Decompositions of λK_n into LOW and OLW Graphs

Derek W. Hein

ABSTRACT. In this paper, we identify LOW and OLW graphs, find the minimum λ for decomposition of λK_n into these graphs, and show that for all viable values of λ, the necessary conditions are sufficient for LOW– and OLW–decompositions using cyclic decompositions from base graphs.

1. Introduction

Decompositions of graphs into subgraphs is a well–known classical problem; for an excellent survey on graph decompositions, see [1]. Recently, several people including Chan [4], El–Zanati, Lapchinda, Tangsupphathawat and Wannasit [5], Hein [**6, 7, 8**], Hurd [**12**], Malick [**13**], Sarvate [**9, 10, 11**], Winter [**15, 16**] and Zhang [**17**] have results on decomposing λK_n into multigraphs. In fact, similar decompositions have been attempted earlier in various papers; see [**14**]. Ternary designs also provide such decompositions; see [**2, 3**].

Hein [**6, 7, 8**] showed how to decompose λK_n into LO, LE, LW and OW graphs. In the sequel, we show how to decompose λK_n into LOW and OLW graphs. Though the main technique used is to construct appropriate base graphs and to develop them cyclically, an additional approach is needed in each type of decomposition.

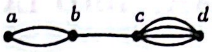
2. Preliminaries

For simplicity of notation, we use the "alphabetic labeling" used in [**6, 7, 8, 9, 10, 11, 15, 16, 17**]:

2000 *Mathematics Subject Classification.* Primary 05C51.
Key words and phrases. Cyclic graph decompositions, LOW graph, OLW graph.

DEFINITION 1. *An* LOW graph *(denoted* $\langle\langle a,b,c,d\rangle\rangle$*) on* $V = \{a,b,c,d\}$ *is a graph with 7 edges where the frequencies of edges* $\{a,b\}$, $\{b,c\}$ *and* $\{c,d\}$ *are 1, 2 and 4 (respectively).*

DEFINITION 2. *An* OLW graph *(denoted* $|||a,b,c,d|||$*) on* $V = \{a,b,c,d\}$ *is a graph with 7 edges where the frequencies of edges* $\{a,b\}$, $\{b,c\}$ *and* $\{c,d\}$ *are 2, 1 and 4 (respectively).*

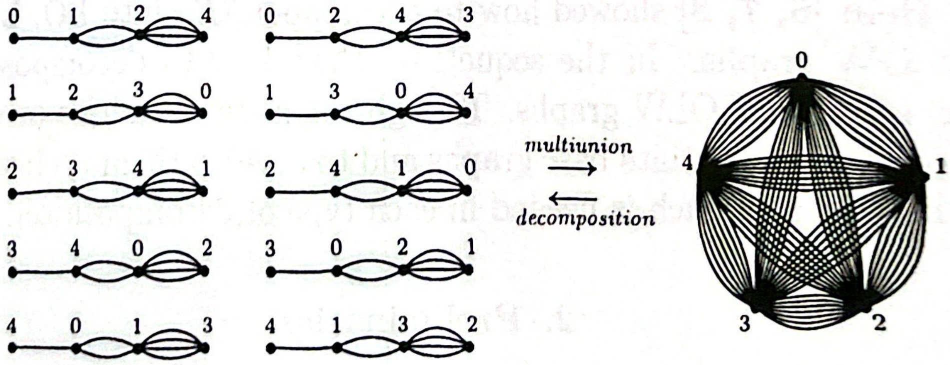


DEFINITION 3. *For positive integers* $n \geq 4$ *and* $\lambda \geq 4$, *an* LOW–decomposition *of* λK_n *(denoted* $LOW(n,\lambda)$*) is a collection of LOW graphs such that the multiunion of their edge sets contains* λ *copies of all edges in a* K_n.

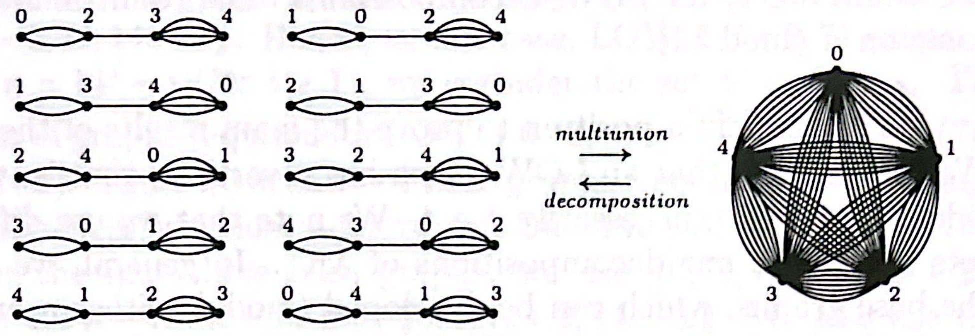
DEFINITION 4. *For positive integers* $n \geq 4$ *and* $\lambda \geq 4$, *an* OLW–decomposition *of* λK_n *(denoted* $OLW(n,\lambda)$*) is a collection of OLW graphs such that the multiunion of their edge sets contains* λ *copies of all edges in a* K_n.

One of the powerful techniques to construct combinatorial designs is based on *difference sets* and *difference families*; see [18] for details. This technique is modified to achieve our decompositions of λK_n — in general, we exhibit the *base graphs*, which can be developed to obtain the decomposition.

EXAMPLE 1. *Considering the set of points to be* $V = \mathbb{Z}_5$, *the LOW base graphs* $\langle\langle 0,1,2,4\rangle\rangle$ *and* $\langle\langle 0,2,4,3\rangle\rangle$ *(when developed modulo 5) constitute an* $LOW(5,7)$.

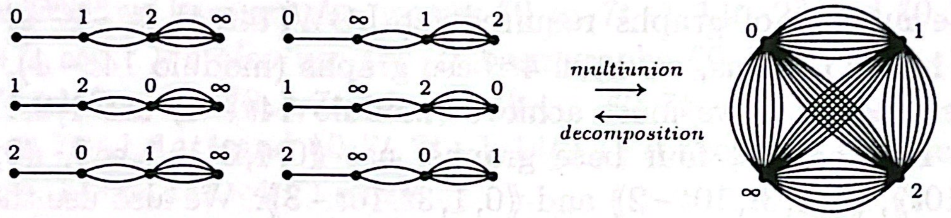


EXAMPLE 2. *Considering the set of points to be* $V = \mathbb{Z}_5$, *the OLW base graphs* $|||0,2,3,4|||$ *and* $|||1,0,2,4|||$ *(when developed modulo 5) constitute an* $OLW(5,7)$.

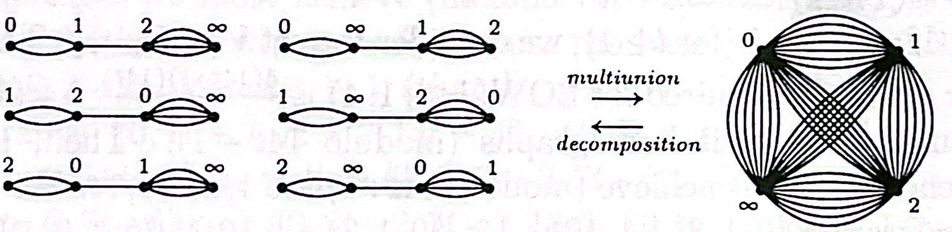


We note that special attention is needed with base graphs containing the "dummy element" ∞; the non-∞ elements are developed, while ∞ is simply rewritten each time.

EXAMPLE 3. *Considering the set of points to be* $V = \mathbb{Z}_3 \cup \{\infty\}$, *the LOW base graphs* $\langle\!\langle 0,1,2,\infty \rangle\!\rangle$ *and* $\langle\!\langle 0,\infty,1,2 \rangle\!\rangle$ *(when developed modulo 3) constitute an* $LOW(4,7)$.



EXAMPLE 4. *Considering the set of points to be* $V = \mathbb{Z}_3 \cup \{\infty\}$, *the OLW base graphs* $|||0,1,2,\infty|||$ *and* $|||0,\infty,1,2|||$ *(when developed modulo 3) constitute an* $OLW(4,7)$.



3. LOW–Decompositions

We first address the minimum values of λ in an LOW(n,λ). Recall that $\lambda \geq 4$.

THEOREM 3.1. *Let* $n \geq 4$. *The minimum values of* λ *for which an* $LOW(n,\lambda)$ *could exist are* $\lambda = 4$ *when* $n \equiv 0,1 \pmod 7$ *and* $\lambda = 7$ *when* $n \not\equiv 0,1 \pmod 7$.

PROOF. Since there are $\frac{\lambda n(n-1)}{2}$ edges in a λK_n, and 7 edges in an LOW graph, we must have that $\lambda n(n-1) \equiv 0 \pmod{14}$ (where

$n \geq 4$ and $\lambda \geq 4$) for LOW–decompositions. The result follows from cases on n (mod 14). ■

We are now in a position to prove the main results of the paper. We first remark that an LOW graph has 4 vertices; that is, we consider $n \geq 4$. Also, necessarily $\lambda \geq 4$. We note that we use difference sets to achieve our decompositions of λK_n. In general, we exhibit the base graphs, which can be developed (modulo either n or $n-1$) to obtain the decomposition. We also note that the frequency of the edges is fixed by position, as per the LOW graph.

THEOREM 3.2. *The minimum number copies of K_n (as given in Theorem 3.1) can be decomposed into LOW graphs.*

PROOF. Let $n \geq 4$. We proceed by cases on n (mod 14).

If $n = 14t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t-1} \cup \{\infty\}$. The number of graphs required for LOW$(14t, 4)$ is $\frac{4(14t)(14t-1)}{14} = 4t(14t-1)$. Thus, we need $4t$ base graphs (modulo $14t-1$). Then, the differences we must achieve (modulo $14t-1$) are $1, 2, \ldots, 7t-1$. For the first four base graphs, use $\langle\langle 0, 1, 3t+1, \infty\rangle\rangle$, $\langle\langle 0, 1, 3t+1, 10t\rangle\rangle$, $\langle\langle 0, 1, 3t, 10t-2\rangle\rangle$ and $\langle\langle 0, 1, 3t, 10t-3\rangle\rangle$. We also use the $4t-4$ base graphs $\langle\langle 0, 2, 3t, 10t-4\rangle\rangle$, $\langle\langle 0, 2, 3t, 10t-5\rangle\rangle$, $\langle\langle 0, 2, 3t-1, 10t-7\rangle\rangle$, $\langle\langle 0, 2, 3t-1, 10t-8\rangle\rangle, \ldots, \langle\langle 0, t, 2t+2, 5t+6\rangle\rangle$, $\langle\langle 0, t, 2t+2, 5t+5\rangle\rangle$, $\langle\langle 0, t, 2t+1, 5t+3\rangle\rangle$ and $\langle\langle 0, t, 2t+1, 5t+2\rangle\rangle$ if necessary. Hence, in this case, LOW$(14t, 4)$ exists.

If $n = 14t+1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t+1}$. The number of graphs required for LOW$(14t+1, 4)$ is $\frac{4(14t+1)(14t)}{14} = 4t(14t+1)$. Thus, we need $4t$ base graphs (modulo $14t+1$). Then, the differences we must achieve (modulo $14t+1$) are $1, 2, \ldots, 7t$. We use the base graphs $\langle\langle 0, 1, 3t+1, 10t+1\rangle\rangle$, $\langle\langle 0, 1, 3t+1, 10t\rangle\rangle$, $\langle\langle 0, 1, 3t, 10t-2\rangle\rangle$, $\langle\langle 0, 1, 3t, 10t-3\rangle\rangle$, $\langle\langle 0, 2, 3t, 10t-4\rangle\rangle$, $\langle\langle 0, 2, 3t, 10t-5\rangle\rangle$, $\langle\langle 0, 2, 3t-1, 10t-7\rangle\rangle$, $\langle\langle 0, 2, 3t-1, 10t-8\rangle\rangle, \ldots, \langle\langle 0, t, 2t+2, 5t+6\rangle\rangle$, $\langle\langle 0, t, 2t+2, 5t+5\rangle\rangle$, $\langle\langle 0, t, 2t+1, 5t+3\rangle\rangle$ and $\langle\langle 0, t, 2t+1, 5t+2\rangle\rangle$. Hence, in this case, LOW$(14t+1, 4)$ exists.

