $\gamma_{dR}(T) \leq 5$. Now, $\gamma_I(T) \neq 2\gamma_{dR}(T)/3$, a contradiction. Thus $\deg(a) \geq 3$, and by symmetry, $\deg(b) \geq 3$. Let T_1 and T_2 be the components of $T - ab$, where $a \in V(T_1)$ and $b \in V(T_2)$. Clearly T_i is a stars of order at least 3, for $i = 1, 2$. Thus $\gamma_I(T_i) = 2$ and $\gamma_{dR}(T_i) = 3$ for $i = 1, 2$. Since $\gamma_I(T_1 - a) \geq \gamma_I(T_1)$, T is obtained from $T_1 \in \mathcal{F}$ by Operation $\mathcal{O}_1$. Thus, we may assume that $diam(T) \geq 4$.

Among all diametrical paths in T, let $x_0x_1...x_d$ be a diametrical path in T such that $\deg(x_{d-1})$ is maximized. We root T at x_0. We consider the following cases:

Case 1: $\deg(x_{d-1}) \geq 3$. Let S be a $\gamma(T)$-set containing any support vertex of T. Then $x_{d-1} \in S$.

Assume that $\deg(x_{d-2}) \geq 3$. Let $T' = T - T_{x_{d-1}}$. Then, $S - \{x_{d-1}\}$ is a dominating set of T', implying that $\gamma(T') \leq \gamma(T) - 1$. On the other hand every dominating set of T' can be extended to a dominating set of T by adding the vertex x_{d-1} to it, implying that $\gamma(T) \leq \gamma(T') + 1$. Consequently, $\gamma(T) = \gamma(T') + 1$. Every DRDF in T' can be extended to a DRDF of T by assigning 3 to x_{d-1} and 0 to any child of x_{d-1}, and so $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 3$. Suppose that $\gamma_I(T') \neq 2\gamma_{dR}(T')/3$. By Corollary 6, $\gamma_{dR}(T') < 3\gamma(T')$ or $\gamma_I(T') < 2\gamma(T')$. If $\gamma_{dR}(T') < 3\gamma(T')$, then $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 3 < 3\gamma(T') + 3 = 3\gamma(T)$, a contradiction, since by Corollary 6, $\gamma_{dR}(T) = 3\gamma(T)$. Thus, $\gamma_{dR}(T') = 3\gamma(T')$, Similarly, $\gamma_I(T') = 2\gamma(T')$. By Corollary 6, $\gamma_I(T') = 2\gamma_{dR}(T')/3$. By the inductive hypothesis, $T' \in \mathcal{F}$. We show that $\gamma_I(T' - x_{d-2}) \geq \gamma_I(T')$. Every IDF for $T' - x_{d-2}$ can be extended to an IDF of T by assigning 2 to x_{d-1} and 0 to each child of x_{d-1}, and so $\gamma_I(T) \leq \gamma_I(T' - x_{d-2}) + 2$. Thus, if $\gamma_I(T'-x_{d-2}) < \gamma_I(T')$, then $\gamma_I(T) \le \gamma_I(T'-x_{d-2})+2 < \gamma_I(T')+2 = 2\gamma(T')+2 = 2\gamma(T) = \gamma_I(T)$, a contradiction. Hence, $\gamma_I(T'-x_{d-2}) \ge \gamma_I(T')$. Therefore T can be obtained from the tree T' by applying Operation $\mathcal{O}_1$. Consequently, $T \in \mathcal{F}$.

