Bounds on the Edge Magic Number for Complete Graphs

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Abstract

An edge-magic total labeling of a graph $G = (V, E)$ is an assignment of integers $1, 2, \ldots, |V| + |E|$ to the vertices and edges of the graph so that the sum of the labels of any edge uv and the labels on vertices u and v is constant. It is known that the class of complete graphs on n vertices, K_n, are not edge-magic for any $n \geq 7$. The edge magic number $M_E(K_n)$ is defined to be the minimum number t of isolated vertices such that $K_n \cup tK_1$ is edge-magic. In this paper we show that, for $n \geq 10$, $M_E(K_n) \leq f_{n+1} + 57 - \frac{n^2+n}{2}$ where f_i is the i^{th} Fibonacci number. With the aid of a computer, we also show that $M_E(K_7) = 4$, $M_E(K_8) = 10$, and $M_E(K_9) = 19$, answering several questions posed by W. D. Wallis.

1 Introduction

For a simple, undirected graph G with vertex set V and edge set E, we define a labeling as an injective map λ from $V \cup E$ to the integers; a *total labeling* is defined as a bijective labeling with range $[|V| + |E|]$. A graph is said to be *edge-magic* if there exists a labeling λ with the property that for each edge uv in G the sum $\lambda(u) + \lambda(uv) + \lambda(v) = \mu$ for a constant μ. In this context, μ will be called the *magic constant* of the labeling. If this labeling is total, we say that G has an *edge magic total labeling* abbreviated EMTL. For example, Figure 1 gives an edge-magic total labeling of $K_4 - \{e\}$ with magic constant 12.

Edge-magic total labelings were introduced by Ringel and Llado in 1996 [1]. In [3], Wallis, Baskoro, Miller, and Slamin enumerated every possible edge-magic total labeling of complete graphs and proved that the complete graph K_n

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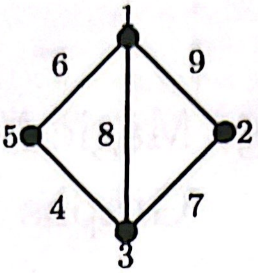

Figure 1: Edge-Magic Total Labeling of $K_4 - \{e\}$

is not edge-magic if $n = 4$ or if $n \geq 7$. In light of this theorem, it is natural to ask for some measurement of how far from edge-magic a given complete graph is. This idea led Wallis [2] to define the *magic number* $M_E(K_n)$ as the minimum number t of isolated vertices such that $K_n \cup tK_1$ is edge-magic. Figure 2 shows an edge-magic total labeling of $K_4 \cup K_1$ using one isolated point, illustrating that $M_E(K_4) = 1$.

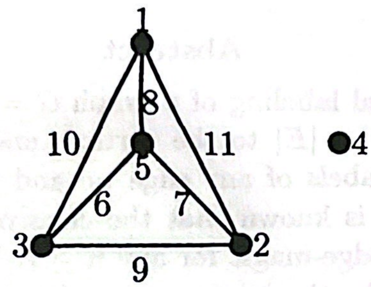

Figure 2: Edge-Magic Total Labeling of $K_4 \cup K_1$

Using the magic number as a measure of how far from edge-magic complete graphs are, Wallis [2] posed two problems: one asking for the magic numbers of K_7 and of K_8 and the other asking for bounds on $M_E(K_n)$. In this paper, we solve an extended form of the first problem and provide a new upper bound for $M_E(K_n)$. Much of our work depends upon the construction of a new element of a particular category of sequences, discussed in the next section.

2 Well-spread Sequences

In this section we will introduce a specific type of sequence that provides candidates for vertex labels in an edge-magic total labeling of a complete graph together with isolated vertices.

A *well-spread sequence* is a strictly increasing sequence $\{a_i\}_{i=1}^{m}$ of integers in which all pairwise sums of elements are distinct.

Lemma 1. *Any edge-magic labeling λ of K_n requires the labels used for the vertices to form a well-spread sequence.*

Proof. Suppose, by way of contradiction, that there is some edge-magic labeling of K_n in which the labels on the vertices do not form a well-spread sequence. Let the magic constant of the labeling be μ. Given μ; the label used on edge uv must be $\mu - (\lambda(u) + \lambda(v))$. Since the sequence of labels on the vertices is not well-spread, the sums $\lambda(u) + \lambda(v)$ are the same for different pairs (u, v) of vertices and it is necessary to use the same label on two distinct edges, violating the bijective requirement of a labeling. The result follows. □

For a well-spread sequence $A = \{a_i\}_{i=1}^n$, let $\rho(A)$ be defined as follows: $\rho(A) = 1$ if $a_1 > 1$, $\rho(A) = 2$ if $a_1 = 1$ and $a_2 > 2$, $\rho(A) = 3$ if $a_1 = 1, a_2 = 2$, and $a_3 > 3$, and $\rho(A) = 4$ if $a_1 = 1, a_2 = 2$, and $a_3 = 3$. Note that there is no well-spread sequence in which $a_1 = 1, a_2 = 2, a_3 = 3$, and $a_4 = 4$ since that would force $a_1 + a_4 = a_2 + a_3$; hence $\rho(A)$ is defined for all well-spread sequences.

We will say a sequence A is *doubly well-spread* if $a_n + a_{n-1} + \rho(A) - a_i - a_j \neq a_k$ for any $i, j, k \in [n]$. Doubly well-spread sequences serve to provide a basis for the labelings used in this paper.

Lemma 2. *The elements of any doubly well-spread sequence $A = \{a_i\}_i^n$ provide candidates for the vertex labels of some edge-magic labeling of K_n (not necessarily a total labeling).*

Proof. Since the sequence A is well-spread, the sum of every pair of vertex labels is distinct, and therefore no edge label will need to be repeated. It remains to show that there exists a labeling using the elements of this sequence as the vertex labels in which no element of the sequence itself is needed as an edge label.

Let a_i be the label on vertex v_i so that a_n is the label on v_n and a_{n-1} is the label on v_{n-1}. Label the edge $v_n v_{n-1}$ with $\rho(A)$. Hence the magic constant must be $\mu = a_n + a_{n-1} + \rho(A)$.

With magic constant μ, the label for an edge $v_i v_k$ is $\mu - a_i - a_k$. Since $\mu = a_n + a_{n-1} + \rho(A)$ and no label may be used more than once, we require $a_n + a_{n-1} + \rho(A) - a_i - a_k$ to be distinct from every term in the sequence, namely $a_n + a_{n-1} + \rho(A) - a_i - a_k \neq a_j$ for all i, j, k such that $1 \leq i, j, k \leq n$. Since A is doubly well-spread, the above inequality holds and the result follows. □

Define $\eta(A) = a_n + a_{n-1} - a_2 - a_1 + \rho(A)$. Note that $\eta(A)$ is actually the largest label required in an edge-magic labeling of K_n that uses doubly well-spread sequence A for the vertex labels.

Lemma 3. *Given a doubly well-spread sequence $A = \{a_i\}_{i=1}^n$, the edge-magic labeling from Lemma 2 can be an extended to an edge-magic total labeling of $K_n \cup tK_1$ with $t = \eta(A) - n - \binom{n}{2}$ and magic constant $\mu = a_n + a_{n-1} + \rho(A)$.*

Proof. We will construct the edge-magic total labeling from the set $L = \{1, 2, \ldots, \eta(A)\}$, which contains the given doubly well-spread sequence A. Let K_n have the labeling from Lemma 2 that has each vertex labeled with some $a_i \in A$. Set $t = \eta(A) - n - \binom{n}{2}$ and append t copies of K_1 to the graph. Assign each unused label from the set L to one of the t K_1's. Since the total number of labels used on the K_n is the n from sequence A and one for each of the $\binom{n}{2}$ edges, the unused labels will fit exactly on the t copies of K_1 and the edge-magic labeling has been extended to a edge-magic total labeling as desired. □

Lemma 2 ensures that finding a doubly well-spread sequence A with n terms coincides with finding an edge-magic labeling of K_n, however, other well-spread sequences may also provide edge-magic labelings of K_n by using a different definition of $\rho(A)$ for the smallest edge label. Lemma 3 allows us to extend the labeling of K_n provided by a doubly well-spread sequence to an edge-magic total labeling of $K_n \cup tk_1$ with $t = \eta(A) - n - \binom{n}{2}$. We will make use of these two properties of doubly well-spread sequences throughout the paper.