If $n = 14t+2$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t+1} \cup \{\infty\}$. The number of graphs required for LOW$(14t+2, 7)$ is $\frac{7(14t+2)(14t+1)}{14} = (7t+1)(14t+1)$. Thus, we need $7t+1$ base graphs (modulo $14t+1$). Then, the differences we must achieve (modulo $14t+1$) are $1, 2, \ldots, 7t$. For the first two base graphs, use $\langle\langle 0, \infty, 7t, 14t\rangle\rangle$ and $\langle\langle 0, 7t, 14t, \infty\rangle\rangle$. We also use the $7t-1$ base graphs $\langle\langle 0, 1, 7t, 7t+1\rangle\rangle$, $\langle\langle 0, 2, 7t, 7t+2\rangle\rangle$,

$\langle\!\langle 0,3,7t,7t+3\rangle\!\rangle,\dots,\langle\!\langle 0,7t-3,7t,14t-3\rangle\!\rangle$, $\langle\!\langle 0,7t-2,7t,14t-2\rangle\!\rangle$ and $\langle\!\langle 0,7t-1,7t,14t-1\rangle\!\rangle$. Hence, in this case, $\mathrm{LOW}(14t+2,7)$ exists.

If $n = 14t+3$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t+3}$. The number of graphs required for $\mathrm{LOW}(14t+3,7)$ is $\frac{7(14t+3)(14t+2)}{14} = (7t+1)(14t+3)$. Thus, we need $7t+1$ base graphs (modulo $14t+3$). Then, the differences we must achieve (modulo $14t+3$) are $1,2,\dots,7t+1$. For the first three base graphs, we use $\langle\!\langle 0,7t,7t+1,14t+2\rangle\!\rangle$, $\langle\!\langle 0,7t+1,14t+1,14t+2\rangle\!\rangle$ and $\langle\!\langle 0,1,7t+2,14t+2\rangle\!\rangle$. We also use the $7t-2$ base graphs $\langle\!\langle 0,2,7t+1,7t+3\rangle\!\rangle$, $\langle\!\langle 0,3,7t+1,7t+4\rangle\!\rangle$, $\langle\!\langle 0,4,7t+1,7t+5\rangle\!\rangle,\dots,\langle\!\langle 0,7t-3,7t+1,14t-2\rangle\!\rangle$, $\langle\!\langle 0,7t-2,7t+1,14t-1\rangle\!\rangle$ and $\langle\!\langle 0,7t-1,7t+1,14t\rangle\!\rangle$. Hence, in this case, $\mathrm{LOW}(14t+3,7)$ exists.

If $n = 14t+4$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+3}\cup\{\infty\}$. The number of graphs required for $\mathrm{LOW}(14t+4,7)$ is $\frac{7(14t+4)(14t+3)}{14} = (7t+2)(14t+3)$. Thus, we need $7t+2$ base graphs (modulo $14t+3$). Then, the differences we must achieve (modulo $14t+3$) are $1,2,\dots,7t+1$. For the first two base graphs, we use $\langle\!\langle 0,\infty,7t+1,14t+2\rangle\!\rangle$ and $\langle\!\langle 0,7t+1,14t+2,\infty\rangle\!\rangle$. We also use the $7t$ base graphs $\langle\!\langle 0,1,7t+1,7t+2\rangle\!\rangle$, $\langle\!\langle 0,2,7t+1,7t+3\rangle\!\rangle$, $\langle\!\langle 0,3,7t+1,7t+4\rangle\!\rangle,\dots,\langle\!\langle 0,7t-2,7t+1,14t-1\rangle\!\rangle$, $\langle\!\langle 0,7t-1,7t+1,14t\rangle\!\rangle$ and $\langle\!\langle 0,7t,7t+1,14t+1\rangle\!\rangle$ if necessary. Hence, in this case, $\mathrm{LOW}(14t+4,7)$ exists.

If $n = 14t+5$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+5}$. The number of graphs required for $\mathrm{LOW}(14t+5,7)$ is $\frac{7(14t+5)(14t+4)}{14} = (7t+2)(14t+5)$. Thus, we need $7t+2$ base graphs (modulo $14t+5$). Then, the differences we must achieve (modulo $14t+5$) are $1,2,\dots,7t+2$. When $t=0$ (that is, when $n=5$), we use the base graphs $\langle\!\langle 0,2,3,4\rangle\!\rangle$ and $\langle\!\langle 1,0,2,4\rangle\!\rangle$. When $t \geq 1$ (that is, when $n \geq 19$), we use the base graphs $\langle\!\langle 0,7t+1,7t+2,14t+4\rangle\!\rangle$, $\langle\!\langle 0,7t+2,14t+3,14t+4\rangle\!\rangle$, $\langle\!\langle 0,1,7t+3,14t+4\rangle\!\rangle$ as well as $\langle\!\langle 0,2,7t+2,7t+4\rangle\!\rangle$, $\langle\!\langle 0,3,7t+2,7t+5\rangle\!\rangle$, $\langle\!\langle 0,4,7t+2,7t+6\rangle\!\rangle,\dots,\langle\!\langle 0,7t-2,7t+2,14t\rangle\!\rangle$, $\langle\!\langle 0,7t-1,7t+2,14t+1\rangle\!\rangle$ and $\langle\!\langle 0,7t,7t+2,14t+2\rangle\!\rangle$. Hence, in this case, $\mathrm{LOW}(14t+5,7)$ exists.

If $n = 14t+6$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+5}\cup\{\infty\}$. The number of graphs required for $\mathrm{LOW}(14t+6,7)$ is $\frac{7(14t+6)(14t+5)}{14} = (7t+3)(14t+5)$. Thus, we need $7t+3$ base graphs (modulo $14t+5$). Then, the differences we must achieve (modulo $14t+5$) are $1,2,\dots,7t+2$. For the first two base graphs, we use $\langle\!\langle 0,\infty,7t+2,14t+4\rangle\!\rangle$ and $\langle\!\langle 0,7t+2,14t+4,\infty\rangle\!\rangle$. We also use the $7t+1$ base graph(s) $\langle\!\langle 0,1,7t+2,7t+3\rangle\!\rangle$, $\langle\!\langle 0,2,7t+2,7t+4\rangle\!\rangle$, $\langle\!\langle 0,3,7t+2,7t+5\rangle\!\rangle,\dots,\langle\!\langle 0,7t-1,7t+2,14t+1\rangle\!\rangle$, $\langle\!\langle 0,7t,7t+2,14t+2\rangle\!\rangle$ and $\langle\!\langle 0,7t+1,7t+2,14t+3\rangle\!\rangle$. Hence, in this case, $\mathrm{LOW}(14t+6,7)$ exists.

If $n = 14t+7$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+6} \cup \{\infty\}$. The number of graphs required for LOW$(14t+7, 4)$ is $\frac{4(14t+7)(14t+6)}{14} = (4t+2)(14t+6)$. Thus, we need $4t+2$ base graphs (modulo $14t+6$). Then, the differences we must achieve (modulo $14t+6$) are $1, 2, \ldots, 7t+3$. When $t = 0$ (that is, when $n = 7$), we use the base graphs $\langle\!\langle 0, 3, 4, \infty \rangle\!\rangle$ and $\langle\!\langle 0, 3, 4, 2 \rangle\!\rangle$. When $t \geq 1$ (that is, when $n \geq 21$), we use the base graphs $\langle\!\langle 0, 7t+3, 7t+4, \infty \rangle\!\rangle$, $\langle\!\langle 0, 7t+3, 7t+4, 7t+6 \rangle\!\rangle$ as well as $\langle\!\langle 0, 3, 3t+5, 10t+7 \rangle\!\rangle$, $\langle\!\langle 0, 3, 3t+5, 10t+6 \rangle\!\rangle$, $\langle\!\langle 0, 3, 3t+4, 10t+4 \rangle\!\rangle$, $\langle\!\langle 0, 3, 3t+4, 10t+3 \rangle\!\rangle$, $\langle\!\langle 0, 4, 3t+4, 10t+2 \rangle\!\rangle$, $\langle\!\langle 0, 4, 3t+4, 10t+1 \rangle\!\rangle$, $\langle\!\langle 0, 4, 3t+3, 10t-1 \rangle\!\rangle$, $\langle\!\langle 0, 4, 3t+3, 10t-2 \rangle\!\rangle, \ldots, \langle\!\langle 0, t+2, 2t+6, 5t+12 \rangle\!\rangle$, $\langle\!\langle 0, t+2, 2t+6, 5t+11 \rangle\!\rangle$, $\langle\!\langle 0, t+2, 2t+5, 5t+9 \rangle\!\rangle$ and $\langle\!\langle 0, t+2, 2t+5, 5t+8 \rangle\!\rangle$. Hence, in this case, LOW$(14t+7, 4)$ exists.

If $n = 14t+8$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+8}$. The number of graphs required for LOW$(14t+8, 4)$ is $\frac{4(14t+8)(14t+7)}{14} = (4t+2)(14t+8)$. Thus, we need $4t+2$ base graphs (modulo $14t+8$). Then, the differences we must achieve (modulo $14t+8$) are $1, 2, \ldots, 7t+4$. When $t = 0$ (that is, when $n = 8$), we use the base graphs $\langle\!\langle 0, 4, 5, 7 \rangle\!\rangle$ and $\langle\!\langle 0, 4, 5, 2 \rangle\!\rangle$. When $t \geq 1$ (that is, when $n \geq 21$), we use the base graphs $\langle\!\langle 0, 7t+4, 7t+5, 7t+7 \rangle\!\rangle$, $\langle\!\langle 0, 7t+4, 7t+5, 7t+8 \rangle\!\rangle$ as well as $\langle\!\langle 0, 4, 3t+7, 10t+10 \rangle\!\rangle$, $\langle\!\langle 0, 4, 3t+7, 10t+9 \rangle\!\rangle$, $\langle\!\langle 0, 4, 3t+6, 10t+7 \rangle\!\rangle$, $\langle\!\langle 0, 4, 3t+6, 10t+6 \rangle\!\rangle$, $\langle\!\langle 0, 5, 3t+6, 10t+5 \rangle\!\rangle$, $\langle\!\langle 0, 5, 3t+6, 10t+4 \rangle\!\rangle$, $\langle\!\langle 0, 5, 3t+5, 10t+2 \rangle\!\rangle$, $\langle\!\langle 0, 5, 3t+5, 10t+1 \rangle\!\rangle, \ldots, \langle\!\langle 0, t+3, 2t+8, 5t+15 \rangle\!\rangle$, $\langle\!\langle 0, t+3, 2t+8, 5t+14 \rangle\!\rangle$, $\langle\!\langle 0, t+3, 2t+7, 5t+12 \rangle\!\rangle$ and $\langle\!\langle 0, t+3, 2t+7, 5t+11 \rangle\!\rangle$. Hence, in this case, LOW$(14t+8, 4)$ exists.

If $n = 14t+9$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+9}$. The number of graphs required for LOW$(14t+9, 7)$ is $\frac{7(14t+9)(14t+8)}{14} = (7t+4)(14t+9)$. Thus, we need $7t+4$ base graphs (modulo $14t+9$). Then, the differences we must achieve (modulo $14t+9$) are $1, 2, \ldots, 7t+4$. For the first three base graphs, we use $\langle\!\langle 0, 7t+4, 7t+5, 14t+8 \rangle\!\rangle$, $\langle\!\langle 0, 7t+3, 14t+7, 14t+8 \rangle\!\rangle$ and $\langle\!\langle 0, 1, 7t+4, 14t+8 \rangle\!\rangle$. We also use the $7t+1$ base graphs $\langle\!\langle 0, 2, 7t+4, 14t+6 \rangle\!\rangle$, $\langle\!\langle 0, 3, 7t+4, 14t+5 \rangle\!\rangle$, $\langle\!\langle 0, 4, 7t+4, 14t+4 \rangle\!\rangle, \ldots, \langle\!\langle 0, 7t, 7t+4, 7t+8 \rangle\!\rangle$, $\langle\!\langle 0, 7t+1, 7t+4, 7t+7 \rangle\!\rangle$ and $\langle\!\langle 0, 7t+2, 7t+4, 7t+6 \rangle\!\rangle$. Hence, in this case, LOW$(14t+9, 7)$ exists.