Next assume that $\deg(x_{d-2}) = 2$. Let $T' = T - T_{x_{d-2}}$. Let D be a $\gamma(T)$-set containing any support vertex of T. Thus $x_{d-1} \in D$. Clearly we may assume that $x_{d-2} \notin D$. Then $D - \{x_{d-1}\}$ is a dominating set of T', and so $\gamma(T') \le \gamma(T) - 1$. On the other hand, every dominating set of T' can be extended to a dominating set of T by adding the vertex x_{d-1} to it, implying that $\gamma(T) \le \gamma(T')+1$. Consequently, $\gamma(T) = \gamma(T')+1$. Every DRDF in T' can be extended to a DRDF of T by assigning 3 to x_{d-1} and 0 to each neighbor of x_{d-1}, and so $\gamma_{dR}(T) \le \gamma_{dR}(T')+3$. If $\gamma_I(T') \neq 2\gamma_{dR}(T')/3$, then by Corollary 6, $\gamma_{dR}(T') < 3\gamma(T')$ or $\gamma_I(T') < 2\gamma(T')$. If $\gamma_{dR}(T') < 3\gamma(T')$, then $\gamma_{dR}(T) \le \gamma_{dR}(T')+3 < 3\gamma(T')+3 = 3\gamma(T)$, a contradiction. Hence $\gamma_{dR}(T') = 3\gamma(T')$. Similarly, $\gamma_I(T') = 2\gamma(T')$. By Corollary 6, $\gamma_I(T') = 2\gamma_{dR}(T')/3$. If $|V(T')| \ge 3$, then by the inductive hypothesis, $T' \in \mathcal{F}$. Hence T is obtained from the tree T' by applying Operation $\mathcal{O}_2$. Consequently, $T \in \mathcal{F}$. Thus assume that $|V(T')| = 2$. Let $T'' = T - T_{x_{d-1}}$. Clearly, T'' is a path of order 3, and clearly $T'' \in \mathcal{F}$. It is obvious that $\gamma_I(T''-x_2) \ge \gamma_I(T'')$. Since $T_{x_{d-1}}$ is a star of order at least three, we deduce that T is obtained from $T'' \in \mathcal{F}$ by Operation $\mathcal{O}_1$. Consequently, $T \in \mathcal{F}$.

Case 2: $\deg(x_{d-1}) = 2$.
We show that $\deg(x_{d-2}) = 2$. Suppose that $\deg(x_{d-2}) \ge 3$. Then any children of x_{d-2} is a support vertex of degree 2 or a leaf.

Assume that x_{d-2} is not a support vertex. Let $T' = T - T_{x_{d-2}}$ and $A = L(T_{x_{d-2}}) \cup \{x_{d-2}\}$. Let D be a minimum dominating set of T containing any support vertex of T. If $x_{d-2} \notin D$, then $D - S(T_{x_{d-2}})$ is a dominating set of T', and so $\gamma(T') \leq \gamma(T) - |S(T_{x_{d-2}})| = \gamma(T) - |A| + 1$. Thus assume that $x_{d-2} \in D$. Then $(D \cup \{x_{d-3}\}) - (S(T_{x_{d-2}}) \cup \{x_{d-2}\})$ is a dominating set of T', and so $\gamma(T') \leq \gamma(T) - |S(T_{x_{d-2}})| = \gamma(T) - |A| + 1$. On the other hand, every dominating set of T' can be extended to a dominating set of T by adding $S(T_{x_{d-2}})$ to it, implying that $\gamma(T) \leq \gamma(T') + |A| - 1$. Consequently, $\gamma(T) = \gamma(T') + |A| - 1$. If D' be a $\gamma(T')$-set, then the function $f = (V(T) - (A \cup D'), A, D')$ is an IDF for T, and so $\gamma_I(T) \leq 2|D'| + |A| = 2|D| - 2|A| + 2 + |A| = 2|D| - |A| + 2 < 2|D| = 2\gamma(T)$, a contradiction by Corollary 6, since $|A| \geq 3$. Next assume that x_{d-2} is a support vertex. Let $T' = T - T_{x_{d-1}}$. Let D be a $\gamma(T)$-set containing any support vertex of T. Then $x_{d-1}, x_{d-2} \in D$, and thus $D - \{x_{d-1}\}$ is a dominating set for T', implying that $\gamma(T') \leq \gamma(T) - 1$. On the other hand every dominating set of T' can be extended to a dominating set of T by adding x_{d-1} to it, implying that $\gamma(T) \leq \gamma(T') + 1$. Consequently, $\gamma(T) = \gamma(T') + 1$. Now assume that D' is a $\gamma(T')$-set containing x_{d-2}. Then the function $f = (V(T) - (D' \cup \{x_d\}), \{x_d\}, D')$ is an IDF for tree T. Hence $\gamma_I(T) \leq w(f) = 2|D'| + 1 = 2\gamma(T') + 1 < 2\gamma(T)$, a contradiction by Corollary 6. We conclude that $\deg(x_{d-2}) = 2$.