3 Bounds for $M_E(K_n)$

In this section we provide two constructive upper bounds on $M_E(K_n)$, answering a question posed by Wallis [2].

3.1 Upper bound for $M_E(K_n)$, $n \geq 7$

First define $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, and $f_{n+1} = f_{n-1} + f_n$, the usual Fibonacci sequence, and define $F_n = \{f_i\}_{i=2}^{n+1}$; we call F_n the *n-term truncated Fibonacci sequence.* The n-term truncated Fibonacci sequence will provide the first bound, holding for all $n \geq 7$.

Claim 1. *For each $n \in \mathbb{Z}^+$, F_n is a doubly well-spread sequence.*

Proof. First we show that F_n is a well-spread sequence. Suppose, by way of contradiction, that it is not: then there exist four distinct elements f_i, f_j, f_k, f_l such that $f_i + f_j = f_k + f_l$.

Suppose without loss of generality that i is the largest index of i, j, k, l; hence $k = i - p$ for some p and $l = i - q$ for some q. Since f_i may be written as a sum of the previous terms in the sequence, we may rewrite f_i as $f_{i-p} + f_{i-q} + W$ where W is the rest of the terms needed to compose f_i. Note that $W \geq 0$. Now we have: $f_{i-p} + f_{i-q} + W + f_j = f_k + f_l$. Since $i - p = k$ and $i - q = l$, this forces $W + f_j = 0$, however no term in F_n is 0 and we have a contradiction.

Next we show that F_n is doubly well-spread. Since $f_2 = 1, f_3 = 2, f_4 = 3$, $\rho(F_n) = 4$. As before, suppose F_n is not doubly well-spread, this forces $f_{n+1} + f_n + \rho(F_n) - f_i - f_j = f_k$ for $i, j, k \leq n$. Without loss of generality, we may assume that $i < j$.

If $f_k = f_{n+1}$, then we have $f_n + 4 = f_i + f_j$. If $f_j = f_{n+1}$ also, then the contradiction is immediate. If instead $f_j < f_{n+1}$, it is possible to rewrite f_n as $f_j + W$ where $W \geq 0$ is zero or a sum of other elements of F_n. This gives $W + 4 = f_i$. If $W = 0$, the contradiction is immediate since 4 is not a Fibonacci number. Furthermore, if $W \neq 0$, then it may be rewritten as $f_i + W'$ with $W' \geq 0$. This gives $W' + 4 = 0$, certainly a contradiction.

If $f_k < f_{n+1}$, then we may follow a similar process to the above, rewriting f_{n+1} as $f_k + W$ and f_n as $f_j + W'$. This forces $W + W' + 4 = f_i$, and we can then rewrite W as $f_i + W''$ and obtain the same contradiction as above. □

The above claim gives a doubly well-spread sequence, which allows us to prove the following upper bound on the magic number of K_n.

Theorem 1. *For any integer $n \geq 7$,*

$$M_E(K_n) \leq \left\lceil \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+2} \right\rceil - \left(\frac{n^2+n-2}{2} \right).$$

Proof. Using F_n, construct an edge-magic total labeling for $K_n \cup tK_1$ by the construction in Lemma 3. Then $t = \eta(F_n) - n - \binom{n}{2}$. Since $M_E(K_n)$ is the minimum t of all labelings, we utilize the closed form of the i^{th} Fibonacci number to obtain:

$$\begin{aligned} M_E(K_n) \leq t = \eta(F_n) - n - \binom{n}{2} \\ &= f_{n+1} + f_n + 4 - f_2 - f_3 - n - \binom{n}{2} \\ &= f_{n+2} + 1 - n - \binom{n}{2} \\ &= \left\lceil \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+2} \right\rceil - \frac{n^2+n-2}{2} \end{aligned}$$

as desired. □

While this gives an upper bound on $M_E(K_n)$ for any $n \geq 7$, it can be improved substantially for larger values of n.

3.2 Upper bound for $M_E(K_n)$, $n \geq 10$

By using the n-term truncated Fibonacci sequence as a base, we can construct a second well-spread sequence that will allow us to manufacture an improved labeling for larger complete graphs.

Let F_n be the n-term truncated Fibonacci sequence from the previous subsection and define $F_n^\dagger = \{g_i\}_{i=1}^n$ as follows: set $g_n = f_{n+1} + 33$, $g_{n-1} = f_{n+1} + 22$, $g_{n-2} = f_{n+1} + 11$, and $g_i = f_{n+1} - f_{n+1-i}$ for $1 \leq i \leq n-3$. We call the sequence $F_n^\dagger$ the *semi-involuted Fibonacci sequence.* In order to construct a labeling with the semi-involuted Fibonacci sequence, we must first show that $F_n^\dagger$ is well-spread. The proof of this fact requires to following observation.

Define a positive integer d to be a *safe difference* for a sequence $\{b_i\}_{i=1}^n$ if d is such that $b_i + b_j \neq b_k - d$ for distinct $i, j, k \in [n]$.

Observation 1. *The numbers* 11, 22, 24, 33, 35, *and* 46 *are all safe differences for the Fibonacci sequence starting at* $f_4 = 3$.

Proof. Let d be one of the given differences and suppose by way of contradiction that there exist $i, j, k \geq 4$ such that $f_k - d = f_i + f_j$ with $f_j > f_k$. Note first that $f_i + f_j < f_k$. Furthermore, $f_i + f_k \leq f_{k-1} + f_{k-3}$ and, if the sum is smaller than that, it is smaller than f_{k-1}. Since Fibonacci numbers grow rapidly, for any $k \geq 14$ ($f_{14} = 377$), $f_k - 46 > f_{k-1} + f_{k-3}$, and hence d is a safe difference whenever $k \geq 14$.

For $k \leq 7$, $f_k - 11 < 2$; certainly no pairs of Fibonacci numbers at least 3 will sum to one of these values, making d a safe difference for all $k \leq 7$.

We now inspect each of the given differences individually for the values of k between 8 and 13.

Case: $d = 11$. Recall that, the assumption is $f_k - d = f_i + f_j$. If $k = 8$, this becomes $10 = f_i + f_j$, and since $f_i > f_j \geq 3$, the contradiction follows. Similarly if $k = 9$, we must have $23 = f_i + f_j$; since $f_{k-1} + f_{k-3} > 23$, the maximum sum is $f_{k-1} = 21$ and the contradiction follows. The case when $k = 10$ follows identically, as $f_{k-1} < f_k - 11 < f_{k-1} + f_{k-3}$. In the cases in which $k = 11, 12$ and $k = 13$, we have that $f_k - 11 > f_{k-1} + f_{k-3}$. This gives that $d = 11$ is a safe difference.

Case: $d = 22$. Since $f_8 < 22$, we need not consider that case. If $k = 9$, the assumption gives $12 = f_i + f_j$, again, certainly a contradiction. If $k = 10$, we obtain $33 = f_i + f_j$, which is also untrue. In the cases of $k = 11, 12$ and 13, we have $f_k - 22 > f_{k-1} + f_{k-3}$. Thus $d = 22$ is a safe difference.

Case: $d = 24$. Since $f_8 < 24$, we need not consider that case. If $k = 9$, the assumption gives $10 = f_i + f_j$, certainly a contradiction. If $k = 10$, we obtain $31 = f_i + f_j$, which is also untrue. If $k = 11$, the assumption gives $65 = f_i + f_j$, certainly a contradiction since no two Fibonacci numbers greater than 3 sum to 65. The case when $k = 12$ follows from the fact that $f_{k-1} < f_k - 24 < f_{k-1} + f_{k-3}$. In the case of $k = 13$, we have $f_k - 24 > f_{k-1} + f_{k-3}$. Hence $d = 24$ is a safe difference.

Case: $d = 33$. Since $f_9 = 34$, the contradiction is trivial. If $k = 10$, the assumption gives $22 = f_i + f_j$, certainly a contradiction. If $k = 11$, we obtain $56 = f_i + f_j$, which is also untrue. If $k = 12$, the assumption gives $111 = f_i + f_j$, certainly a contradiction since no two Fibonacci numbers greater than 3 sum to 111. In the case of $k = 13$, we have $f_k - 24 > f_{k-1} + f_{k-3}$. Hence $d = 33$ is a safe difference.