If $n = 14t+10$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+9} \cup \{\infty\}$. The number of graphs required for LOW$(14t+10, 7)$ is $\frac{7(14t+10)(14t+9)}{14} = (7t+5)(14t+9)$. Thus, we need $7t+5$ base graphs (modulo $14t+9$). Then, the differences we must achieve (modulo $14t+9$) are $1, 2, \ldots, 7t+4$. For the first two base graphs, we use $\langle\!\langle 0, \infty, 7t+$

$4, 14t+8\rangle\!\rangle$ and $\langle\!\langle 0, 7t+4, 14t+8, \infty\rangle\!\rangle$. We also use the $7t+3$ base graphs $\langle\!\langle 0, 1, 7t+4, 7t+5\rangle\!\rangle$, $\langle\!\langle 0, 2, 7t+4, 7t+6\rangle\!\rangle$, $\langle\!\langle 0, 3, 7t+4, 7t+7\rangle\!\rangle, \ldots, \langle\!\langle 0, 7t+1, 7t+4, 14t+5\rangle\!\rangle$, $\langle\!\langle 0, 7t+2, 7t+4, 14t+6\rangle\!\rangle$ and $\langle\!\langle 0, 7t+3, 7t+4, 14t+7\rangle\!\rangle$. Hence, in this case, LOW$(14t+10, 7)$ exists.

If $n = 14t + 11$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+11}$. The number of graphs required for LOW$(14t + 11, 7)$ is $\frac{7(14t+11)(14t+10)}{14} = (7t + 5)(14t + 11)$. Thus, we need $7t + 5$ base graphs (modulo $14t + 11$). Then, the differences we must achieve (modulo $14t + 11$) are $1, 2, \ldots, 7t + 5$. For the first three base graphs, we use $\langle\!\langle 0, 7t + 5, 7t + 6, 14t + 10\rangle\!\rangle$, $\langle\!\langle 0, 7t + 4, 14t + 9, 14t + 10\rangle\!\rangle$ and $\langle\!\langle 0, 1, 7t + 5, 14t + 10\rangle\!\rangle$. We also use the $7t + 2$ base graphs $\langle\!\langle 0, 2, 7t + 5, 7t + 7\rangle\!\rangle$, $\langle\!\langle 0, 3, 7t + 5, 7t + 8\rangle\!\rangle, \ldots, \langle\!\langle 0, 7t + 2, 7t + 5, 14t + 7\rangle\!\rangle$ and $\langle\!\langle 0, 7t + 3, 7t + 5, 14t + 8\rangle\!\rangle$. Hence, in this case, LOW$(14t + 11, 7)$ exists.

If $n = 14t + 12$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+11} \cup \{\infty\}$. The number of graphs required for LOW$(14t+12, 7)$ is $\frac{7(14t+12)(14t+11)}{14} = (7t + 6)(14t + 11)$. Thus, we need $7t + 6$ base graphs (modulo $14t+11$). Then, the differences we must achieve (modulo $14t+11$) are $1, 2, \ldots, 7t+5$. For the first two base graphs, we use $\langle\!\langle 0, \infty, 7t+5, 14t+10\rangle\!\rangle$ and $\langle\!\langle 0, 7t + 5, 14t + 10, \infty\rangle\!\rangle$. We also use the $7t + 4$ base graphs $\langle\!\langle 0, 1, 7t+5, 7t+6\rangle\!\rangle$, $\langle\!\langle 0, 2, 7t+5, 7t+7\rangle\!\rangle$, $\langle\!\langle 0, 3, 7t+5, 7t+8\rangle\!\rangle, \ldots, \langle\!\langle 0, 7t+2, 7t+5, 14t+7\rangle\!\rangle$, $\langle\!\langle 0, 7t+3, 7t+5, 14t+8\rangle\!\rangle$ and $\langle\!\langle 0, 7t+4, 7t+5, 14t+9\rangle\!\rangle$. Hence, in this case, LOW$(14t + 12, 7)$ exists.

If $n = 14t + 13$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+13}$. The number of graphs required for LOW$(14t + 13, 7)$ is $\frac{7(14t+13)(14t+12)}{14} = (7t + 6)(14t + 13)$. Thus, we need $7t + 6$ base graphs (modulo $14t + 13$). Then, the differences we must achieve (modulo $14t + 13$) are $1, 2, \ldots, 7t + 6$. For the first three base graphs, we use $\langle\!\langle 0, 7t + 6, 7t + 7, 14t + 12\rangle\!\rangle$, $\langle\!\langle 0, 7t+5, 14t+11, 14t+12\rangle\!\rangle$ and $\langle\!\langle 0, 1, 7t+6, 14t+12\rangle\!\rangle$. We also use the $7t+3$ base graphs $\langle\!\langle 0, 2, 7t+6, 7t+8\rangle\!\rangle$, $\langle\!\langle 0, 3, 7t+6, 7t+9\rangle\!\rangle$, $\langle\!\langle 0, 4, 7t+6, 7t+10\rangle\!\rangle, \ldots, \langle\!\langle 0, 7t+2, 7t+6, 14t+8\rangle\!\rangle$, $\langle\!\langle 0, 7t+3, 7t+6, 14t+9\rangle\!\rangle$ and $\langle\!\langle 0, 7t + 4, 7t + 6, 14t + 10\rangle\!\rangle$. Hence, in this case, LOW$(14t + 13, 7)$ exists. ∎

We now address the sufficiency of existence of LOW(n, λ).

THEOREM 3.3. *Let $n \geq 4$ and $\lambda \geq 4$. For existence of LOW(n, λ), the necessary condition for n is that $n \equiv 0, 1 \pmod 7$ when $\lambda \not\equiv 0 \pmod 7$. There is no condition for n when $\lambda \equiv 0 \pmod 7$.*

PROOF. Similar to the proof of Theorem 3.1, but by cases on $\lambda \pmod{14}$. ∎

LEMMA 3.1. *There exists an $LOW(n,4)$ for necessary $n \geq 4$.*

PROOF. From Theorem 3.3, the necessary condition is $n \equiv 0, 1, 7, 8 \pmod{14}$. In these cases, LOW$(n,4)$ exists from Theorem 3.2. ∎

LEMMA 3.2. *There does not exist an $LOW(n,5)$.*

PROOF. The only edge frequencies in an LOW graph are 1, 2 and 4. The only ways to write $\lambda = 5$ as a sum of 1s, 2s and 4s are as $5 = 4+1$, $5 = 2+2+1$, $5 = 2+1+1+1$ and $5 = 1+1+1+1+1$. In an LOW$(n,5)$, the number of times each edge needs to occur with frequency 4 is always the same as the number of times it needs to occur with frequency 1. Every other way to realize $\lambda = 5$ using edge frequencies of 2 will contribute at least one more unmatched edge frequency of 1. Thus, such a decomposition is not possible. ∎

LEMMA 3.3. *There exists an $LOW(n,6)$ for necessary $n \geq 4$.*

PROOF. From Theorem 3.3, the necessary condition is $n \equiv 0, 1, 7, 8 \pmod{14}$.

If $n = 14t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t-1} \cup \{\infty\}$. The number of graphs required for LOW$(14t,6)$ is $\frac{6(14t)(14t-1)}{14} = 6t(14t-1)$. Thus, we need $6t$ base graphs (modulo $14t-1$). The differences we must achieve (modulo $14t-1$) are $1, 2, \ldots, 7t-1$. We use the base graphs $\langle\langle 0,1,\infty,2\rangle\rangle$, $\langle\langle 0,1,7t,8t+1\rangle\rangle$, $\langle\langle 0,1,7t-1,8t+1\rangle\rangle$, $\langle\langle 0,1,7t-2,8t+1\rangle\rangle$, $\langle\langle 0,1,7t-3,8t+1\rangle\rangle$, $\langle\langle 0,1,7t-4,8t+1\rangle\rangle, \ldots,$ $\langle\langle 0,t,2t+6,9t\rangle\rangle$, $\langle\langle 0,t,2t+5,9t\rangle\rangle$, $\langle\langle 0,t,2t+4,9t\rangle\rangle$, $\langle\langle 0,t,2t+3,9t\rangle\rangle$, $\langle\langle 0,t,2t+2,9t\rangle\rangle$ and $\langle\langle 0,t,2t+1,9t\rangle\rangle$. Hence, in this case, LOW$(14t,6)$ exists.

If $n = 14t+1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t+1}$. The number of graphs required for LOW$(14t+1,6)$ is $\frac{6(14t+1)(14t)}{14} = 6t(14t+1)$. Thus, we need $6t$ base graphs (modulo $14t+1$). The differences we must achieve (modulo $14t+1$) are $1, 2, \ldots, 7t$. We use the base graphs $\langle\langle 0,1,7t+1,8t+2\rangle\rangle$, $\langle\langle 0,1,7t,8t+2\rangle\rangle$, $\langle\langle 0,1,7t-1,8t+2\rangle\rangle$, $\langle\langle 0,1,7t-2,8t+2\rangle\rangle$, $\langle\langle 0,1,7t-3,8t+2\rangle\rangle$, $\langle\langle 0,1,7t-4,8t+2\rangle\rangle, \ldots,$ $\langle\langle 0,t,2t+6,9t+1\rangle\rangle$, $\langle\langle 0,t,2t+5,9t+1\rangle\rangle$, $\langle\langle 0,t,2t+4,9t+1\rangle\rangle$, $\langle\langle 0,t,2t+3,9t+1\rangle\rangle$, $\langle\langle 0,t,2t+2,9t+1\rangle\rangle$ and $\langle\langle 0,t,2t+1,9t+1\rangle\rangle$. Hence, in this case, LOW$(14t+1,6)$ exists.

If $n = 14t+7$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+6} \cup \{\infty\}$. The number of graphs required for LOW$(14t+7,6)$ is $\frac{6(14t+7)(14t+6)}{14} = (6t+3)(14t+6)$. Thus, we need $6t+3$ base graphs (modulo $14t+6$). The differences we must achieve (modulo $14t+6$) are $1, 2, \ldots, 7t+3$. For the first three base graphs, we use $\langle\langle 0,7t+3,14t+4,\infty\rangle\rangle$,

$\langle\!\langle \infty, 0, 7t+3, 14t+4 \rangle\!\rangle$ and $\langle\!\langle \infty, 0, 7t+2, 14t+4 \rangle\!\rangle$. We also use the $6t$ base graphs $\langle\!\langle 0, 1, 7t+1, 8t+2 \rangle\!\rangle$, $\langle\!\langle 0, 1, 7t, 8t+2 \rangle\!\rangle$, $\langle\!\langle 0, 1, 7t-1, 8t+2 \rangle\!\rangle$, $\langle\!\langle 0, 1, 7t-2, 8t+2 \rangle\!\rangle$, $\langle\!\langle 0, 1, 7t-3, 8t+2 \rangle\!\rangle$, $\langle\!\langle 0, 1, 7t-4, 8t+2 \rangle\!\rangle, \ldots, \langle\!\langle 0, t, 2t+6, 9t+1 \rangle\!\rangle$, $\langle\!\langle 0, t, 2t+5, 9t+1 \rangle\!\rangle$, $\langle\!\langle 0, t, 2t+4, 9t+1 \rangle\!\rangle$, $\langle\!\langle 0, t, 2t+3, 9t+1 \rangle\!\rangle$, $\langle\!\langle 0, t, 2t+2, 9t+1 \rangle\!\rangle$ and $\langle\!\langle 0, t, 2t+1, 9t+1 \rangle\!\rangle$ if necessary. Hence, in this case, LOW$(14t+7, 6)$ exists.

