Discovery of Some New Classes of Graceful Unicyclic Graphs

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Abstract

Graceful graphs were first studied by Rosa [17]. A graceful labeling f of a graph G is a one-to-one map from the set of vertices of G to the set $\{0, 1, \cdots, |E(G)|\}$, where for edges xy, the induced edge labels $|f(x) - f(y)|$ form the set $\{1, 2, \cdots, |E(G)|$, with no label repeated. In this paper, we investigate the set of labels taken by the central vertex of the star in the graph $K_{1,m-1} \oplus C_n$ for each graceful labeling. We also study gracefulness of certain unicyclic graphs where paths P_3, P_2 are pendant at vertices of the cycle. For these unicyclic graphs, the deletion of any edge of the cycle does not result in a caterpillar.

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1 Introduction

For a simple graph $G = (V, E)$, a graceful labeling f is a one-to-one map from V to the set $\{0, 1, \cdots, |E(G)|\}$ such that when an edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct and form the set $\{1, 2, \cdots, |E(G)|\}$. A graph G which admits at least one graceful labeling is a graceful graph. The notion of graceful labeling was introduced by Rosa [17] and used to attempt to resolve the conjecture of Ringel [16] that states that given any tree T with n edges, the complete graph K_{2n+1} can be edge-decomposed into $2n + 1$ copies of T. Rosa [17] showed that if a tree T with n edges is graceful, then K_{2n+1} can be so edge-decomposed. Thus Ringel's conjecture is reduced to proving gracefulness of all trees.

Besides trees, gracefulness of other types of graphs has also been investigated. Gallian's extensive survey [10] presents details for cycle-related graphs, product related graphs, complete graphs, disconnected graphs and joins of graphs. In this paper, we study gracefulness of certain unicyclic graphs.

Several authors have investigated gracefulness of unicyclic graphs. Rosa [17] proved that the n-cycle C_n is graceful if and only if n is congruent to 0 or 3 modulo 4. Truszczyński [18] conjectured that the non-graceful unicyclic graphs are only cycles C_n with the size n congruent to 1 or 2 modulo 4. Doma [7], in his master's thesis, investigated the gracefulness of some unicyclic graphs where the cycle has up to 9 edges. Barientos [5] proved that a unicyclic graph in which the deletion of any edge on the cycle results in a caterpillar is graceful. Bagga et al. [1, 2, 3, 4, 14] investigated algorithms to generate all graceful labelings of several classes of unicyclic graphs. In this paper, we study properties of graceful labelings of a class of unicyclic graphs, gracefulness of certain classes of unicyclic graphs.

The organization of the paper is as follows. We present definitions and notation in Section 2. In Section 3 we give a summary of the related known work on unicyclic graceful graphs. In section 4 we discuss the set of labels taken by the central vertex of the star in the graph $K_{1,m-1} \oplus C_n$. In section 5, we study the gracefulness of certain unicyclic graphs with pendant P_2's and P_3's attached to vertices of the cycle, where the deletion of one edge of the cycle does not result in a caterpillar. We build software code to visualize our graceful unicyclic graphs in the tool Graphviz in the section 6. Section 7 presents a summary.

2 Definitions and Notation

In this section, we give definitions and notation used in the paper. For all other standard terminology, we follow Diestel [6]. We consider only connected graphs. Unicyclic graphs are graphs with exactly one cycle. For uniyclic graphs, the number of vertices is equal to the number of edges. A C_n-unicyclic graph is a unicyclic graph where the cycle has n vertices, $n \geq 3$. A caterpillar is a tree that is a P_1, a P_2 or has more than two vertices and the deletion of its leaves results in a path. This path is called the spine of the caterpillar. A caterpillar R with spine $P_n = v_0v_1 \cdots v_{n-1}$ is also denoted $R(v_0v_1 \cdots v_{n-1})$. A cycle with a pendant caterpillar is obtained by identifying a vertex of the cycle with a leaf of $R(v_0v_1 \cdots v_{n-1})$ that is adjacent to v_0 (or v_{n-1}).

Rosa [17] defined four types of labelings (he called them *valuations*) to study graph decompositions of complete graphs into isomorphic subgraphs. For our purposes, we only need two of these: α-valuation and β-valuation. We define these below. For a graph $G = (V, E)$ and a one-to-one mapping $f : V(G) \to \{0, 1, \cdots, E(G)\}$, we define an induced mapping on the edge set of G as follows: for an edge xy of G, $f(xy) = |f(x) - f(y)|$.

- f as above is an α-valuation of G if there exists a label l in $L = \{0, 1, \cdots, |E(G)|\}$ such that for each edge xy, we have either $f(x) \leq l < f(y)$ or $f(y) \leq l < f(x)$. l has been called the *critical value* by Figueroa-Centeno et al. [8]; while Mavronicolas et al. [13] used term *strength* of the labeling, and they used the name strongly graceful labeling for an α-labeling.