Case: $d = 35$. Since $f_9 = 34$, the contradiction is trivial. If $k = 10$, the assumption gives $20 = f_i + f_j$, certainly a contradiction. If $k = 11$, we obtain $54 = f_i + f_j$, which is also untrue. If $k = 12$, the assumption gives $109 = f_i + f_j$, certainly a contradiction since no two Fibonacci numbers greater than 3 sum to 109. In the case of $k = 13$, we have $f_k - 24 > f_{k-1} + f_{k-3}$. Hence $d = 35$ is a safe difference.

Case: $d = 46$. Since $f_9 = 34$, the contradiction is trivial. If $k = 10$, the assumption gives $9 = f_i + f_j$, certainly a contradiction. If $k = 11$, we obtain $43 = f_i + f_j$, which is also untrue. If $k = 12$, the assumption gives $98 = f_i + f_j$, certainly a contradiction since no two Fibonacci numbers greater than 3 sum to 98. In the case of $k = 13$, we have $187 = f_i + f_j$, also not a sum that can be formed from Fibonacci numbers. Hence $d = 46$ is a safe difference. □

With these safe differences, we can now prove that the semi-involuted Fibonacci sequence is well-spread.

Claim 2. *The sequence $F_n^\dagger$ is well-spread.*

Proof. We must show that $g_i + g_j \neq g_k + g_l$ for distinct i, j, k, l. Without loss of generality we may assume that $i > j, k, l$ and $k > l$. The proof is split into cases depending upon i.

Case: $i \leq n-3$. Suppose, by way of contradiction, that $g_i + g_j = g_k + g_l$ for some $i, j, k, l \in [n-3]$. By definition of the terms, we have:

$$f_{n+1} - f_{n+1-i} + f_{n+1} - f_{n+1-k=j} = f_{n+1} - f_{n+1-k} + f_{n+1} - f_{n+1-l}$$
$$f_{n+1-i} + f_{n+1-k} = f_{n+1-j} + f_{n+1-l}.$$

Since each of i, j, k, l are distinct and the n-term truncated Fibonacci sequences is well-spread, this is certainly a contradiction.

Case: $i = n$. As before, suppose by way of contradiction that the sequence is not well-spread. To attain a series of contradictions, we consider the possibilities for j and k.

If $j = n-1$, the result is trivial since g_n and g_{n-1} are the largest terms.

If $j = n-2$, then $k = n-1$, otherwise the result is also trivial. Suppose that this is the case and that there is some l such that $g_n + g_{n-2} = g_{n-1} + g_l$. By the definition of the semi-involuted Fibonacci sequence we have:

$$f_{n+1} + 33 + f_{n+1} + 11 = f_{n+1} + 22 + f_{n+1} - f_{n+1-l}$$
$$22 = -f_{n+1-l}$$

which is certainly a contradiction.

If instead, $j \leq n-3$ we condition upon k.

Consider the case in which $k = n-1$. This gives:

$$g_n + g_j = g_{n-1} + g_l$$
$$f_{n+1} + 33 + f_{n+1} - f_{n+1-j} = f_{n+1} + 22 + g_l$$
$$f_{n+1} + 11 = g_l + f_{n+1-j}.$$

If $l \leq n-3$, we have $11 = f_{n+1-j} - f_{n+1-l}$. Since the only Fibonacci numbers whose difference is 11 are 13 and 2, and since $l \leq n-3$, $f_{n+1-l} \geq f_4 = 3$, this gives the needed contradiction.

If instead, $l = n-2$, we have $0 = f_{n+1-j}$, a straightforward contradiction.

Now consider the case in which $k = n-2$. This gives:

$$g_n + g_j = g_{n-2} + g_l$$
$$f_{n+1} + 33 + f_{n+1} - f_{n+1-j} = f_{n+1} + 11 + f_{n+1} - f_{n+1-l}$$
$$f_{n+1-l} + 22 = f_{n+1-j}.$$

No Fibonacci numbers have a difference of 22, resulting in the needed contradiction.

Finally consider the case in which $k \leq n-3$. In this instance, we have:

$$f_{n+1} + 33 + f_{n+1} - f_{n+1-j} = f_{n+1} - f_{n+1-k} + f_{n+1} - f_{n+1-l}$$
$$f_{n+1-k} + f_{n+1-l} = f_{n+1-j} - 33.$$

Together with Observation 1, this gives the contradiction.

Case: $i = n-1$. As above, suppose by way of contradiction that the sequence is not well-spread. To attain a series of contradictions, we consider the possibilities for j and k.

If $j = n-2$, the result is trivial, since $i \geq j, k, l$.

If $j \leq n-3$ and $k = n-2$, then we have:

$$g_{n-1} + g_j = g_{n-2} + g_l$$
$$f_{n+1} + 22 + f_{n+1} - f_{n+1-j} = f_{n+1} + 11 + f_{n+1} - f_{n+1-l}$$
$$11 = f_{n+1-j} - f_{n+1-l}.$$

Just as before, no two members of the Fibonacci sequence greater than 3, have difference 11 and the contradiction follows.

If instead, $j, k, l \leq n-3$, we have:

$$g_{n-1} + g_j = g_k + g_l$$
$$f_{n+1} + 22 + f_{n+1} - f_{n+1-j} = f_{n+1} - f_{n+1-k} + f_{n+1} - f_{n+1-l}$$
$$f_{n+1-j} - 22 = f_{n+1-k} + f_{n+1-l}.$$

By Observation 1, the contradiction follows.

Case: $i = n-2$. Finally we consider the case in which $i = n-2$ and all of $j, k, l \leq n-3$. As before, suppose that the sequence is not well-spread. This gives:

$$g_{n-2} + g_j = g_k + g_l$$
$$f_{n+1} + 11 + f_{n+1} - f_{n-j} = f_{n+1} - f_{n+1-k} + f_{n+1} - f_{n+1-l}$$
$$f_{n+1-j} - 11 = f_{n+1-k} + f_{n+1-l}.$$

By Observation 1, we obtain a contradiction and the result follows.

Thus the semi-involuted Fibonacci sequence is well-spread. □

Unfortunately, the semi-involuted Fibonacci sequence is not doubly well-spread with the usual definition of $\rho(F_n^\dagger) = 1$. However, by instead setting $\rho(F_n^\dagger) = 2$ the sequence becomes doubly well-spread. The proof requires a quick observation.

Observation 2. *Let x be one of* 13,, 24, 35. *Then any $k \leq n$, $f_{n+1} - f_k \neq x$ provided $n \geq 10$.*

Proof. Suppose by way of contradiction that $f_{n+1} - f_k = x$ for some $k \leq n$. We consider two cases, $k = n$ and $k < n$.

Case: $k = n$. If $k = n$, then we have $f_{n-1} = x$. Since $n \geq 10$, $n - 1 \geq 9$ and so $f_{n-1} \geq 34$. The contradiction follows since 35 is not a Fibonacci number.

Case: $k < n$. If $k < n$, then we have $f_n + W = x$, where W is the remainder from the fact that $f_{n-1} = f_k + W$. Since $f_n > 55$, the contradiction is immediate. □

Claim 3. *Let $F_n^\dagger = \{g_i\}_{i=1}^n$ be the n-term semi-involuted Fibonacci sequence with $n \geq 10$ and set $\rho(F_n^\dagger) = 2$. Then $F_n^\dagger$ is doubly well-spread, namely $g_n + g_{n-1} + 2 - g_i - g_j \neq g_k$ for any $i, j, k \in [n]$.*

Proof. As in the proof of Claim 2, assume without loss of generality that $i > j$. We first observe that:

$$\begin{aligned} g_n + g_{n-1} + 2 &= f_{n+1} + 33 + f_{n+1} + 22 + 2 \\ &= 2f_{n+1} + 57. \end{aligned}$$

As before, we split the proof into cases depending on the value of i and pursue a contradiction in each instance.

Case: $i \leq n-3$. Suppose by way of contradiction that $g_n + g_{n-1} + 2 - g_i - g_j = g_k$ for some $i, j \leq n-3$ and $k \in [n]$. Then:

$$\begin{aligned} 2f_{n+1} + 57 - g_i - g_j &= g_k \\ 2f_{n+1} + 57 - (f_{n+1} - f_{n+1-i}) - (f_{n+1} - f_{n+1-j}) &= g_k \\ f_{n+1-i} + f_{n+1-j} + 57 &= g_k \end{aligned}$$

If $k = n$, then the last line becomes $f_{n+1-i} + f_{n+1-j} = f_{n+1} - 24$, a contradiction by Observation 1.