If $n = 14t+8$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+8}$. The number of graphs required for LOW$(14t+8, 6)$ is $\frac{6(14t+8)(14t+7)}{14} = (6t+3)(14t+8)$. Thus, we need $6t+3$ base graphs (modulo $14t+8$). The differences we must achieve (modulo $14t+8$) are $1, 2, \ldots, 7t+4$. For the first three base graphs, we use $\langle\!\langle 0, 7t+4, 14t+6, 14t+7 \rangle\!\rangle$, $\langle\!\langle 0, 1, 7t+5, 14t+7 \rangle\!\rangle$ and $\langle\!\langle 0, 1, 7t+4, 14t+7 \rangle\!\rangle$. We also use the $6t$ base graphs $\langle\!\langle 0, 2, 7t+3, 8t+5 \rangle\!\rangle$, $\langle\!\langle 0, 2, 7t+2, 8t+5 \rangle\!\rangle$, $\langle\!\langle 0, 2, 7t+1, 8t+5 \rangle\!\rangle$, $\langle\!\langle 0, 2, 7t, 8t+5 \rangle\!\rangle$, $\langle\!\langle 0, 2, 7t-1, 8t+5 \rangle\!\rangle$, $\langle\!\langle 0, 2, 7t-2, 8t+5 \rangle\!\rangle, \ldots, \langle\!\langle 0, t+1, 2t+8, 9t+4 \rangle\!\rangle$, $\langle\!\langle 0, t+1, 2t+7, 9t+4 \rangle\!\rangle$, $\langle\!\langle 0, t+1, 2t+6, 9t+4 \rangle\!\rangle$, $\langle\!\langle 0, t+1, 2t+5, 9t+4 \rangle\!\rangle$, $\langle\!\langle 0, t+1, 2t+4, 9t+4 \rangle\!\rangle$ and $\langle\!\langle 0, t+1, 2t+3, 9t+4 \rangle\!\rangle$ if necessary. Hence, in this case, LOW$(14t+8, 6)$ exists. ■

LEMMA 3.4. *There exists an LOW$(n, 7)$ for any $n \geq 4$.*

PROOF. From Theorem 3.3, there is no condition for n. We consider cases when $n \geq 4$ is odd or even.

If $n = 2t+1$ (for $t \geq 2$), we consider the set V as $\mathbb{Z}_{2t+1}$. The number of graphs required for LOW$(2t+1, 7)$ is $\frac{7(2t+1)(2t)}{14} = t(2t+1)$. Thus, we need t base graphs (modulo $2t+1$). The differences we must achieve (modulo $2t+1$) are $1, 2, \ldots, t$. When $t = 2$ (that is, when $n = 5$), we use the base graphs $\langle\!\langle 0, 1, 3, 4 \rangle\!\rangle$ and $\langle\!\langle 0, 2, 3, 1 \rangle\!\rangle$. When $t \geq 3$ (that is, when $n \geq 7$), we use the base graphs $\langle\!\langle 0, 1, t+1, 2t \rangle\!\rangle$ and $\langle\!\langle 0, 2, 3, t+3 \rangle\!\rangle$ as well as $\langle\!\langle 0, 3, 5, t+3 \rangle\!\rangle$, $\langle\!\langle 0, 4, 7, t+4 \rangle\!\rangle$, $\langle\!\langle 0, 5, 9, t+5 \rangle\!\rangle, \ldots, \langle\!\langle 0, t-2, 2t-5, 2t-2 \rangle\!\rangle$, $\langle\!\langle 0, t-1, 2t-3, 2t-1 \rangle\!\rangle$ and $\langle\!\langle 0, t, 2t-1, 2t \rangle\!\rangle$. Hence, in this case, LOW$(2t+1, 7)$ exists.

If $n = 2t$ (for $t \geq 2$), we consider the set V as $\mathbb{Z}_{2t-1} \cup \{\infty\}$. The number of graphs required for LOW$(2t, 7)$ is $\frac{7(2t)(2t-1)}{14} = t(2t-1)$. Thus, we need t base graphs (modulo $2t-1$). The differences we must achieve (modulo $2t-1$) are $1, 2, \ldots, t-1$. When $t = 2$ (that is, when $n = 4$), we use the base graphs $\langle\!\langle \infty, 0, 1, 2 \rangle\!\rangle$ and $\langle\!\langle 0, 1, \infty, 2 \rangle\!\rangle$. When $t \geq 3$ (that is, when $n \geq 6$), we use the base graphs $\langle\!\langle \infty, 0, t-1, 2t-2 \rangle\!\rangle$, $\langle\!\langle 0, t-1, \infty, 1 \rangle\!\rangle$ as well as $\langle\!\langle 0, 1, t-1, t \rangle\!\rangle$, $\langle\!\langle 0, 2, t-1, t+1 \rangle\!\rangle, \ldots, \langle\!\langle 0, t-3, t-1, 2t-4 \rangle\!\rangle$ and $\langle\!\langle 0, t-2, t-1, 2t-3 \rangle\!\rangle$. Hence, in this case, LOW$(2t, 7)$ exists. ■

The following examples play important roles in the sequel.

EXAMPLE 5. *The LOW graphs* $⟪1,2,3,4⟫$, $⟪1,3,5,6⟫$, $⟪1,4,6,7⟫$, $⟪1,5,4,7⟫$, $⟪2,1,3,6⟫$, $⟪2,1,4,7⟫$, $⟪2,3,1,6⟫$, $⟪2,4,5,7⟫$, $⟪2,7,3,4⟫$, $⟪3,2,1,7⟫$, $⟪3,2,6,4⟫$, $⟪3,5,6,1⟫$, $⟪3,6,4,2⟫$, $⟪3,7,2,4⟫$, $⟪4,3,2,7⟫$, $⟪4,5,6,7⟫$, $⟪4,6,2,5⟫$, $⟪5,2,1,3⟫$, $⟪6,1,5,4⟫$, $⟪6,2,7,1⟫$, $⟪6,3,2,5⟫$, $⟪6,3,7,5⟫$, $⟪6,5,3,7⟫$, $⟪7,1,2,6⟫$, $⟪7,4,1,5⟫$, $⟪7,5,1,4⟫$ *and* $⟪7,6,3,5⟫$ *constitute an example of an* $LOW(7,9)$ *with point set* $V = \{1,\dots,7\}$.

EXAMPLE 6. *The LOW graphs* $⟪1,2,3,5⟫$, $⟪1,4,6,7⟫$, $⟪1,4,6,8⟫$, $⟪1,5,6,8⟫$, $⟪1,7,2,3⟫$, $⟪1,8,2,6⟫$, $⟪2,3,4,7⟫$, $⟪2,5,4,3⟫$, $⟪2,7,3,6⟫$, $⟪2,8,1,5⟫$, $⟪3,1,6,5⟫$, $⟪3,5,8,7⟫$, $⟪3,7,4,8⟫$, $⟪3,7,8,4⟫$, $⟪3,8,1,2⟫$, $⟪4,1,6,3⟫$, $⟪4,2,5,7⟫$, $⟪4,3,1,2⟫$, $⟪4,5,8,2⟫$, $⟪4,6,7,1⟫$, $⟪5,2,7,1⟫$, $⟪5,2,8,3⟫$, $⟪5,6,2,4⟫$, $⟪5,6,7,2⟫$, $⟪5,8,7,3⟫$, $⟪6,1,5,3⟫$, $⟪6,2,3,1⟫$, $⟪6,3,4,5⟫$, $⟪6,5,1,8⟫$, $⟪6,8,5,7⟫$, $⟪7,3,1,4⟫$, $⟪7,4,5,2⟫$, $⟪7,5,8,3⟫$, $⟪7,6,2,4⟫$, $⟪8,4,1,6⟫$ *and* $⟪8,7,4,6⟫$ *constitute an example of an* $LOW(8,9)$ *with point set* $V = \{1,\dots,8\}$.

EXAMPLE 7. *The LOW graphs* $⟪a,1,b,2⟫$, $⟪a,3,b,4⟫$, $⟪a,5,b,6⟫$, $⟪a,7,b,1⟫$, $⟪b,3,a,2⟫$, $⟪b,4,a,6⟫$, $⟪b,5,a,4⟫$, $⟪b,7,a,1⟫$, $⟪1,b,3,a⟫$, $⟪2,a,5,b⟫$, $⟪2,b,1,a⟫$, $⟪2,b,5,a⟫$, $⟪2,b,7,a⟫$, $⟪4,a,6,b⟫$, $⟪6,a,3,b⟫$, $⟪6,a,4,b⟫$, $⟪6,a,7,b⟫$ *and* $⟪6,b,2,a⟫$ *constitute an example of an LOW–decomposition of* $9K_{\{a,b\},\{1,2,3,4,5,6,7\}}$.

Sarvate, Winter and Zhang [**15, 16**] have obtained several results on such multigraph decompositions of bipartite graphs.

LEMMA 3.5. *There exists an* $LOW(n,9)$ *for necessary* $n \geq 4$.

PROOF. From Theorem 3.3, the necessary condition is $n \equiv 0,1,7,8 \pmod{14}$.

If $n = 14t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t-1} \cup \{\infty\}$. The number of graphs required for LOW$(14t,9)$ is $\frac{9(14t)(14t-1)}{14} = 9t(14t-1)$. Thus, we need $9t$ base graphs (modulo $14t-1$). The differences we must achieve (modulo $14t-1$) are $1,2,\dots,7t-1$. We use the base graphs $⟪0,\infty,1,2⟫$, $⟪0,\infty,1,3⟫$, $⟪0,\infty,1,4⟫$, $⟪0,1,3,7t⟫$, $⟪0,2,5,7t+3⟫$, $⟪0,3,4,7t+3⟫$, $⟪0,7t-3,7t,14t-2⟫$, $⟪0,7t-2,7t-1,14t-2⟫$, $⟪0,7t-1,7t+1,14t-2⟫$; $⟪0,7t-4,14t-8,14t-4⟫$, $⟪0,7t-4,14t-8,14t-3⟫$, $⟪0,7t-4,14t-8,14t-2⟫$, $⟪0,4,9,7t+2⟫$, $⟪0,5,11,7t+5⟫$, $⟪0,6,10,7t+5⟫$, $⟪0,7t-7,7t-1,14t-7⟫$, $⟪0,7t-6,7t-2,14t-7⟫$, $⟪0,7t-5,7t,14t-7⟫,\dots,⟪0,3t+4,6t+8,9t+6⟫$, $⟪0,3t+4,6t+8,9t+7⟫$, $⟪0,3t+4,6t+8,9t+8⟫$, $⟪0,3t-2,6t-3,9t-2⟫$, $⟪0,3t-1,6t-1,9t+1⟫$,

$\langle\!\langle 0, 3t, 6t-2, 9t+1\rangle\!\rangle$, $\langle\!\langle 0, 3t+1, 6t+1, 9t+3\rangle\!\rangle$, $\langle\!\langle 0, 3t+2, 6t, 9t+3\rangle\!\rangle$ and $\langle\!\langle 0, 3t+3, 6t+2, 9t+3\rangle\!\rangle$. Hence, in this case, LOW$(14t, 9)$ exists.