- f is a β-valuation of G if the induced edge labels are distinct. A β-valuation is also called a graceful labeling. We observe that an α-valuation is also a β-valuation.

Rosa [17] proved that every caterpillar has an α-valuation. Since we shall use Rosa's canonical graceful labeling of a caterpillar, we describe it next. Suppose R is a caterpillar of order p. If $p = 2$, then label the vertices 1 and 0. For $p \geq 3$, let $R = R(v_0v_1 \cdots v_{n-1})$ ($n \geq 1$). Partition the vertices of R in two partite sets A and B where A consists of vertices that are at an even distance from v_0, and B consists of vertices that are at an odd distance from v_0. Clearly $v_i \in A$ for even i and $v_i \in B$ for odd i. Label v_0 with label $p - 1$. Label the neighbors of v_0 with labels $0, 1, 2 \cdots$, such that the neighbor v_1 gets the largest of these labels. Label the unlabeled neighbors of v_1 with labels $p - 2, p - 3, p - 4, \cdots$ such that the neighbor v_2 gets the smallest of these labels. Continue in this fashion until all vertices have labels. This process assigns consecutive labels $p-1, p-2, p-3, \cdots$ to vertices in A, and $0, 1, 2, \cdots$ to those in B. It is easy to check that this results in an α-valuation and hence a graceful labeling. In figure 1, we present the Rosa's canonical labeling of a caterpillar, d_i is the degree of v_i, $t = \sum_{i=0}^{\lceil (n-2)/2 \rceil} d_{2i} - \lceil n/2 \rceil$ when n is even and $t = \sum_{i=0}^{\lceil (n-2)/2 \rceil} d_{2i} - \lfloor n/2 \rfloor$ when n is odd.

In the figure 2, we present an example of a graceful labeling obtained after a canonical labeling. We next discuss some graphs that are constructed by identifying some vertices of two vertex disjoint graphs. Let G and H be two vertex

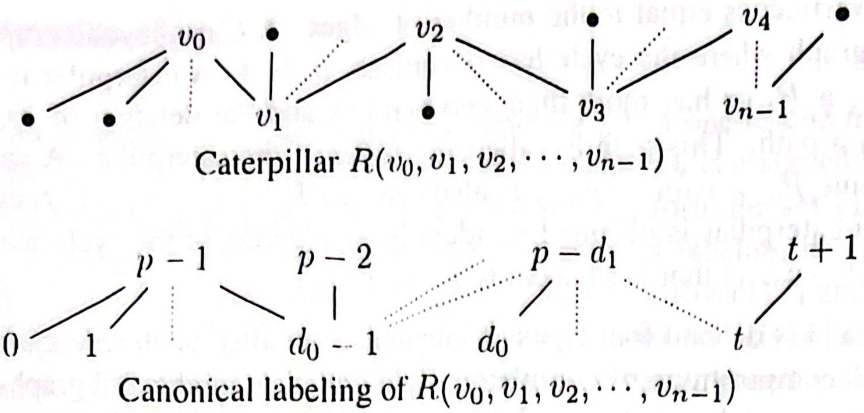


Figure 1: Canonical labeling of $R(v_0, v_1, v_2, \cdots, v_{n-1})$

disjoint graphs. For a vertex x of G and a vertex y of H, $G(x) \oplus H(y)$ denotes the graph obtained by identifying x with y. Thus the vertices of $G(x) \oplus H(y)$ are those of G and H except x and y, and a new vertex z. $G(x) \oplus H(y)$ has all the edges G and H that are not incident at x or y. The edges incident at z come from those edges in G and H that are incident at x or y.

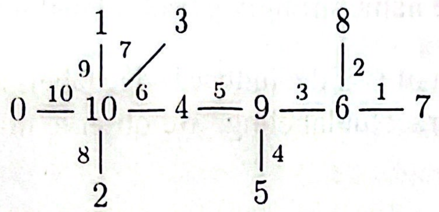
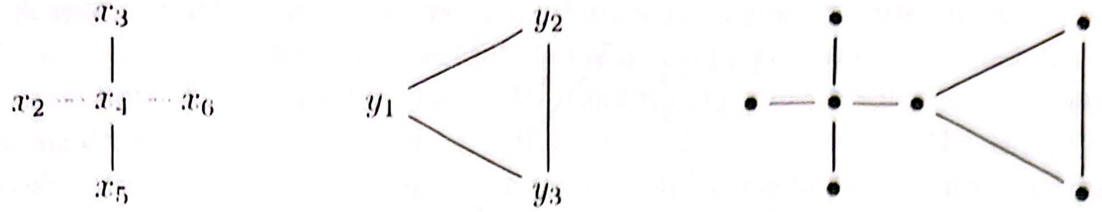