If $k = n - 1$, then the last line becomes $f_{n+1-i} + f_{n+1-j} = f_{n+1} - 35$, a contradiction by Observation 1.

If $k = n - 2$, then the last line becomes $f_{n+1-i} + f_{n+1-j} = f_{n+1} - 46$, a contradiction by Observation 1.

If $k \leq n-3$, then the last line becomes $f_{n+1-i} + f_{n+1-j} = f_{n+1} - f_{n+1-k} - 57$. Since $n \geq 10$, it can readily be observed that $f_{n+1} - 57 \neq f_x + f_y + f_z$ for any $x, y, z \leq n$, giving us the necessary contradiction.

Case: $i = n$. Suppose by way of contradiction that $g_n + g_{n-1} + 2 - g_n - g_j = g_k$ for some $j \leq n - 1$ and $k \in [n]$. Then we have $f_{n+1} + 24 - g_j = g_k$.

If $k = n$, then this becomes $g_j = -11$; certainly a contradiction.

If $k = n - 1$, then this becomes $g_j = 2$, also a contradiction since the minimum value of any $g_j = g_1 = f_{n+1} - f_n > 2$ for all $n \geq 10$.

If $k = n - 2$, then this becomes $g_j = 13$. Observation 2 gives us the contradiction.

Finally, if $k \leq n - 3$, then we have: $24 = g_j - f_{n+1-k}$. If $j \leq n - 3$, then we have a contradiction since 24 is a safe difference. If instead, $j = n - 1$ or $j = n - 2$, then we have either $f_{n+1-k} = f_{n+1} - 2$ or $f_{n+1} - 13$ respectively, also contradictions by Observation 2.

Case: $i = n-1$. Suppose by way of contradiction that $g_n + g_{n-1} + 2 - g_{n-1} - g_j = g_k$ for some $j \leq n-2$ and $k \in [n]$. Then we have $f_{n+1} + 35 - g_j = g_k$.

If $k = n$, then this becomes $g_j = 2$; a contradiction since the minimum value of any g_j is $g_1 = f_{n+1} - f_n > 2$ for all $n \geq 10$.

If $k = n-1$, then this becomes $g_j = 13$. Observation 2 gives us the contradiction.

If $k = n-2$, then this becomes $g_j = 24$. Observation 2 gives us the contradiction.

Finally, if $k \leq n-3$, then we have: $35 = g_j - f_{n+1-k}$. If $j \leq n-3$, then we have a contradiction since 35 is a safe difference. If instead, $j = n-2$, then Observation 2 gives us the contradiction.

Case: $i = n-2$. Suppose by way of contradiction that $g_n + g_{n-1} + 2 - g_{n-2} - g_j = g_k$ for some $j \leq n-3$ and $k \in [n]$. Then we have $46 + f_{n+1-j} = g_k$.

If $k = n$, then this becomes $f_{n+1-j} = f_{n+1} - 13$; a contradiction by Observation 2.

If $k = n-1$, then this becomes $f_{n+1-j} = f_{n+1} - 24$. Observation 2 gives us the contradiction.

If $k = n-2$, then this becomes $f_{n+1-j} = f_{n+1} - 35$. Observation 2 gives us the contradiction.

Finally, if $k \leq n-3$, then we have: $f_{n+1} - 35 = f_{n+1-k} + f_{n+1-k}$. Since 35 is a safe difference, this provides the desired contradiction and the result follows. □

Using the n-term semi-involuted Fibonacci sequence, we can construct an edge magic labeling for K_n, by Lemma 2 and an EMTL of $K_n \cup tK_1$ with $t = g_n + g_{n-1} + 2 - g_1 - g_2 - n - \binom{n}{2}$ by Lemma 3, using $\rho(F_n^\dagger) = 2$ as the smallest edge label. This new construction allows the following bound for $M_E(K_n)$.

Theorem 2. *For $n \geq 10$, we have:*

$$M_E(K_n) \leq \left\lceil \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} \right\rceil + 57 - \frac{n^2+n}{2}.$$

Proof. By Lemma 3, using the n-term semi-involuted Fibonacci sequence $F_n^\dagger = \{g_i\}_{i=1}^n$ with $\rho(F_n^\dagger) = 2$ we may construct an EMTL for $K_n \cup tK_1$. In the lemma,

t is exactly equal to the magic number of the labeling, so we have that:

$$\begin{aligned}
M_E(K_n) \leq t &= g_n + g_{n-1} + 2 - g_1 - g_2 - n - \binom{n}{2} \\
&= (f_{n+1} + 33) + (f_{n+1} + 22) + 2 \\
&\quad - (f_{n+1} - f_{n+1-1}) - (f_{n+1} - f_{n+1-2}) - n - \binom{n}{2} \\
&= f_n + f_{n-1} + 57 - n - \binom{n}{2} \\
&= f_{n+1} + 57 - \frac{n^2 + n}{2} \\
&= \left\lfloor \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} \right\rceil + 57 - \frac{n^2+n}{2}
\end{aligned}$$

□

Whenever $n \geq 10$, this bound is strictly an improvement on that given by the n-term truncated Fibonacci sequence.

4 Magic Numbers for Specific Complete Graphs

In this section we prove that $M_E(K_7) = 4$, $M_E(K_8) = 10$ and $M_E(K_9) = 19$. The lower bounds for much of this work depend on the nonexistence of well-spread sequences satisfying the necessary conditions. This nonexistence was shown using exhaustive computer searches created with C. The complete source code for these searches is available by emailing the corresponding author, the source code for the K_9 problem is included in Appendix A.

Observation 3. *For K_4 we have $M_E(K_4) = 1$.*

A labeling using one isolated vertex is illustrated in Figure 2, giving the upper bound. For the lower bound, Corollary 2.1.1 in [2] states that the complete graph K_n is not edge-magic when $n \equiv 4 \pmod 8$.

Theorem 3. *The magic number of K_7 is 4.*

Proof. For the upper bound, observe that the sequence

$$A = \{10, 18, 23, 26, 28, 29, 30\}$$

is a doubly well-spread sequence with $\rho(A) = 1$. Labeling the vertices with the numbers from this sequence and giving the edge $\{29, 30\}$ the label 1 gives magic constant $\mu = 60$ as in Lemma 3. The magic number for this labeling can be calculated by finding the maximum label used and subtracting off the number of labels used on the K_7 itself, as the rest of the labels will need to be placed on the isolated vertices to make a total labeling. In this case, since $\mu = 60$ the maximum label will be $60 - (10 + 18) = 32$ used on the $\{10, 18\}$ edge. So $M_E(K_7) \leq 32 - 28 = 4$.

The lower bound comes from a computer search showing that there are no doubly well-spread sequences that provide candidates for an EMTL of $K_7 \cup tK_1$ when $t \leq 3$. □

Theorem 4. *The magic number of K_8 is 10.*

Proof. To prove the upper bound, notice that $A = \{19, 25, 31, 36, 41, 43, 44, 45\}$ is a doubly well-spread sequence with $\rho(A) = 1$. Constructing the labeling using Lemma 3 gives magic constant $\mu = 90$ ans $t = 46 - 36 = 10$. Hence, $M_E(K_8) \leq 10$.

The lower bound comes from a computer search showing that there are no doubly well-spread sequences that provide candidates for an EMTL of $K_8 \cup tK_1$ when $t \leq 9$. □

Theorem 5. *The magic number of K_9 is 19.*

Proof. For the upper bound, notice that $A = \{26, 33, 40, 49, 54, 58, 59, 60, 62\}$ is a doubly well-spread sequence with $\rho(A) = 1$. Constructing the labeling using Lemma 3 gives magic constant $\mu = 123$ and $t = 64 - 45 = 19$ and thus, $M_E(K_9) \leq 19$.

The lower bound comes from a computer search showing that there are no doubly well-spread sequences that provide candidates for an EMTL of $K_9 \cup tK_1$ when $t \leq 18$. □

The patterns observed in the computer produced sequences and the basic bounds on well-spread sequences lead us to the following conjecture:

Conjecture 1. *It is possible to find an n-term doubly well-spread sequence with largest term no larger than $n^2 - 19$.*

5 Acknowledgments

We are grateful to our institutions, Norwich University and the University of Rhode Island, for supporting this research and to Dr. Nancy Eaton of the University of Rhode Island for her valuable feedback. We also wish to thank the anonymous referees for their helpful comments.