If $n = 14t+1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t+1}$. The number of graphs required for LOW$(14t+1, 9)$ is $\frac{9(14t+1)(14t)}{14} = 9t(14t+1)$. Thus, we need $9t$ base graphs (modulo $14t+1$). The differences we must achieve (modulo $14t+1$) are $1, 2, \ldots, 7t$. We use the base graphs $\langle\!\langle 0, 1, 7t+1, 14t\rangle\!\rangle$, $\langle\!\langle 0, 2, 7t+2, 14t\rangle\!\rangle$, $\langle\!\langle 0, 3, 7t+3, 14t\rangle\!\rangle$, $\langle\!\langle 0, 7t-3, 7t, 14t-2\rangle\!\rangle$, $\langle\!\langle 0, 7t-2, 7t-1, 14t-2\rangle\!\rangle$, $\langle\!\langle 0, 7t-1, 7t+1, 14t-2\rangle\!\rangle$, $\langle\!\langle 0, 7t, 7t+1, 7t+4\rangle\!\rangle$, $\langle\!\langle 0, 7t, 7t+2, 7t+4\rangle\!\rangle$, $\langle\!\langle 0, 7t, 7t+3, 7t+4\rangle\!\rangle$; $\langle\!\langle 0, 4, 7t, 14t-5\rangle\!\rangle$, $\langle\!\langle 0, 5, 7t+1, 14t-5\rangle\!\rangle$, $\langle\!\langle 0, 6, 7t+2, 14t-5\rangle\!\rangle$, $\langle\!\langle 0, 7t-7, 7t-1, 14t-7\rangle\!\rangle$, $\langle\!\langle 0, 7t-6, 7t-2, 14t-7\rangle\!\rangle$, $\langle\!\langle 0, 7t-5, 7t, 14t-7\rangle\!\rangle$, $\langle\!\langle 0, 7t-4, 7t, 7t+6\rangle\!\rangle$, $\langle\!\langle 0, 7t-4, 7t+1, 7t+6\rangle\!\rangle$, $\langle\!\langle 0, 7t-4, 7t+2, 7t+6\rangle\!\rangle, \ldots, \langle\!\langle 0, 3t-2, 6t+2, 9t+5\rangle\!\rangle$, $\langle\!\langle 0, 3t-1, 6t+3, 9t+5\rangle\!\rangle$, $\langle\!\langle 0, 3t, 6t+4, 9t+5\rangle\!\rangle$, $\langle\!\langle 0, 3t+1, 6t+1, 9t+3\rangle\!\rangle$, $\langle\!\langle 0, 3t+2, 6t, 9t+3\rangle\!\rangle$, $\langle\!\langle 0, 3t+3, 6t+2, 9t+3\rangle\!\rangle$, $\langle\!\langle 0, 3t+4, 6t+2, 9t+2\rangle\!\rangle$, $\langle\!\langle 0, 3t+4, 6t+3, 9t+2\rangle\!\rangle$ and $\langle\!\langle 0, 3t+4, 6t+4, 9t+2\rangle\!\rangle$. Hence, in this case, LOW$(14t+1, 9)$ exists.

If $n = 14t+7$ (for $t \geq 0$), we consider the set V as $\{a_1, a_2, \ldots, a_{14t}, b_1, b_2, \ldots, b_7\}$. To obtain an LOW$(14t+7, 9)$, we use an LOW$(14t, 9)$ on $\{a_1, a_2, \ldots, a_{14t}\}$ (given two cases above) if necessary, an LOW$(7, 9)$ on $\{b_1, b_2, \ldots, b_7\}$ (as in Example 5), and an LOW–decomposition of $9K_{\{a_{2i-1}, a_{2i}\}, \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}}$ for all $i = 1, 2, \ldots, 7t$ (as in Example 7) if necessary. Hence, in this case, LOW$(14t+7, 9)$ exists.

If $n = 14t+8$ (for $t \geq 0$), we consider the set V as $\{a_1, a_2, \ldots, a_8, b_1, b_2, \ldots, b_{14t}\}$. To obtain an LOW$(8+14t, 9)$, we use an LOW$(8, 9)$ on $\{a_1, a_2, \ldots, a_8\}$ (as in Example 6), an LOW$(14t, 9)$ on $\{b_1, b_2, \ldots, b_{14t}\}$ (given three cases above) if necessary, and an LOW–decomposition of $9K_{\{a_{2i-1}, a_{2i}\}, \{b_{7j-6}, b_{7j-5}, b_{7j-4}, b_{7j-3}, b_{7j-2}, b_{7j-1}, b_{7j}\}}$ for all $i = 1, 2, 3, 4$ and for all $j = 1, 2, \ldots, 2t$ (as in Example 7) if necessary. Hence, in this case, LOW$(14t+8, 9)$ exists. ■

THEOREM 3.4. *An LOW(n, λ) exists for all $\lambda \geq 4$ except $\lambda = 5$ (according to Lemma 3.2), for corresponding necessary $n \geq 4$.*

PROOF. We proceed by cases on λ (mod 7).

For $\lambda \equiv 0 \pmod 7$ (so that $\lambda = 7t$ for $t \geq 1$), by taking t copies of an LOW$(n, 7)$ (given in Lemma 3.4), we have an LOW$(n, 7t)$.

For $\lambda \equiv 1 \pmod 7$ (so that $\lambda = 7t+1 = 7(t-1)+8$ for $t \geq 1$), we first take two copies of an LOW$(n, 4)$ (given in Lemma 3.1). (This gives us $\lambda = 8$ thus far.) We then adjoin this to $t-1$ copies of an

LOW$(n,7)$ (given in Lemma 3.4) if necessary. Hence, we have an LOW$(n,7t+1)$.

For $\lambda \equiv 2 \pmod 7$ (so that $\lambda = 7t+2 = 7(t-1)+9$ for $t \geq 1$), we first take an LOW$(n,9)$ (given in Lemma 3.5). (This gives us $\lambda = 9$ thus far.) We then adjoin this to $t-1$ copies of an LOW$(n,7)$ (given in Lemma 3.4) if necessary. Hence, we have an LOW$(n,7t+2)$.

For $\lambda \equiv 3 \pmod 7$ (so that $\lambda = 7t+3 = 7(t-1)+10$ for $t \geq 1$), we first take an LOW$(n,6)$ (given in Lemma 3.3) and an LOW$(n,4)$ (given in Lemma 3.1). (This gives us $\lambda = 10$ thus far.) We then adjoin this to $t-1$ copies of an LOW$(n,7)$ (given in Lemma 3.4) if necessary. Hence, we have an LOW$(n,7t+3)$.

For $\lambda \equiv 4 \pmod 7$ (so that $\lambda = 7t+4$ for $t \geq 0$), we first take an LOW$(n,4)$ (given in Lemma 3.1). (This gives us $\lambda = 4$ thus far.) We then adjoin this to t copies of an LOW$(n,7)$ (given in Lemma 3.4) if necessary. Hence, we have an LOW$(n,7t+4)$.

For $\lambda \equiv 5 \pmod 7$ (so that $\lambda = 7t+5 = 7(t-1)+12$ for $t \geq 1$), we first take two copies of an LOW$(n,6)$ (given in Lemma 3.3). (This gives us $\lambda = 12$ thus far.) We then adjoin this to $t-1$ copies of an LOW$(n,7)$ (given in Lemma 3.4) if necessary. Hence, we have an LOW$(n,7t+5)$.

For $\lambda \equiv 6 \pmod 7$ (so that $\lambda = 7t+6$ for $t \geq 0$), we first take an LOW$(n,6)$ (given in Lemma 3.3). (This gives us $\lambda = 6$ thus far.) We then adjoin this to t copies of an LOW$(n,7)$ (given in Lemma 3.4) if necessary. Hence, we have an LOW$(n,7t+6)$. ■

4. OLW–Decompositions

We first address the minimum values of λ in an OLW(n,λ). Recall that $\lambda \geq 4$.

THEOREM 4.1. *Let $n \geq 4$. The minimum values of λ for which an OLW(n,λ) could exist are $\lambda = 4$ when $n \equiv 0,1 \pmod 7$ and $\lambda = 7$ when $n \not\equiv 0,1 \pmod 7$.*

PROOF. Since there are $\frac{\lambda n(n-1)}{2}$ edges in a λK_n, and 7 edges in an OLW graph, we must have that $\lambda n(n-1) \equiv 0 \pmod{14}$ (where $n \geq 4$ and $\lambda \geq 4$) for OLW–decompositions. The result follows from cases on $n \pmod{14}$. ■

We are now in a position to prove the main results of the paper. We first remark that an OLW graph has 4 vertices; that is, we consider $n \geq 4$. Also, necessarily $\lambda \geq 4$. We note that we use difference sets to achieve our decompositions of λK_n. In general, we exhibit the base graphs, which can be developed (modulo either n or $n-1$) to obtain the decomposition. We also note that the frequency of the edges is fixed by position, as per the OLW graph.

THEOREM 4.2. *The minimum number copies of K_n (as given in Theorem 4.1) can be decomposed into OLW graphs.*

PROOF. Let $n \geq 4$. We proceed by cases on n (mod 14).

If $n = 14t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t-1} \cup \{\infty\}$. The number of graphs required for OLW$(14t, 4)$ is $\frac{4(14t)(14t-1)}{14} = 4t(14t-1)$. Thus, we need $4t$ base graphs (modulo $14t-1$). Then, the differences we must achieve (modulo $14t-1$) are $1, 2, \ldots, 7t-1$. For the first four base graphs, use $|||0, 3t, 3t+1, \infty|||$, $|||0, 3t, 3t+1, 10t|||$, $|||0, 3t-1, 3t, 10t-2|||$ and $|||0, 3t-1, 3t, 10t-3|||$. We also use the $4t-4$ base graphs $|||0, 3t-2, 3t, 10t-4|||$, $|||0, 3t-2, 3t, 10t-5|||$, $|||0, 3t-3, 3t-1, 10t-7|||$, $|||0, 3t-3, 3t-1, 10t-8|||, \ldots, |||0, t+2, 2t+2, 5t+6|||$, $|||0, t+2, 2t+2, 5t+5|||$, $|||0, t+1, 2t+1, 5t+3|||$ and $|||0, t+1, 2t+1, 5t+2|||$ if necessary. Hence, in this case, OLW$(14t, 4)$ exists.

If $n = 14t+1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t+1}$. The number of graphs required for OLW$(14t+1, 4)$ is $\frac{4(14t+1)(14t)}{14} = 4t(14t+1)$. Thus, we need $4t$ base graphs (modulo $14t+1$). Then, the differences we must achieve (modulo $14t+1$) are $1, 2, \ldots, 7t$. We use the base graphs $|||0, 3t, 3t+1, 10t+1|||$, $|||0, 3t, 3t+1, 10t|||$, $|||0, 3t-1, 3t, 10t-2|||$, $|||0, 3t-1, 3t, 10t-3|||$, $|||0, 3t-2, 3t, 10t-4|||$, $|||0, 3t-2, 3t, 10t-5|||$, $|||0, 3t-3, 3t-1, 10t-7|||$, $|||0, 3t-3, 3t-1, 10t-8|||, \ldots, |||0, t+2, 2t+2, 5t+6|||$, $|||0, t+2, 2t+2, 5t+5|||$, $|||0, t+1, 2t+1, 5t+3|||$ and $|||0, t+1, 2t+1, 5t+2|||$. Hence, in this case, OLW$(14t+1, 4)$ exists.

If $n = 14t+2$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t+1} \cup \{\infty\}$. The number of graphs required for OLW$(14t+2, 7)$ is $\frac{7(14t+2)(14t+1)}{14} = (7t+1)(14t+1)$. Thus, we need $7t+1$ base graphs (modulo $14t+1$). Then, the differences we must achieve (modulo $14t+1$) are $1, 2, \ldots, 7t$. For the first two base graphs, use $|||0, \infty, 7t, 14t|||$ and $|||0, 7t, 14t, \infty|||$. We also use the $7t-1$ base graphs $|||0, 1, 7t, 7t+1|||$, $|||0, 2, 7t, 7t+2|||$, $|||0, 3, 7t, 7t+3|||, \ldots, |||0, 7t-3, 7t, 14t-3|||$, $|||0, 7t-2, 7t, 14t-2|||$ and $|||0, 7t-1, 7t, 14t-1|||$. Hence, in this case, OLW$(14t+2, 7)$ exists.

If $n = 14t+3$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t+3}$. The number of graphs required for OLW$(14t+3, 7)$ is $\frac{7(14t+3)(14t+2)}{14} = (7t+1)(14t+3)$. Thus, we need $7t+1$ base graphs (modulo $14t+3$). Then, the differences we must achieve (modulo $14t+3$) are $1, 2, \ldots, 7t+$

1. For the first three base graphs, we use $|||0, 7t, 7t+1, 14t+2|||$, $|||0, 7t+1, 14t+1, 14t+2|||$ and $|||0, 1, 7t+2, 14t+2|||$. We also use the $7t-2$ base graphs $|||0, 2, 7t+1, 7t+3|||$, $|||0, 3, 7t+1, 7t+4|||$, $|||0, 4, 7t+1, 7t+5|||, \ldots, |||0, 7t-3, 7t+1, 14t-2|||$, $|||0, 7t-2, 7t+1, 14t-1|||$ and $|||0, 7t-1, 7t+1, 14t|||$. Hence, in this case, OLW$(14t+3, 7)$ exists.