Let $T' = T - T_{x_{d-2}}$ and D be a minimum dominating set of T containing any support vertex of T. Thus, $x_{d-1} \in D$, and we may assume that $x_{d-2} \notin D$. Then $D - \{x_{d-1}\}$ is a dominating set of T', implying that $\gamma(T') \leq \gamma(T) - 1$. On the other hand, every dominating set of T' can be extended to a dominating set of T by adding x_{d-1} to it, implying that $\gamma(T) \leq \gamma(T') + 1$. Consequently, $\gamma(T) = \gamma(T') + 1$. Every DRDF in T' can be extended to a DRDF of T by assigning 3 to x_{d-1} and 0 to the neighbors of x_{d-1}, and so $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 3$. If $\gamma_I(T') \neq 2\gamma_{dR}(T')/3$, then by Corollary 6, $\gamma_{dR}(T') < 3\gamma(T')$ or $\gamma_I(T') < 2\gamma(T')$. If $\gamma_{dR}(T') < 3\gamma(T')$, then $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 3 < 3\gamma(T') + 3 = 3\gamma(T)$, a contradiction to the Corollary 6. Hence $\gamma_{dR}(T') = 3\gamma(T')$. Similarly $\gamma_I(T') = 2\gamma(T')$. Hence, by Corollary 6, $\gamma_I(T') = 2\gamma_{dR}(T')/3$.

If $|V(T')| = 2$, then $T = P_5$, and clearly, $\gamma_I(T) \neq 2\gamma_{dR}(T)/3$. Thus, $|V(T')| \geq 3$. By the inductive hypothesis, $T' \in \mathcal{F}$. Hence, T is obtained from T' by Operation $\mathcal{O}_2$. Consequently, $T \in \mathcal{F}$. ∎

Acknowledgements:
We would like to thank the referee for his/her careful review of the paper.

References

[1] R.A. Beeler, T.W. Haynes, S.T. Hedetniemi, Double Roman domination, *Discrete Appl. Math.* 211 (2016), 23-29.

[2] M. Blidia, M. Chellali, L. Volkmann, Bounds on the 2-domination number of graphs, *Utilitas Math.* 71 (2006), 209-216.

[3] M. Chellali, T. Haynes, S.T. Hedetniemi, A. McRae, Roman {2}-domination, *Discrete Appl. Math.* 204 (2016), 22–28.

[4] E.J. Cockayne, P.A. Dreyer, S.M. Hedetniemi, S.T. Hedetniemi, Roman domination in graphs, *Discrete Math.* 278 (2004), 11–22.

[5] J.F. Fink and M.S. Jacobson, *n-Domination in graphs, Graph Theory with Applications to Algorithms and Computer Science*, John Wiley and Sons, New York, 283–300, 1985.

[6] A. Hansberg, L. Volkmann, On graphs with equal domination and 2-domination numbers, *Discrete Math.* 308 (2008), 2277–2281.

[7] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[8] M.A. Henning, W.F. Klostermeyer, Italian domination in trees, *Discrete Appl. Math.* 217 (2017), 557–564.

[9] W. Klostermeyer and G. MacGillivray, *Roman, Italian, and 2-domination*, J. Comb. Math. Comb. Comput., to appear.