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[2] A. M. Marr and W. D. Wallis, *Magic Graphs*, Birkhäuser, Boston, MA, 2012.

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A Source code for confirming nonexistence of well-spread sequences needed in Section 4

```
// Well-spread Sequence producing code for K9 possible labelings

// Outline of Algorithm
// Initialize all the variables, especially important are M,
// the magic number, mu, the magic constant, toplabel, the
// largest label used in the labeling,
//and found, the number of sequences found.

// We declare x_1 to be the label of vertex 1,
//  x_2 to be label of vertex 2, and so on.
// The top loop runs through all possible values for M
//between 0 and the upper bound
//given by the Fibonacci sequence;
// it breaks out upon finding all of the sequences satisfying
//the first M value reached.
// The x_i loops check each sequence in the order:
// (1, 2, 3, . . .,9)
//then (1, 3, 4, . .),  and (1, 4, . . .) . . . finally resetting
// to (2, 3, 4. . .) and going through all possible sequences.
// each loop incrementing one of the numbers in the sequence
// at a time. So in the first run through: we have the sequence
//(1,2,3,4,5,6,7,8,9), the
// innermost loop increments the 9 until it reaches 'toplabel',
//before kicking it back to the next (x_8) loop which
//changes the 8 to an 9 giving (1,2,3,4,5,6,9,10) and
// repeating the (x_9) loop. Once all of these have been checked,
// it kicks back to the x_7 and so on, checking every possible
//sequence of 9 numbers to see if it is doubly well-spread.
// Whenever a sequence is found, the found counter increments.
// Once this counter is nonzero, the loop is broken out of since
// we have found a possible candidate for a labeling as needed.
#include <iostream>
#include <string>

int main()
{ int mu=0; //declare magic constant to be 0.
  int toplabel; //initialize the toplabel value to 0.
  int M; //declare M(9) to start at 0
  int found =0; //declare sequences found counter to be 0.
  for(M=0; M <=20; M++){
    toplabel = M+45; //set upper toplabel on label numbers to
```

```
    max + deadpoints.
    for(int x_1 = 2; x_1 <=toplabel-8; x_1++){
    //Declare x_1 to be the first vertex label, run through
    // all 1's, then all 2's, etc.
      for(int x_2 = x_1+1; x_2<=toplabel-7; x_2++){
      //Declare x_2 for v2 label test all 2's, then 3's and so on
int s_12=x_1+x_2;
//declare s12 to be edge sum for v1 v2 edge.
for(int x_3 = x_2+1; x_3 <=toplabel-6; x_3++){
//checking all versions of the v3 label with x3
  int s_13 = x_1+x_3; //declare more edge sums
  int s_23 = x_2+x_3;
    for(int x_4= x_3+1; x_4 <=toplabel-5; x_4++){
      int s_14=x_1+x_4;
      int s_24= x_2 + x_4; //more sums
      int s_34 = x_3+x_4;
for(int x_5 = x_4+1; x_5 <=toplabel-4; x_5++){
  int s_15=x_1 + x_5;
  int s_25=x_2+x_5;
  int s_35=x_3+x_5;
  int s_45=x_4+x_5;
    for(int x_6 =x_5+1; x_6 <=toplabel-3; x_6++){
      int s_16=x_1+x_6;
      int s_26=x_2+x_6;
      int s_36=x_3+x_6;
      int s_46 = x_4+x_6;
      int s_56 = x_5+x_6;
for(int x_7=x_6+1; x_7 <=toplabel-2; x_7++){
  int s_17= x_1+x_7;
  int s_27= x_2+x_7;
  int s_37= x_3+x_7;
  int s_47= x_4+x_7;
  int s_57= x_5+x_7;
  int s_67 = x_6+x_7;
    for(int x_8=x_7+1; x_8 <=toplabel-1; x_8++){
      int s_18=x_1+x_8;
      int s_28=x_2+x_8;
      int s_38=x_3+x_8;
      int s_48=x_4+x_8;
      int s_58=x_5+x_8;
      int s_68=x_6+x_8;
      int s_78=x_7+x_8;
      for(int x_9=x_8+1; x_9 <=toplabel; x_9++){

int s_19=x_1+x_9;
int s_29=x_2+x_9;
```

```
int s_39=x_3+x_9;
int s_49=x_4+x_9;
int s_59=x_5+x_9;
int s_69=x_6+x_9;
int s_79=x_7+x_9;
int s_89=x_8+x_9;
//compare each sum with every other sum, insuring
//they are not equal
if (
  (s_12 != s_23)&(s_12!=s_24)&(s_12 !=s_25)
 &(s_12 != s_26)&(s_12 !=s_27)&(s_12 != s_28)
 &(s_12 != s_29) &

  (s_13 != s_23)&(s_13!=s_24)&(s_13 !=s_25)
 &(s_13 != s_26)&(s_13 !=s_27)&(s_13 != s_28)
 &(s_13 != s_29) &

  (s_14 != s_23)&(s_14!=s_24)&(s_14 !=s_25)
 &(s_14 != s_26) &(s_14 !=s_27) &(s_14 != s_28)
 &(s_14 != s_29) &

  (s_15 != s_23)&(s_15!=s_24)&(s_15 !=s_25)
 &(s_15 != s_26)&(s_15 !=s_27)&(s_15 != s_28)
 &(s_15 != s_29) &

  (s_16 != s_23)&(s_16!=s_24)&(s_16 !=s_25)
 &(s_16 != s_26)&(s_16 !=s_27)&(s_16 != s_28)
 &(s_16 != s_29) &

  (s_17 != s_23)&(s_17!=s_24)&(s_17 !=s_25)
 &(s_17 != s_26)&(s_17 !=s_27)&(s_17 != s_28)
 &(s_17 != s_29) &

  (s_18 != s_23)&(s_18!=s_24)&(s_18 !=s_25)
 &(s_18 != s_26) &(s_18 !=s_27)&(s_18 != s_28)
 &(s_18 != s_29) &

  (s_19 != s_23)&(s_19!=s_24)&(s_19 !=s_25)
 &(s_19 != s_26)&(s_19 !=s_27)&(s_19 != s_28)
 &(s_19 != s_29) &

  (s_12 != s_34)&(s_12!=s_35)&(s_12 !=s_36)