If $n = 14t+4$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+3} \cup \{\infty\}$. The number of graphs required for OLW$(14t+4, 7)$ is $\frac{7(14t+4)(14t+3)}{14} = (7t+2)(14t+3)$. Thus, we need $7t+2$ base graphs (modulo $14t+3$). Then, the differences we must achieve (modulo $14t+3$) are $1, 2, \ldots, 7t+1$. For the first two base graphs, we use $|||0, \infty, 7t+1, 14t+2|||$ and $|||0, 7t+1, 14t+2, \infty|||$. We also use the $7t$ base graphs $|||0, 1, 7t+1, 7t+2|||$, $|||0, 2, 7t+1, 7t+3|||$, $|||0, 3, 7t+1, 7t+4|||, \ldots, |||0, 7t-2, 7t+1, 14t-1|||$, $|||0, 7t-1, 7t+1, 14t|||$ and $|||0, 7t, 7t+1, 14t+1|||$ if necessary. Hence, in this case, OLW$(14t+4, 7)$ exists.

If $n = 14t+5$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+5}$. The number of graphs required for OLW$(14t+5, 7)$ is $\frac{7(14t+5)(14t+4)}{14} = (7t+2)(14t+5)$. Thus, we need $7t+2$ base graphs (modulo $14t+5$). Then, the differences we must achieve (modulo $14t+5$) are $1, 2, \ldots, 7t+2$. When $t = 0$ (that is, when $n = 5$), we use the base graphs $|||0, 2, 3, 4|||$ and $|||1, 0, 2, 4|||$. When $t \geq 1$ (that is, when $n \geq 19$), we use the base graphs $|||0, 7t+1, 7t+2, 14t+4|||$, $|||0, 7t+2, 14t+3, 14t+4|||$, $|||0, 1, 7t+3, 14t+4|||$ as well as $|||0, 2, 7t+2, 7t+4|||$, $|||0, 3, 7t+2, 7t+5|||$, $|||0, 4, 7t+2, 7t+6|||, \ldots, |||0, 7t-2, 7t+2, 14t|||$, $|||0, 7t-1, 7t+2, 14t+1|||$ and $|||0, 7t, 7t+2, 14t+2|||$. Hence, in this case, OLW$(14t+5, 7)$ exists.

If $n = 14t+6$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+5} \cup \{\infty\}$. The number of graphs required for OLW$(14t+6, 7)$ is $\frac{7(14t+6)(14t+5)}{14} = (7t+3)(14t+5)$. Thus, we need $7t+3$ base graphs (modulo $14t+5$). Then, the differences we must achieve (modulo $14t+5$) are $1, 2, \ldots, 7t+2$. For the first two base graphs, we use $|||0, \infty, 7t+2, 14t+4|||$ and $|||0, 7t+2, 14t+4, \infty|||$. We also use the $7t+1$ base graph(s) $|||0, 1, 7t+2, 7t+3|||$, $|||0, 2, 7t+2, 7t+4|||$, $|||0, 3, 7t+2, 7t+5|||, \ldots, |||0, 7t-1, 7t+2, 14t+1|||$, $|||0, 7t, 7t+2, 14t+2|||$ and $|||0, 7t+1, 7t+2, 14t+3|||$. Hence, in this case, OLW$(14t+6, 7)$ exists.

If $n = 14t+7$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+6} \cup \{\infty\}$. The number of graphs required for OLW$(14t+7, 4)$ is $\frac{4(14t+7)(14t+6)}{14} = (4t+2)(14t+6)$. Thus, we need $4t+2$ base graphs (modulo $14t+6$). Then, the differences we must achieve (modulo $14t+6$) are $1, 2, \ldots, 7t+3$. When $t = 0$ (that is, when $n = 7$), we use the base graphs $|||0, 1, 4, \infty|||$ and $|||0, 1, 4, 2|||$. When $t \geq 1$ (that is, when $n \geq 21$), we use the base graphs $|||0,1,7t+4,\infty|||$, $|||0,1,7t+4,7t+6|||$ as well as $|||0,3t+2,3t+5,10t+7|||$, $|||0,3t+2,3t+5,10t+6|||$, $|||0,3t+1,3t+4,10t+4|||$, $|||0,3t+1,3t+4,10t+3|||$, $|||0,3t,3t+4,10t+2|||$, $|||0,3t,3t+4,10t+1|||$, $|||0,3t-1,3t+3,10t-1|||$, $|||0,3t-1,3t+3,10t-2|||,\ldots,|||0,t+4,2t+6,5t+12|||$, $|||0,t+4,2t+6,5t+11|||$, $|||0,t+3,2t+5,5t+9|||$ and $|||0,t+3,2t+5,5t+8|||$. Hence, in this case, $\mathrm{OLW}(14t+7,4)$ exists.

If $n=14t+8$ (for $t\geq 0$), we consider the set V as $\mathbb{Z}_{14t+8}$. The number of graphs required for $\mathrm{OLW}(14t+8,4)$ is $\frac{4(14t+8)(14t+7)}{14}=(4t+2)(14t+8)$. Thus, we need $4t+2$ base graphs (modulo $14t+8$). Then, the differences we must achieve (modulo $14t+8$) are $1,2,\ldots,7t+4$. When $t=0$ (that is, when $n=8$), we use the base graphs $|||0,1,5,7|||$ and $|||0,1,5,2|||$. When $t\geq 1$ (that is, when $n\geq 21$), we use the base graphs $|||0,1,7t+5,7t+7|||$, $|||0,1,7t+5,7t+8|||$ as well as $|||0,3t+3,3t+7,10t+10|||$, $|||0,3t+3,3t+7,10t+9|||$, $|||0,3t+2,3t+6,10t+7|||$, $|||0,3t+2,3t+6,10t+6|||$, $|||0,3t+1,3t+6,10t+5|||$, $|||0,3t+1,3t+6,10t+4|||$, $|||0,3t,3t+5,10t+2|||$, $|||0,3t,3t+5,10t+1|||,\ldots,|||0,t+5,2t+8,5t+15|||$, $|||0,t+5,2t+8,5t+14|||$, $|||0,t+4,2t+7,5t+12|||$ and $|||0,t+4,2t+7,5t+11|||$. Hence, in this case, $\mathrm{OLW}(14t+8,4)$ exists.

If $n=14t+9$ (for $t\geq 0$), we consider the set V as $\mathbb{Z}_{14t+9}$. The number of graphs required for $\mathrm{OLW}(14t+9,7)$ is $\frac{7(14t+9)(14t+8)}{14}=(7t+4)(14t+9)$. Thus, we need $7t+4$ base graphs (modulo $14t+9$). Then, the differences we must achieve (modulo $14t+9$) are $1,2,\ldots,7t+4$. For the first three base graphs, we use $|||0,7t+4,7t+5,14t+8|||$, $|||0,7t+3,14t+7,14t+8|||$ and $|||0,1,7t+4,14t+8|||$. We also use the $7t+1$ base graphs $|||0,2,7t+4,14t+6|||$, $|||0,3,7t+4,14t+5|||$, $|||0,4,7t+4,14t+4|||,\ldots,|||0,7t,7t+4,7t+8|||$, $|||0,7t+1,7t+4,7t+7|||$ and $|||0,7t+2,7t+4,7t+6|||$. Hence, in this case, $\mathrm{OLW}(14t+9,7)$ exists.

If $n=14t+10$ (for $t\geq 0$), we consider the set V as $\mathbb{Z}_{14t+9}\cup\{\infty\}$. The number of graphs required for $\mathrm{OLW}(14t+10,7)$ is $\frac{7(14t+10)(14t+9)}{14}=(7t+5)(14t+9)$. Thus, we need $7t+5$ base graphs (modulo $14t+9$). Then, the differences we must achieve (modulo $14t+9$) are $1,2,\ldots,7t+4$. For the first two base graphs, we use $|||0,\infty,7t+4,14t+8|||$ and $|||0,7t+4,14t+8,\infty|||$. We also use the $7t+3$ base graphs $|||0,1,7t+4,7t+5|||$, $|||0,2,7t+4,7t+6|||$, $|||0,3,7t+4,7t+7|||,\ldots,|||0,7t+1,7t+4,14t+5|||$, $|||0,7t+2,7t+4,14t+6|||$ and $|||0,7t+3,7t+4,14t+7|||$. Hence, in this case, $\mathrm{OLW}(14t+10,7)$ exists.

If $n=14t+11$ (for $t\geq 0$), we consider the set V as $\mathbb{Z}_{14t+11}$. The number of graphs required for $\mathrm{OLW}(14t+11,7)$ is $\frac{7(14t+11)(14t+10)}{14}=$

$(7t+5)(14t+11)$. Thus, we need $7t+5$ base graphs (modulo $14t+11$). Then, the differences we must achieve (modulo $14t+11$) are $1, 2, \ldots, 7t+5$. For the first three base graphs, we use $|||0, 7t+5, 7t+6, 14t+10|||$, $|||0, 7t+4, 14t+9, 14t+10|||$ and $|||0, 1, 7t+5, 14t+10|||$. We also use the $7t+2$ base graphs $|||0, 2, 7t+5, 7t+7|||$, $|||0, 3, 7t+5, 7t+8|||, \ldots, |||0, 7t+2, 7t+5, 14t+7|||$ and $|||0, 7t+3, 7t+5, 14t+8|||$. Hence, in this case, OLW$(14t+11, 7)$ exists.

If $n = 14t+12$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+11} \cup \{\infty\}$. The number of graphs required for OLW$(14t+12, 7)$ is $\frac{7(14t+12)(14t+11)}{14}$ $= (7t+6)(14t+11)$. Thus, we need $7t+6$ base graphs (modulo $14t+11$). Then, the differences we must achieve (modulo $14t+11$) are $1, 2, \ldots, 7t+5$. For the first two base graphs, we use $|||0, \infty, 7t+5, 14t+10|||$ and $|||0, 7t+5, 14t+10, \infty|||$. We also use the $7t+4$ base graphs $|||0, 1, 7t+5, 7t+6|||$, $|||0, 2, 7t+5, 7t+7|||$, $|||0, 3, 7t+5, 7t+8|||, \ldots, |||0, 7t+2, 7t+5, 14t+7|||$, $|||0, 7t+3, 7t+5, 14t+8|||$ and $|||0, 7t+4, 7t+5, 14t+9|||$. Hence, in this case, OLW$(14t+12, 7)$ exists.

If $n = 14t+13$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+13}$. The number of graphs required for OLW$(14t+13, 7)$ is $\frac{7(14t+13)(14t+12)}{14} =$ $(7t+6)(14t+13)$. Thus, we need $7t+6$ base graphs (modulo $14t+13$). Then, the differences we must achieve (modulo $14t+13$) are $1, 2, \ldots, 7t+6$. For the first three base graphs, we use $|||0, 7t+6, 7t+7, 14t+12|||$, $|||0, 7t+5, 14t+11, 14t+12|||$ and $|||0, 1, 7t+6, 14t+12|||$. We also use the $7t+3$ base graphs $|||0, 2, 7t+6, 7t+8|||$, $|||0, 3, 7t+6, 7t+9|||$, $|||0, 4, 7t+6, 7t+10|||, \ldots, |||0, 7t+2, 7t+6, 14t+8|||$, $|||0, 7t+3, 7t+6, 14t+9|||$ and $|||0, 7t+4, 7t+6, 14t+10|||$. Hence, in this case, OLW$(14t+13, 7)$ exists. ■

We now address the sufficiency of existence of OLW(n, λ).