Figure 2: Graceful labeling obtained after canonical labeling

More generally, given a cycle $C_n = (y_1, y_2, \cdots, y_n)$ and n graphs H_i $(i = 1, 2, \cdots, n)$, the graph $C_n \oplus (H_1(y_1); H_2(y_2); \cdots; H_n(y_n))$ is obtained by identifying each vertex y_i of the cycle to a vertex of the graph H_i. If y_i is a peripheral vertex of H_i, then the notation is simplified to $C_n \oplus (H_1; H_2; \cdots; H_i; H_{i+1}; \cdots; H_n)$ when the peripheral vertices are indifferent. In figure 3, we present a unicyclic graph obtained by identifying an end vertex of a caterpillar R_1 and a vertex of the cycle C_3. Similarly in case of paths, the notation $C_n \oplus (P_2(y_1); P_2(y_2); \cdots; P_2(y_n))$ is simplified to $C_n \oplus (P_2; P_2; \cdots; P_2)$.

3 Known results

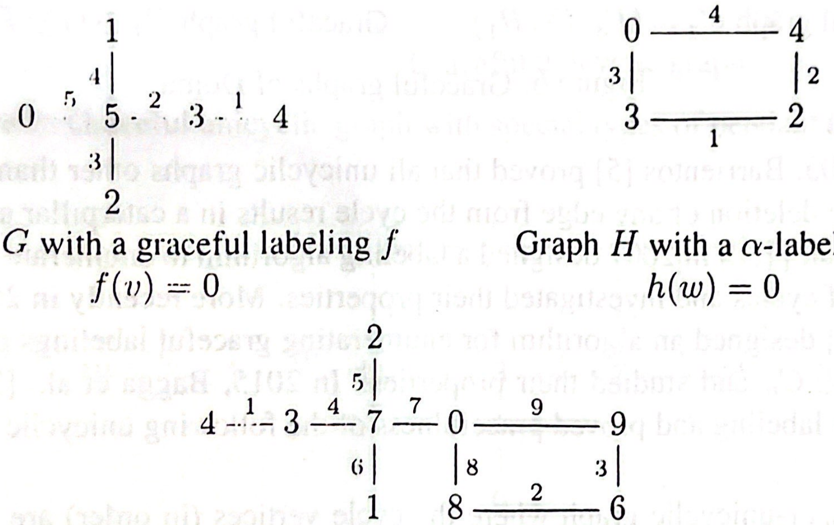
Several authors have published results dealing with the gracefulness of graphs with at least one cycle. We focus our work on unicyclic graphs. In 1984, Truszczyński [18] proved that for two graphs G and H with disjoint set of vertices, if G is graceful under some graceful labeling f, $v \in V(G)$ and $f(v) = 0$, and H has a



Caterpillar R_1 Cycle C_3 $R_1(x_6) \oplus C_3(y_1)$

Figure 3: $R_1(x_6) \oplus C_3(y_1)$

α-labeling h, $w \in V(H)$ and $h(w) = 0$, then $G(v) \oplus H(w)$ is graceful. In figure 4, we present an example of the graceful graph described by Truszczyński [18]. Truszczyński also studied gracefulness of unicyclic graphs where a caterpillar or a path is attached on a vertex of the cycle. He stated without proof that the unicyclic graph $C_n \oplus R(u)$, where u is a vertex adjacent to a peripheral vertex of the caterpillar R, $n \geq 3$, $|V(R)| \geq 3$, is graceful. He also stated without proof that the graph $C_n \oplus P_m(u)$, where u is a peripheral of the path P_m, $n \geq 3$, $m \geq 2$, is graceful.



Graph G with a graceful labeling f $f(v) = 0$

Graph H with a α-labeling h $h(w) = 0$

The graph $G(v) \oplus H(w)$ obtained is a graceful graph.

Figure 4: Graceful graph obtained from a graceful graph G and a graph H having a α-labeling

Later in 1991, Doma [7] showed that all unicyclic graphs with size up to nine are graceful. He worked on C_n-unicyclic graphs, $3 \leq n \leq 9$, with one caterpillar attached to a vertex or with two caterpillars attached to two adjacent vertices. His main idea was to remove one edge of the cycle such that the remaining graph is a caterpillar. Then he did a modified canonical labeling of the remaining graph. For the graphs where the modified canonical labeling does not give a graceful labeling, he gave ad hoc graceful labelings. In figure 5, we present an ad hoc graceful labeling of C_6-unicyclic graph with a caterpillar attached to a vertex of

C_6. We present some graceful graphs studied by Doma [7] in figure 6.