 &(s_12 != s_37)&(s_12 != s_38)&(s_12 != s_39) &

  (s_13 != s_34)&(s_13!=s_35)&(s_13 !=s_36)
 &(s_13 != s_37)&(s_13 != s_38)&(s_13 != s_39) &
```

```
 (s_14 != s_34)&(s_14!=s_35)&(s_14 !=s_36)
&(s_14 != s_37)&(s_14 != s_38)&(s_14 != s_39) &

 (s_15 != s_34)&(s_15!=s_35)&(s_15 !=s_36)
&(s_15 != s_37)&(s_15 != s_38)&(s_15 != s_39) &

 (s_16 != s_34)&(s_16!=s_35)&(s_16 !=s_36)
&(s_16 != s_37) &(s_16 != s_38)&(s_16 != s_39) &

 (s_17 != s_34)&(s_17!=s_35)&(s_17 !=s_36)
&(s_17 != s_37)&(s_17 != s_38)&(s_17 != s_39) &

 (s_18 != s_34)&(s_18!=s_35)&(s_18 !=s_36)
&(s_18 != s_37)&(s_18 != s_38)&(s_18 != s_39) &

 (s_19 != s_34)&(s_19!=s_35)&(s_19 !=s_36)
&(s_19 != s_37)&(s_19 != s_38)&(s_19 != s_39) &

 (s_12 != s_45)&(s_12!=s_46)&(s_12 !=s_47)
&(s_12 != s_48)&(s_12 != s_49) &

 (s_13 != s_45)&(s_13!=s_46)&(s_13 !=s_47)
&(s_13 != s_48)&(s_13 != s_49) &

 (s_14 != s_45)&(s_14!=s_46)&(s_14 !=s_47)
&(s_14 != s_48) &(s_14 != s_49) &

 (s_15 != s_45)&(s_15!=s_46)&(s_15 !=s_47)
&(s_15 != s_48) &(s_15 != s_49) &

 (s_16 != s_45)&(s_16!=s_46)&(s_16 !=s_47)
&(s_16 != s_48)&(s_16 != s_49) &

 (s_17 != s_45)&(s_17!=s_46)&(s_17 !=s_47)
&(s_17 != s_48) &(s_17 != s_49) &

 (s_18 != s_45)&(s_18!=s_46)&(s_18 !=s_47)
&(s_18 != s_48) &(s_18 != s_49) &

 (s_19 != s_45)&(s_19!=s_46)&(s_19 !=s_47)
&(s_19 != s_48)&(s_19 != s_49) &

 (s_12 != s_56)&(s_12!=s_57)&(s_12 !=s_58)
&(s_12 != s_59) &
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 (s_13 != s_56)&(s_13!=s_57)&(s_13 !=s_58)
&(s_13 != s_59) &

 (s_14 != s_56)&(s_14!=s_57)&(s_14 !=s_58)
&(s_14 != s_59) &

 (s_15 != s_56)&(s_15!=s_57)&(s_15 !=s_58)
&(s_15 != s_59) &

 (s_16 != s_56)&(s_16!=s_57)&(s_16 !=s_58)
&(s_16 != s_59) &

 (s_17 != s_56)&(s_17!=s_57)&(s_17 !=s_58)
&(s_17 != s_59) &

 (s_18 != s_56)&(s_18 != s_57)&(s_18 != s_58)
&(s_18 != s_59) &

 (s_19 != s_56)&(s_19 != s_57)&(s_19 != s_58)
&(s_19 != s_59) &

 (s_12 != s_67)&(s_12 != s_68)&(s_12 != s_69) &
 (s_13 != s_67)&(s_13 != s_68)&(s_13 != s_69) &
 (s_14 != s_67)&(s_14 != s_68)&(s_14 != s_69) &
 (s_15 != s_67)&(s_15 != s_68)&(s_15 != s_69) &
 (s_16 != s_67)&(s_16 != s_68)&(s_16 != s_69) &
 (s_17 != s_67)&(s_17 != s_68)&(s_17 != s_69) &
 (s_18 != s_67)&(s_18 != s_68)&(s_18 != s_69) &
 (s_19 != s_67)&(s_19 != s_68)&(s_19 != s_69) &
 (s_12 != s_78)&(s_12 != s_79) &
 (s_13 != s_78)&(s_13 != s_79) &
 (s_14 != s_78)&(s_14 != s_79) &
 (s_15 != s_78)&(s_15 != s_79) &
 (s_16 != s_78)&(s_16 != s_79) &
 (s_17 != s_78)&(s_17 != s_79) &
 (s_18 != s_78)&(s_18 != s_79) &
 (s_19 != s_78)&(s_19 != s_79) &
 (s_12 != s_89) &
 (s_13 != s_89) &
 (s_14 != s_89) &
 (s_15 != s_89) &
 (s_16 != s_89) &
 (s_17 != s_89) &
 (s_18 != s_89) &
 (s_19 != s_89) &
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 (s_23 != s_34)&(s_23!=s_35)&(s_23 !=s_36)
&(s_23 !=s_37)&(s_23 != s_38)&(s_23 != s_39) &

 (s_24 != s_34)&(s_24!=s_35)&(s_24 !=s_36)
&(s_24 !=s_37)&(s_24 != s_38)&(s_24 != s_39) &

 (s_25 != s_34)&(s_25!=s_35)&(s_25 !=s_36)
&(s_25 !=s_37)&(s_25 != s_38)&(s_25 != s_39) &

 (s_26 != s_34)&(s_26!=s_35)&(s_26 !=s_36)
&(s_26 !=s_37) &(s_26 != s_38)&(s_26 != s_39) &

 (s_27 != s_34)&(s_27!=s_35)&(s_27 !=s_36)
&(s_27 !=s_37)&(s_27 != s_38)&(s_27 != s_39) &

 (s_28 != s_34)&(s_28!=s_35)&(s_28 !=s_36)
&(s_28 !=s_37)&(s_28 != s_38)&(s_28 != s_39) &

 (s_29 != s_34)&(s_29!=s_35)&(s_29 !=s_36)
&(s_29 !=s_37)&(s_29 != s_38)&(s_29 != s_39) &

 (s_23 != s_45)&(s_23!=s_46)&(s_23 !=s_47)
&(s_23 != s_48)&(s_23 != s_49) &

 (s_24 != s_45)&(s_24!=s_46)&(s_24 !=s_47)
&(s_24 != s_48)&(s_24 != s_49) &

 (s_25 != s_45)&(s_25!=s_46)&(s_25 !=s_47)
&(s_25 != s_48)&(s_25 != s_49) &

 (s_26 != s_45)&(s_26!=s_46)&(s_26 !=s_47)
&(s_26 != s_48) &(s_26 != s_49) &

 (s_27 != s_45)&(s_27!=s_46)&(s_27 !=s_47)
&(s_27 != s_48)&(s_27 != s_49) &

 (s_28 != s_45)&(s_28!=s_46)&(s_28 !=s_47)
&(s_28 != s_48)&(s_28 != s_49) &

 (s_29 != s_45)&(s_29!=s_46)&(s_29 !=s_47)
&(s_29 != s_48)&(s_29 != s_49) &

 (s_23 != s_56)&(s_23!=s_57)&(s_23 !=s_58)
&(s_23 != s_59) &
 (s_24 != s_56)&(s_24!=s_57)&(s_24 !=s_58)
&(s_24 != s_59) &
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 (s_25 != s_56)&(s_25!=s_57)&(s_25 !=s_58)
&(s_25 != s_59) &
 (s_26 != s_56)&(s_26!=s_57)&(s_26 !=s_58)
&(s_26 != s_59) &
 (s_27 != s_56)&(s_27!=s_57)&(s_27 !=s_58)
&(s_27 != s_59) &
 (s_28 != s_56)&(s_28!=s_57)&(s_28 !=s_58)
&(s_28 != s_59) &
 (s_29 != s_56)&(s_29!=s_57)&(s_29 !=s_58)
&(s_29 != s_59) &

 (s_23 != s_67)&(s_23 != s_68)&(s_23 != s_69) &
 (s_24 != s_67)&(s_24 != s_68)&(s_24 != s_69) &
 (s_25 != s_67)&(s_25 != s_68)&(s_25 != s_69) &
 (s_26 != s_67)&(s_26 != s_68)&(s_26 != s_69) &
 (s_27 != s_67)&(s_27 != s_68)&(s_27 != s_69) &
 (s_28 != s_67)&(s_28 != s_68)&(s_28 != s_69) &