THEOREM 4.3. *Let $n \geq 4$ and $\lambda \geq 4$. For existence of OLW(n, λ), the necessary condition for n is that $n \equiv 0, 1 \pmod 7$ when $\lambda \not\equiv 0 \pmod 7$. There is no condition for n when $\lambda \equiv 0 \pmod 7$.*

PROOF. Similar to the proof of Theorem 4.1, but by cases on $\lambda \pmod{14}$. ■

LEMMA 4.1. *There exists an OLW$(n, 4)$ for necessary $n \geq 4$.*

PROOF. From Theorem 4.3, the necessary condition is $n \equiv 0, 1, 7, 8 \pmod{14}$. In these cases, OLW$(n, 4)$ exists from Theorem 4.2. ■

LEMMA 4.2. *There does not exist an OLW$(n, 5)$.*

PROOF. The only edge frequencies in an OLW graph are 1, 2 and 4. The only ways to write $\lambda = 5$ as a sum of 1s, 2s and 4s are as $5 = 4 + 1$, $5 = 2 + 2 + 1$, $5 = 2 + 1 + 1 + 1$ and $5 = 1 + 1 + 1 + 1 + 1$. In an OLW$(n, 5)$, the number of times each edge needs to occur with frequency 4 is always the same as the number of times it needs to occur with frequency 1. Every other way to realize $\lambda = 5$ using edge frequencies of 2 will contribute at least one more unmatched edge frequency of 1. Thus, such a decomposition is not possible. ■

LEMMA 4.3. *There exists an OLW$(n, 6)$ for necessary $n \geq 4$.*

PROOF. From Theorem 4.3, the necessary condition is $n \equiv 0, 1, 7, 8 \pmod{14}$.

If $n = 14t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t-1} \cup \{\infty\}$. The number of graphs required for OLW$(14t, 6)$ is $\frac{6(14t)(14t-1)}{14} = 6t(14t-1)$. Thus, we need $6t$ base graphs (modulo $14t - 1$). The differences we must achieve (modulo $14t-1$) are $1, 2, \ldots, 7t-1$. We use the base graphs $|||\infty, 0, 1, t+2|||$, $|||0, 7t-1, 7t, \infty|||$, $|||0, 7t-2, 7t-1, 8t+1|||$, $|||0, 7t-3, 7t-2, 8t+1|||$, $|||0, 7t-4, 7t-3, 8t+1|||$, $|||0, 7t-5, 7t-4, 8t+1|||, \ldots$, $|||0, t+6, 2t+6, 9t|||$, $|||0, t+5, 2t+5, 9t|||$, $|||0, t+4, 2t+4, 9t|||$, $|||0, t+3, 2t+3, 9t|||$, $|||0, t+2, 2t+2, 9t|||$ and $|||0, t+1, 2t+1, 9t|||$. Hence, in this case, OLW$(14t, 6)$ exists.

If $n = 14t + 1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t+1}$. The number of graphs required for OLW$(14t + 1, 6)$ is $\frac{6(14t+1)(14t)}{14} = 6t(14t + 1)$. Thus, we need $6t$ base graphs (modulo $14t + 1$). The differences we must achieve (modulo $14t + 1$) are $1, 2, \ldots, 7t$. We use the base graphs $|||0, 7t, 7t+1, 8t+2|||$, $|||0, 7t-1, 7t, 8t+2|||$, $|||0, 7t-2, 7t-1, 8t+2|||$, $|||0, 7t-3, 7t-2, 8t+2|||$, $|||0, 7t-4, 7t-3, 8t+2|||$, $|||0, 7t-5, 7t-4, 8t+2|||, \ldots$, $|||0, t+6, 2t+6, 9t+1|||$, $|||0, t+5, 2t+5, 9t+1|||$, $|||0, t+4, 2t+4, 9t+1|||$, $|||0, t+3, 2t+3, 9t+1|||$, $|||0, t+2, 2t+2, 9t+1|||$ and $|||0, t+1, 2t+1, 9t+1|||$. Hence, in this case, OLW$(14t+1, 6)$ exists.

If $n = 14t + 7$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+6} \cup \{\infty\}$. The number of graphs required for OLW$(14t+7, 6)$ is $\frac{6(14t+7)(14t+6)}{14} = (6t+3)(14t+6)$. Thus, we need $6t+3$ base graphs (modulo $14t+6$). The differences we must achieve (modulo $14t + 6$) are $1, 2, \ldots, 7t + 3$. For the first three base graphs, we use $|||\infty, 0, 7t+3, 14t+5|||$, $|||0, 7t+2, 7t+3, \infty|||$ and $|||0, 7t+3, 7t+4, 7t+5|||$. We also use the $6t$ base graphs $|||0, 7t, 7t+1, 8t+2|||$, $|||0, 7t-1, 7t, 8t+2|||$, $|||0, 7t-2, 7t-1, 8t+2|||$, $|||0, 7t-3, 7t-2, 8t+2|||$, $|||0, 7t-4, 7t-3, 8t+2|||$, $|||0, 7t-5, 7t-4, 8t+2|||, \ldots$, $|||0, t+6, 2t+6, 9t+1|||$, $|||0, t+5, 2t+5, 9t+1|||$,

$|||0,t+4,2t+4,9t+1|||$, $|||0,t+3,2t+3,9t+1|||$, $|||0,t+2,2t+2,9t+1|||$ and $|||0,t+1,2t+1,9t+1|||$ if necessary. Hence, in this case, OLW$(14t+7,6)$ exists.

If $n = 14t + 8$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{14t+8}$. The number of graphs required for OLW$(14t + 8, 6)$ is $\frac{6(14t+8)(14t+7)}{14} = (6t+3)(14t+8)$. Thus, we need $6t+3$ base graphs (modulo $14t+8$). The differences we must achieve (modulo $14t+8$) are $1, 2, \ldots, 7t+4$. For the first three base graphs, we use $|||0, 7t + 2, 14t + 6, 14t + 7|||$, $|||0, 7t+4, 7t+5, 14t+7|||$ and $|||0, 7t+3, 7t+4, 14t+7|||$. We also use the $6t$ base graphs $|||0, 7t + 1, 7t + 3, 8t + 5|||$, $|||0, 7t, 7t + 2, 8t + 5|||$, $|||0, 7t-1, 7t+1, 8t+5|||$, $|||0, 7t-2, 7t, 8t+5|||$, $|||0, 7t-3, 7t-1, 8t+5|||$, $|||0, 7t-4, 7t-2, 8t+5|||, \ldots, |||0, t+7, 2t+8, 9t+4|||$, $|||0, t+6, 2t+7, 9t+4|||$, $|||0, t+5, 2t+6, 9t+4|||$, $|||0, t+4, 2t+5, 9t+4|||$, $|||0, t+3, 2t+4, 9t+4|||$ and $|||0, t+2, 2t+3, 9t+4|||$ if necessary. Hence, in this case, OLW$(14t+8, 6)$ exists. ■

LEMMA 4.4. *There exists an OLW$(n,7)$ for any $n \geq 4$.*

PROOF. From Theorem 4.3, there is no condition for n. We consider cases when $n \geq 4$ is odd or even.

If $n = 2t + 1$ (for $t \geq 2$), we consider the set V as $\mathbb{Z}_{2t+1}$. The number of graphs required for OLW$(2t+1, 7)$ is $\frac{7(2t+1)(2t)}{14} = t(2t+1)$. Thus, we need t base graphs (modulo $2t + 1$). The differences we must achieve (modulo $2t + 1$) are $1, 2, \ldots, t$. When $t = 2$ (that is, when $n = 5$), we use the base graphs $|||0, 1, 3, 4|||$ and $|||0, 2, 3, 1|||$. When $t \geq 3$ (that is, when $n \geq 7$), we use the base graphs $|||0, t, t + 1, 2t|||$ and $|||0, 1, 3, t + 3|||$ as well as $|||0, 2, 5, t + 3|||$, $|||0, 3, 7, t + 4|||$, $|||0, 4, 9, t+5|||, \ldots, |||0, t-3, 2t-5, 2t-2|||$, $|||0, t-2, 2t-3, 2t-1|||$ and $|||0, t-1, 2t-1, 2t|||$. Hence, in this case, OLW$(2t+1, 7)$ exists.

If $n = 2t$ (for $t \geq 2$), we consider the set V as $\mathbb{Z}_{2t-1} \cup \{\infty\}$. The number of graphs required for OLW$(2t, 7)$ is $\frac{7(2t)(2t-1)}{14} = t(2t-1)$. Thus, we need t base graphs (modulo $2t-1$). The differences we must achieve (modulo $2t-1$) are $1, 2, \ldots, t-1$. When $t = 2$ (that is, when $n = 4$), we use the base graphs $|||0, \infty, 1, 2|||$ and $|||0, 1, 2, \infty|||$. When $t \geq 3$ (that is, when $n \geq 6$), we use the base graphs $|||0, \infty, 1, t|||$, $|||0, t-1, 2t-2, \infty|||$ as well as $|||0, t-2, t-1, t|||$, $|||0, t-3, t-1, t+1|||, \ldots, |||0, 2, t-1, 2t-4|||$ and $|||0, 1, t-1, 2t-3|||$. Hence, in this case, OLW$(2t, 7)$ exists. ■

The following examples play important roles in the sequel.

EXAMPLE 8. *The OLW graphs* $|||1,2,4,5|||$, $|||1,5,2,4|||$, $|||1,6,7,3|||$, $|||1,7,4,6|||$, $|||1,7,5,6|||$, $|||2,1,6,3|||$, $|||2,3,1,7|||$, $|||2,3,7,4|||$, $|||2,4,1,3|||$, $|||2,5,1,4|||$, $|||2,6,7,4|||$, $|||3,4,5,2|||$, $|||3,4,6,1|||$, $|||3,5,2,7|||$, $|||3,6,2,1|||$, $|||4,1,2,6|||$, $|||4,1,7,6|||$, $|||4,2,7,3|||$, $|||4,5,1,3|||$, $|||4,5,2,3|||$, $|||5,3,4,6|||$, $|||5,6,3,4|||$, $|||5,6,7,2|||$, $|||6,1,5,7|||$, $|||6,2,3,5|||$, $|||6,3,5,7|||$ *and* $|||7,6,5,1|||$ *constitute an example of an* $OLW(7,9)$ *with point set* $V = \{1,\dots,7\}$.

EXAMPLE 9. *The OLW graphs* $|||1,2,3,4|||$, $|||1,4,2,8|||$, $|||1,5,4,6|||$, $|||1,6,3,4|||$, $|||1,6,3,7|||$, $|||1,7,4,5|||$, $|||1,7,5,8|||$, $|||2,1,4,5|||$, $|||2,3,1,4|||$, $|||2,3,7,4|||$, $|||2,7,1,3|||$, $|||2,7,6,5|||$, $|||2,7,8,1|||$, $|||3,1,5,2|||$, $|||3,5,8,1|||$, $|||3,5,8,6|||$, $|||3,6,4,8|||$, $|||3,8,4,2|||$, $|||4,1,3,8|||$, $|||4,6,3,5|||$, $|||4,6,5,7|||$, $|||4,8,2,6|||$, $|||4,8,3,7|||$, $|||5,1,6,7|||$, $|||5,6,7,8|||$, $|||5,7,2,3|||$, $|||5,8,1,7|||$, $|||6,3,1,2|||$, $|||6,3,5,2|||$, $|||6,5,2,8|||$, $|||7,2,1,5|||$, $|||7,5,8,6|||$, $|||7,6,2,4|||$, $|||7,8,6,1|||$, $|||8,3,4,7|||$ *and* $|||8,7,6,2|||$ *constitute an example of an* $OLW(8,9)$ *with point set* $V = \{1,\dots,8\}$.

EXAMPLE 10. *The OLW graphs* $|||a,1,b,2|||$, $|||a,1,b,5|||$, $|||a,5,b,6|||$, $|||a,7,b,1|||$, $|||b,1,a,7|||$, $|||b,7,a,3|||$, $|||b,7,a,5|||$, $|||b,7,a,6|||$, $|||2,a,3,b|||$, $|||2,a,4,b|||$, $|||3,b,2,a|||$, $|||3,b,4,a|||$, $|||4,a,5,b|||$, $|||4,a,6,b|||$, $|||4,b,3,a|||$, $|||4,b,6,a|||$, $|||5,a,2,b|||$ *and* $|||7,b,1,a|||$ *constitute an example of an OLW–decomposition of* $9K_{\{a,b\},\{1,2,3,4,5,6,7\}}$.

LEMMA 4.5. *There exists an* $OLW(n,9)$ *for necessary* $n \geq 4$.

PROOF. From Theorem 4.3, the necessary condition is $n \equiv 0,1,7,8 \pmod{14}$.