 (s_29 != s_67)&(s_29 != s_68)&(s_29 != s_69) &
 (s_23 != s_78)&(s_23 != s_79) &
 (s_24 != s_78)&(s_24 != s_79) &
 (s_25 != s_78)&(s_25 != s_79) &
 (s_26 != s_78)&(s_26 != s_79) &
 (s_27 != s_78)&(s_27 != s_79) &
 (s_28 != s_78)&(s_28 != s_79) &
 (s_29 != s_78)&(s_29 != s_79) &
 (s_23 != s_89) &
 (s_24 != s_89) &
 (s_25 != s_89) &
 (s_26 != s_89) &
 (s_27 != s_89) &
 (s_28 != s_89) &
 (s_29 != s_89) &

 (s_34 != s_45)&(s_34 != s_46)&(s_34 != s_47)
&(s_34 != s_48)&(s_34 != s_49) &
 (s_35 != s_45)&(s_35 != s_46)&(s_35 != s_47)
&(s_35 != s_48)&(s_35 != s_49) &
 (s_36 != s_45)&(s_36 != s_46)&(s_36 != s_47)
&(s_36 != s_48)&(s_36 != s_49) &
 (s_37 != s_45)&(s_37 != s_46)&(s_37 != s_47)
&(s_37 != s_48)&(s_37 != s_49) &
 (s_38 != s_45)&(s_38 != s_46)&(s_38 != s_47)
&(s_38 != s_48)&(s_38 != s_49) &
 (s_39 != s_45)&(s_39 != s_46)&(s_39 != s_47)
&(s_39 != s_48)&(s_39 != s_49) &
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 (s_34 != s_56)&(s_34 != s_57)&(s_34 != s_58)
&(s_34 != s_59) &
 (s_35 != s_56)&(s_35 != s_57)&(s_35 != s_58)
&(s_35 != s_59) &
 (s_36 != s_56)&(s_36 != s_57)&(s_36 != s_58)
&(s_36 != s_59) &
 (s_37 != s_56)&(s_37 != s_57)&(s_37 != s_58)
&(s_37 != s_59) &
 (s_38 != s_56)&(s_38 != s_57)&(s_38 != s_58)
&(s_38 != s_59) &
 (s_39 != s_56)&(s_39 != s_57)&(s_39 != s_58)
&(s_39 != s_59) &

 (s_34 != s_67)&(s_34 != s_68)&(s_34 != s_69) &
 (s_35 != s_67)&(s_35 != s_68)&(s_35 != s_69) &
 (s_36 != s_67)&(s_36 != s_68)&(s_36 != s_69) &
 (s_37 != s_67)&(s_37 != s_68)&(s_37 != s_69) &
 (s_38 != s_67)&(s_38 != s_68)&(s_38 != s_69) &
 (s_39 != s_67)&(s_39 != s_68)&(s_39 != s_69) &
 (s_34 != s_78)&(s_34 != s_79) &
 (s_35 != s_78)&(s_35 != s_79) &
 (s_36 != s_78)&(s_36 != s_79) &
 (s_37 != s_78)&(s_37 != s_79) &
 (s_38 != s_78)&(s_38 != s_79) &
 (s_39 != s_78)&(s_39 != s_79) &
 (s_34 != s_89) &
 (s_35 != s_89) &
 (s_36 != s_89) &
 (s_37 != s_89) &
 (s_38 != s_89) &
 (s_39 != s_89) &

 (s_45 != s_56)&(s_45 != s_57)&(s_45 != s_58)
&(s_45 != s_59) &
 (s_46 != s_56)&(s_46 != s_57)&(s_46 != s_58)
&(s_46 != s_59) &
 (s_47 != s_56)&(s_47 != s_57)&(s_47 != s_58)
&(s_47 != s_59) &
 (s_48 != s_56)&(s_48 != s_57)&(s_48 != s_58)
&(s_48 != s_59) &
 (s_49 != s_56)&(s_49 != s_57)&(s_49 != s_58)
&(s_49 != s_59) &

 (s_45 != s_67)&(s_45 != s_68)&(s_45 != s_69) &
 (s_46 != s_67)&(s_46 != s_68)&(s_46 != s_69) &
 (s_47 != s_67)&(s_47 != s_68)&(s_47 != s_69) &
```

```
 (s_48 != s_67)&(s_48 != s_68)&(s_48 != s_69) &
 (s_49 != s_67)&(s_49 != s_68)&(s_49 != s_69) &
 (s_45 != s_78)&(s_45 != s_79) &
 (s_46 != s_78)&(s_46 != s_79) &
 (s_47 != s_78)&(s_47 != s_79) &
 (s_48 != s_78)&(s_48 != s_79) &
 (s_49 != s_78)&(s_49 != s_79) &
 (s_45 != s_89) &
 (s_46 != s_89) &
 (s_47 != s_89) &
 (s_48 != s_89) &
 (s_49 != s_89) &
 (s_56 != s_67)&(s_56 != s_68)&(s_56 != s_69) &
 (s_57 != s_67)&(s_57 != s_68)&(s_57 != s_69) &
 (s_58 != s_67)&(s_58 != s_68)&(s_58 != s_69) &
 (s_59 != s_67)&(s_59 != s_68)&(s_59 != s_69) &
 (s_56 != s_78)&(s_56 != s_79) &
 (s_57 != s_78)&(s_57 != s_79) &
 (s_58 != s_78)&(s_58 != s_79) &
 (s_59 != s_78)&(s_59 != s_79) &
 (s_56 != s_89) &
 (s_57 != s_89) &
 (s_58 != s_89) &
 (s_59 != s_89) &
 (s_67 != s_78)&(s_67 != s_79) &
 (s_68 != s_78)&(s_68 != s_79) &
 (s_69 != s_78)&(s_69 != s_79) &
 (s_67 != s_89) &
 (s_68 != s_89) &
 (s_69 != s_89) &
 (s_78 != s_89) &
 (s_79 != s_89)
){
    mu=s_89+1;
//check for the smallest value of edge label for edge 89.
//check to make sure that we can fill in each gap and none
// of the edge labels equal the vertex labels.
    if(
(mu-s_12 <=toplabel) &
(mu-s_12!= x_1)&(mu-s_12!=x_2)&(mu-s_12!=x_3)
& (mu-s_12!= x_4)&(mu-s_12!=x_5)&(mu-s_12!= x_6)
& (mu-s_12!=x_7)&(mu-s_12!= x_8)&(mu-s_12!= x_9) &

(mu-s_13!= x_1)&(mu-s_13!=x_2)&(mu-s_13!=x_3)
& (mu-s_13!= x_4)&(mu-s_13!=x_5)&(mu-s_13!= x_6)
& (mu-s_13!=x_7)&(mu-s_13!= x_8)&(mu-s_13!= x_9) &
```

```
(mu-s_14!= x_1)&(mu-s_14!=x_2)&(mu-s_14!=x_3)
& (mu-s_14!= x_4)&(mu-s_14!=x_5)&(mu-s_14!= x_6)
& (mu-s_14!=x_7)&(mu-s_14!= x_8)&(mu-s_14!= x_9) &

(mu-s_15!= x_1)&(mu-s_15!=x_2)&(mu-s_15!=x_3)
& (mu-s_15!= x_4)&(mu-s_15!=x_5)&(mu-s_15!= x_6)
& (mu-s_15!=x_7)&(mu-s_15!= x_8)&(mu-s_15!= x_9) &

(mu-s_16!= x_1)&(mu-s_16!=x_2)&(mu-s_16!=x_3)
& (mu-s_16!= x_4)&(mu-s_16!=x_5)&(mu-s_16!= x_6)
& (mu-s_16!=x_7)&(mu-s_16!= x_8)&(mu-s_16!= x_9) &

(mu-s_17!= x_1)&(mu-s_17!=x_2)&(mu-s_17!=x_3)
& (mu-s_17!= x_4)&(mu-s_17!=x_5)&(mu-s_17!= x_6)
& (mu-s_17!=x_7)&(mu-s_17!= x_8)&(mu-s_17!= x_9) &

(mu-s_18!= x_1)&(mu-s_18!=x_2)&(mu-s_18!=x_3)
& (mu-s_18!= x_4)&(mu-s_18!=x_5)& (mu-s_18!= x_6)
&(mu-s_18!=x_7)&(mu-s_18!= x_8)&(mu-s_18!= x_9) &

(mu-s_19!= x_1)&(mu-s_19!=x_2)&(mu-s_19!=x_3)
& (mu-s_19!= x_4)&(mu-s_19!=x_5)&(mu-s_19!= x_6)
& (mu-s_19!=x_7)&(mu-s_19!= x_8)&(mu-s_19!= x_9) &