If $n = 14t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t-1} \cup \{\infty\}$. The number of graphs required for OLW$(14t,9)$ is $\frac{9(14t)(14t-1)}{14} = 9t(14t-1)$. Thus, we need $9t$ base graphs (modulo $14t-1$). The differences we must achieve (modulo $14t-1$) are $1,2,\dots,7t-1$. We use the base graphs $|||0,\infty,1,2|||$, $|||0,\infty,1,3|||$, $|||0,\infty,1,4|||$, $|||0,2,3,7t|||$, $|||0,3,5,7t+3|||$, $|||0,1,4,7t+3|||$, $|||0,3,7t,14t-2|||$, $|||0,1,7t-1,14t-2|||$, $|||0,2,7t+1,14t-2|||$; $|||0,7t-4,14t-8,14t-4|||$, $|||0,7t-4,14t-8,14t-3|||$, $|||0,7t-4,14t-8,14t-2|||$, $|||0,5,9,7t+2|||$, $|||0,6,11,7t+5|||$, $|||0,4,10,7t+5|||$, $|||0,6,7t-1,14t-7|||$, $|||0,4,7t-2,14t-7|||$, $|||0,5,7t,14t-7|||,\dots,|||0,3t+4,6t+8,9t+6|||$, $|||0,3t+4,6t+8,9t+7|||$, $|||0,3t+4,6t+8,9t+8|||$, $|||0,3t-1,6t-3,9t-2|||$, $|||0,3t,6t-1,9t+1|||$, $|||0,3t-2,6t-2,9t+1|||$, $|||0,3t,6t+1,9t+3|||$, $|||0,3t-2,6t,9t+3|||$ and $|||0,3t-1,6t+2,9t+3|||$. Hence, in this case, OLW$(14t,9)$ exists.

If $n = 14t+1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{14t+1}$. The number of graphs required for OLW$(14t+1,9)$ is $\frac{9(14t+1)(14t)}{14} = 9t(14t+1)$.

Thus, we need $9t$ base graphs (modulo $14t+1$). The differences we must achieve (modulo $14t+1$) are $1, 2, \ldots, 7t$. We use the base graphs $|||0, 7t, 7t+1, 14t|||$, $|||0, 7t, 7t+2, 14t|||$, $|||0, 7t, 7t+3, 14t|||$, $|||0, 3, 7t, 14t-2|||$, $|||0, 1, 7t-1, 14t-2|||$, $|||0, 2, 7t+1, 14t-2|||$, $|||0, 1, 7t+1, 7t+4|||$, $|||0, 2, 7t+2, 7t+4|||$, $|||0, 3, 7t+3, 7t+4|||$; $|||0, 7t-4, 7t, 14t-5|||$, $|||0, 7t-4, 7t+1, 14t-5|||$, $|||0, 7t-4, 7t+2, 14t-5|||$, $|||0, 6, 7t-1, 14t-7|||$, $|||0, 4, 7t-2, 14t-7|||$, $|||0, 5, 7t, 14t-7|||$, $|||0, 4, 7t, 7t+6|||$, $|||0, 5, 7t+1, 7t+6|||$, $|||0, 6, 7t+2, 7t+6|||, \ldots, |||0, 3t+4, 6t+2, 9t+5|||$, $|||0, 3t+4, 6t+3, 9t+5|||$, $|||0, 3t+4, 6t+4, 9t+5|||$, $|||0, 3t, 6t+1, 9t+3|||$, $|||0, 3t-2, 6t, 9t+3|||$, $|||0, 3t-1, 6t+2, 9t+3|||$, $|||0, 3t-2, 6t+2, 9t+2|||$, $|||0, 3t-1, 6t+3, 9t+2|||$ and $|||0, 3t, 6t+4, 9t+2|||$. Hence, in this case, OLW$(14t+1, 9)$ exists.

If $n = 14t+7$ (for $t \geq 0$), we consider the set V as $\{a_1, a_2, \ldots, a_{14t}, b_1, b_2, \ldots, b_7\}$. To obtain an OLW$(14t+7, 9)$, we use an OLW$(14t, 9)$ on $\{a_1, a_2, \ldots, a_{14t}\}$ (given two cases above) if necessary, an OLW$(7, 9)$ on $\{b_1, b_2, \ldots, b_7\}$ (as in Example 8), and an OLW–decomposition of $9K_{\{a_{2i-1}, a_{2i}\}, \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}}$ for all $i = 1, 2, \ldots, 7t$ (as in Example 10) if necessary. Hence, in this case, OLW$(14t+7, 9)$ exists.

If $n = 14t+8$ (for $t \geq 0$), we consider the set V as $\{a_1, a_2, \ldots, a_8, b_1, b_2, \ldots, b_{14t}\}$. To obtain an OLW$(8+14t, 9)$, we use an OLW$(8, 9)$ on $\{a_1, a_2, \ldots, a_8\}$ (as in Example 9), an OLW$(14t, 9)$ on $\{b_1, b_2, \ldots, b_{14t}\}$ (given three cases above) if necessary, and an OLW–decomposition of $9K_{\{a_{2i-1}, a_{2i}\}, \{b_{7j-6}, b_{7j-5}, b_{7j-4}, b_{7j-3}, b_{7j-2}, b_{7j-1}, b_{7j}\}}$ for all $i = 1, 2, 3, 4$ and for all $j = 1, 2, \ldots, 2t$ (as in Example 10) if necessary. Hence, in this case, OLW$(14t+8, 9)$ exists. ■

THEOREM 4.4. *An OLW(n, λ) exists for all $\lambda \geq 4$ except $\lambda = 5$ (according to Lemma 4.2), for corresponding necessary $n \geq 4$.*

PROOF. We proceed by cases on λ (mod 7).

For $\lambda \equiv 0$ (mod 7) (so that $\lambda = 7t$ for $t \geq 1$), by taking t copies of an OLW$(n, 7)$ (given in Lemma 4.4), we have an OLW$(n, 7t)$.

For $\lambda \equiv 1$ (mod 7) (so that $\lambda = 7t+1 = 7(t-1)+8$ for $t \geq 1$), we first take two copies of an OLW$(n, 4)$ (given in Lemma 4.1). (This gives us $\lambda = 8$ thus far.) We then adjoin this to $t-1$ copies of an OLW$(n, 7)$ (given in Lemma 4.4) if necessary. Hence, we have an OLW$(n, 7t+1)$.

For $\lambda \equiv 2$ (mod 7) (so that $\lambda = 7t+2 = 7(t-1)+9$ for $t \geq 1$), we first take an OLW$(n, 9)$ (given in Lemma 4.5). (This gives us $\lambda = 9$ thus far.) We then adjoin this to $t-1$ copies of an OLW$(n, 7)$ (given in Lemma 4.4) if necessary. Hence, we have an OLW$(n, 7t+2)$.

For $\lambda \equiv 3 \pmod 7$ (so that $\lambda = 7t + 3 = 7(t-1) + 10$ for $t \geq 1$), we first take an OLW$(n, 6)$ (given in Lemma 4.3) and an OLW$(n, 4)$ (given in Lemma 4.1). (This gives us $\lambda = 10$ thus far.) We then adjoin this to $t - 1$ copies of an OLW$(n, 7)$ (given in Lemma 4.4) if necessary. Hence, we have an OLW$(n, 7t + 3)$.

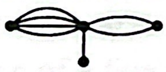
For $\lambda \equiv 4 \pmod 7$ (so that $\lambda = 7t + 4$ for $t \geq 0$), we first take an OLW$(n, 4)$ (given in Lemma 4.1). (This gives us $\lambda = 4$ thus far.) We then adjoin this to t copies of an OLW$(n, 7)$ (given in Lemma 4.4) if necessary. Hence, we have an OLW$(n, 7t + 4)$.

For $\lambda \equiv 5 \pmod 7$ (so that $\lambda = 7t + 5 = 7(t-1) + 12$ for $t \geq 1$), we first take two copies of an OLW$(n, 6)$ (given in Lemma 4.3). (This gives us $\lambda = 12$ thus far.) We then adjoin this to $t - 1$ copies of an OLW$(n, 7)$ (given in Lemma 4.4) if necessary. Hence, we have an OLW$(n, 7t + 5)$.

For $\lambda \equiv 6 \pmod 7$ (so that $\lambda = 7t + 6$ for $t \geq 0$), we first take an OLW$(n, 6)$ (given in Lemma 4.3). (This gives us $\lambda = 6$ thus far.) We then adjoin this to t copies of an OLW$(n, 7)$ (given in Lemma 4.4) if necessary. Hence, we have an OLW$(n, 7t + 6)$. ■

5. Conclusion

We have identified LOW and OLW graphs, found the minimum λ for decomposition of λK_n into these graphs, and showed that for all viable values of λ, the necessary conditions are sufficient for LOW- and OLW-decompositions.

We leave it as an open problem to find cyclic decompositions of λK_n into so-called LWO graphs  and into (the unnamed) graphs 

References

[1] P. Adams, D. Bryant, and M. Buchanan, A survey on the existence of G-Designs, *J. Combin. Designs* **16** (2008), 373–410.

[2] E. J. Billington, Balanced n-ary designs: a combinatorial survey and some new results, *Ars Combin.* **17** (1984), A, 37–72.

[3] E. J. Billington, Designs with repeated elements in blocks: a survey and some recent results, eighteenth Manitoba conference on numerical mathematics and computing (Winnipeg, MB, 1988), *Congr. Numer.* **68** (1989), 123–146.

[4] H. Chan and D. G. Sarvate, Stanton graph decompositions, *Bulletin of the ICA* **64** (2012), 21–29.

[5] S. El-Zanati, W. Lapchinda, P. Tangsupphathawat and W. Wannasit, The spectrum for the Stanton 3-cycle, *Bulletin of the ICA* **69** (2013), 79–88.

[6] D. W. Hein, A new construction for decompositions of λK_n into LE graphs, *J. Combin. Math. Combin. Comput.* **100** (2017), 37–43.

[7] D. W. Hein, Decompositions of λK_n into LW and OW graphs, *J. Combin. Math. Combin. Comput.* **102** (2017), to appear.

[8] D. W. Hein, Generalized Stanton–type graphs, *J. Combin. Math. Combin. Comput.* **101** (2017), 59–71.

[9] D. W. Hein and D. G. Sarvate, Decompositions of λK_n into LEO and ELO graphs, *J. Combin. Math. Combin. Comput.* **98** (2016), 125–137.

[10] D. W. Hein and D. G. Sarvate, Decompositions of λK_n into $S(4,3)$'s, *J. Combin. Math. Combin. Comput.* **94** (2015), 3–14.

[11] D. W. Hein and D. G. Sarvate, Decompositions of λK_n using Stanton–type graphs, *J. Combin. Math. Combin. Comput.* **90** (2014), 185–195.

[12] S. P. Hurd and D. G. Sarvate, Graph decompositions of $K(v,\lambda)$ into modified triangles using Langford and Skolem sequences, *Ars Combin.* **129** (2016), 403–416.

[13] S. Malick and D. G. Sarvate, Decomposition of λK_v into multigraphs with four vertices and five edges, *J. Combin. Math. Combin. Comput.* **86** (2013), 221-237.

[14] M. Priesler and M. Tarsi, Multigraph decomposition into stars and into multistars, *Discrete Math.* **296** (2005), no. 2–3, 235–244.

[15] D. G. Sarvate, P. A. Winter and L. Zhang, A fundamental theorem of multigraph decomposition of a $\lambda K_{m,n}$, *J. Combin. Math. Combin. Comput.* **96** (2016), 201–216.

[16] D. G. Sarvate, P. A. Winter and L. Zhang, Decomposition of a $\lambda K_{m,n}$ into graphs on four vertices and five edges, *J. Combin. Math. Combin. Comput.* **98** (2016), 139–150.

[17] D. G. Sarvate and L. Zhang, Decompositions of λK_n into LOE and OLE graphs, *Ars Combinatoria* (2017), accepted.

[18] D. R. Stinson, Combinatorial designs: constructions and analysis, Springer, New York, 2004.

Southern Utah University, Dept. of Math., Cedar City, UT, 84720
E-mail address: hein@suu.edu