(mu-s_23!= x_1)&(mu-s_23!=x_2)&(mu-s_23!=x_3)
& (mu-s_23!= x_4)&(mu-s_23!=x_5)&(mu-s_23!= x_6)
& (mu-s_23!=x_7)&(mu-s_23!= x_8)&(mu-s_23!= x_9) &

(mu-s_24!= x_1)&(mu-s_24!=x_2)&(mu-s_24!=x_3)
& (mu-s_24!= x_4)&(mu-s_24!=x_5)&(mu-s_24!= x_6)
& (mu-s_24!=x_7)&(mu-s_24!= x_8)&(mu-s_24!= x_9) &

(mu-s_25!= x_1)&(mu-s_25!=x_2)&(mu-s_25!=x_3)
& (mu-s_25!= x_4)&(mu-s_25!=x_5)&(mu-s_25!= x_6)
& (mu-s_25!=x_7)&(mu-s_25!= x_8)&(mu-s_25!= x_9) &

(mu-s_26!= x_1)&(mu-s_26!=x_2)&(mu-s_26!=x_3)
& (mu-s_26!= x_4)&(mu-s_26!=x_5)&(mu-s_26!= x_6)
& (mu-s_26!=x_7)&(mu-s_26!= x_8)&(mu-s_26!= x_9) &

(mu-s_27!= x_1)&(mu-s_27!=x_2)&(mu-s_27!=x_3)
& (mu-s_27!= x_4)&(mu-s_27!=x_5)&(mu-s_27!= x_6)
& (mu-s_27!=x_7)&(mu-s_27!= x_8)&(mu-s_27!= x_9) &

(mu-s_28!= x_1)&(mu-s_28!=x_2)&(mu-s_28!=x_3)
```

```
& (mu-s_28!= x_4)&(mu-s_28!=x_5)&(mu-s_28!= x_6)
& (mu-s_28!=x_7)&(mu-s_28!= x_8)&(mu-s_28!= x_9) &

(mu-s_29!= x_1)&(mu-s_29!=x_2)&(mu-s_29!=x_3)
& (mu-s_29!= x_4)&(mu-s_29!=x_5)&(mu-s_29!= x_6)
& (mu-s_29!=x_7)&(mu-s_29!= x_8)&(mu-s_29!= x_9) &

(mu-s_34!= x_1)&(mu-s_34!=x_2)&(mu-s_34!=x_3)
& (mu-s_34!= x_4)&(mu-s_34!=x_5)&(mu-s_34!= x_6)
& (mu-s_34!=x_7)&(mu-s_34!= x_8)&(mu-s_34!= x_9)&

(mu-s_35!= x_1)&(mu-s_35!=x_2)&(mu-s_35!=x_3)
& (mu-s_35!= x_4)&(mu-s_35!=x_5)&(mu-s_35!= x_6)
& (mu-s_35!=x_7)&(mu-s_35!= x_8)&(mu-s_35!= x_9) &

(mu-s_36!= x_1)&(mu-s_36!=x_2)&(mu-s_36!=x_3)
& (mu-s_36!= x_4)&(mu-s_36!=x_5)&(mu-s_36!= x_6)
& (mu-s_36!=x_7)&(mu-s_36!= x_8)&(mu-s_36!= x_9) &

(mu-s_37!= x_1)&(mu-s_37!=x_2)&(mu-s_37!=x_3)
& (mu-s_37!= x_4)&(mu-s_37!=x_5)&(mu-s_37!= x_6)
& (mu-s_37!=x_7)&(mu-s_37!= x_8)&(mu-s_37!= x_9) &

(mu-s_38!= x_1)&(mu-s_38!=x_2)&(mu-s_38!=x_3)
& (mu-s_38!= x_4)&(mu-s_38!=x_5)&(mu-s_38!= x_6)
& (mu-s_38!=x_7)&(mu-s_38!= x_8)&(mu-s_38!= x_9) &

(mu-s_39!= x_1)&(mu-s_39!=x_2)&(mu-s_39!=x_3)
& (mu-s_39!= x_4)&(mu-s_39!=x_5)&(mu-s_39!= x_6)
& (mu-s_39!=x_7)&(mu-s_39!= x_8)&(mu-s_39!= x_9) &

(mu-s_45!= x_1)&(mu-s_45!=x_2)&(mu-s_45!=x_3)
& (mu-s_45!= x_4)&(mu-s_45!=x_5)&(mu-s_45!= x_6)
& (mu-s_45!=x_7)&(mu-s_45!= x_8)&(mu-s_45!= x_9)&

(mu-s_46!= x_1)&(mu-s_46!=x_2)&(mu-s_46!=x_3)
& (mu-s_46!= x_4)&(mu-s_46!=x_5)&(mu-s_46!= x_6)
& (mu-s_46!=x_7)&(mu-s_46!= x_8)&(mu-s_46!= x_9)&

(mu-s_47!= x_1)&(mu-s_47!=x_2)&(mu-s_47!=x_3)
& (mu-s_47!= x_4)&(mu-s_47!=x_5)&(mu-s_47!= x_6)
& (mu-s_47!=x_7)&(mu-s_47!= x_8)&(mu-s_47!= x_9)&

(mu-s_48!= x_1)&(mu-s_48!=x_2)&(mu-s_48!=x_3)
& (mu-s_48!= x_4)&(mu-s_48!=x_5)&(mu-s_48!= x_6)
& (mu-s_48!=x_7)&(mu-s_48!= x_8)&(mu-s_48!= x_9) &
```

```
(mu-s_49!= x_1)&(mu-s_49!=x_2)&(mu-s_49!=x_3)
& (mu-s_49!= x_4)&(mu-s_49!=x_5)&(mu-s_49!= x_6)
& (mu-s_49!=x_7)&(mu-s_49!= x_8)&(mu-s_49!= x_9)&

(mu-s_56!= x_1)&(mu-s_56!=x_2)&(mu-s_56!=x_3)
& (mu-s_56!= x_4)&(mu-s_56!=x_5)&(mu-s_56!= x_6)
& (mu-s_56!=x_7)&(mu-s_56!= x_8)&(mu-s_56!= x_9)&

(mu-s_57!= x_1)&(mu-s_57!=x_2)&(mu-s_57!=x_3)
& (mu-s_57!= x_4)&(mu-s_57!=x_5)&(mu-s_57!= x_6)
& (mu-s_57!=x_7)&(mu-s_57!= x_8)&(mu-s_57!= x_9)&

(mu-s_58!= x_1)&(mu-s_58!=x_2)&(mu-s_58!=x_3)
& (mu-s_58!= x_4)&(mu-s_58!=x_5)&(mu-s_58!= x_6)
& (mu-s_58!=x_7)&(mu-s_58!= x_8)&(mu-s_58!= x_9) &

(mu-s_59!= x_1)&(mu-s_59!=x_2)&(mu-s_59!=x_3)
& (mu-s_59!= x_4)&(mu-s_59!=x_5)&(mu-s_59!= x_6)
& (mu-s_59!=x_7)&(mu-s_59!= x_8)&(mu-s_59!= x_9)&

(mu-s_67!= x_1)&(mu-s_67!=x_2)&(mu-s_67!=x_3)
& (mu-s_67!= x_4)&(mu-s_67!=x_5)&(mu-s_67!= x_6)
& (mu-s_67!=x_7)&(mu-s_67!= x_8)&(mu-s_67!= x_9)&

(mu-s_68!= x_1)&(mu-s_68!=x_2)&(mu-s_68!=x_3)
& (mu-s_68!= x_4)&(mu-s_68!=x_5)&(mu-s_68!= x_6)
& (mu-s_68!=x_7)&(mu-s_68!= x_8)&(mu-s_68!= x_9)&

(mu-s_69!= x_1)&(mu-s_69!=x_2)&(mu-s_69!=x_3)
& (mu-s_69!= x_4)&(mu-s_69!=x_5)&(mu-s_69!= x_6)
& (mu-s_69!=x_7)&(mu-s_69!= x_8)&(mu-s_69!= x_9)&

(mu-s_78!= x_1)&(mu-s_78!=x_2)&(mu-s_78!=x_3)
& (mu-s_78!= x_4)&(mu-s_78!=x_5)&(mu-s_78!= x_6)
& (mu-s_78!=x_7)&(mu-s_78!= x_8)&(mu-s_78!= x_9)&

(mu-s_79!= x_1)&(mu-s_79!=x_2)&(mu-s_79!=x_3)
& (mu-s_79!= x_4)&(mu-s_79!=x_5)&(mu-s_79!= x_6)
& (mu-s_79!=x_7)&(mu-s_79!= x_8)&(mu-s_79!= x_9)&

(mu-s_89!= x_1)&(mu-s_89!=x_2)&(mu-s_89!=x_3)
& (mu-s_89!= x_4)&(mu-s_89!=x_5)&(mu-s_89!= x_6)
& (mu-s_89!=x_7)&(mu-s_89!= x_8)&(mu-s_89!= x_9)
){
```

```
found = found +1;
//increment found counter if a sequence satisfies the
//conditions
std::cout <<"mu=" << mu << " M=" << M <<
" label:{" <<x_1 << ","<<x_2 << ","<< x_3 << ","<<x_4
<< "," << x_5 << "," << x_6 <<","<<x_7 <<"," <<x_8
<<"," <<x_9 <<"}\n";}
  //print sequence of labels

}
}
  }
}
      }
    }
 }
        }
       }
    }
  if(found >0){break;}
  //breaks loop after finding all sequences that
  // work for the first number of isolated vertices
  //to have a sequence.
  else{std::cout <<"M="<<M<<" none\n";}

  }
}
```