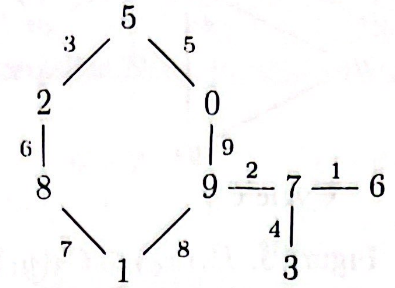
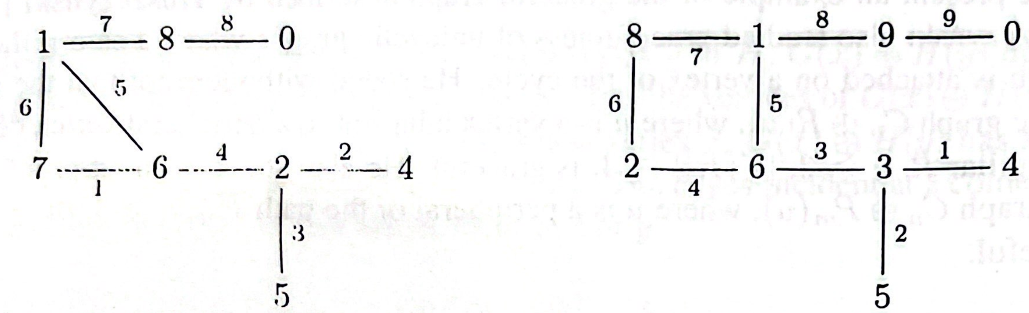


Figure 5: Ad hoc graceful labeling of a C_6-unicyclic graph



Graceful graph $C_3 \oplus (P_3; P_1; R_1)$ Graceful graph $C_4 \oplus (P_3; P_1; P_1; R_2)$

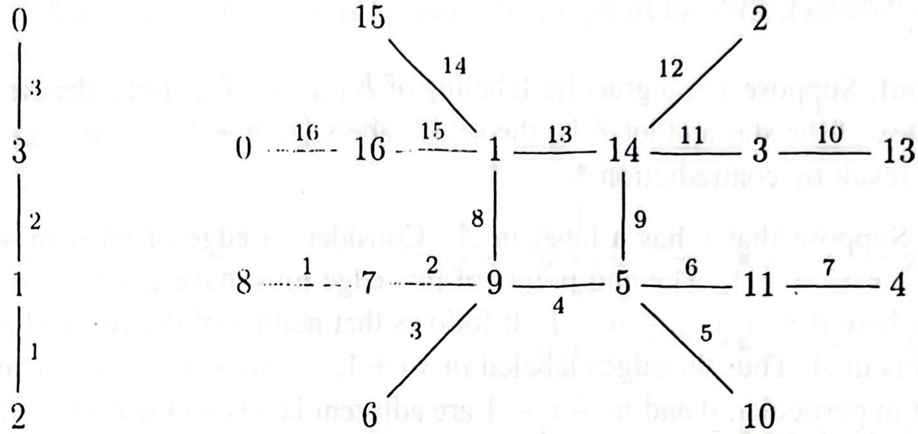
Figure 6: Graceful graphs of Doma

In 2005, Barrientos [5] proved that all unicyclic graphs other than a cycle for which the deletion of any edge from the cycle results in a caterpillar are graceful. Bagga et al. [1, 2] in 2007 designed a labeling algorithm to enumerate graceful labelings of cycles and investigated their properties. More recently in 2014, Pambe et al. [14] designed an algorithm for enumerating graceful labelings of the graph $K_{1,m-1} \oplus C_n$ and studied their properties. In 2015, Bagga et al. [3] modified canonical labeling and proved gracefulness of the following unicyclic graphs:

- C_{2n+1}-unicyclic graph where the cycle vertices (in order) are x_0, x_1, x_2, $\cdots$, x_{2n}, such that two adjacent vertices on the cycle (say x_0 and x_{2n}) have pendant caterpillars R_1 and R_2, while every other vertex on the cycle has any number of pendant P_2's, and $|E(R_i)| \geq n + t - 1, i \in \{1, 2\}$ where t is the total number of pendant P_2's at vertices $x_1, x_2, x_3, \cdots, x_{2n-3}$.

- C_{2n}-unicyclic graph where the cycle vertices (in order) are x_0, x_1, x_2, x_3, $\cdots$, x_{2n-1}, such that two adjacent vertices on the cycle (say x_0 and x_{2n-1}) have pendant caterpillars R_1 and R_2 respectively, while every other vertex on the cycle has any number of pendant P_2's, and $|E(R_1)| \geq n+t-2$ and $|E(R_2)| = n+t-2$, where t is the total number of pendant P_2's at vertices $x_1, x_2, x_3, \cdots, x_{2n-3}$.

Still in 2015, Figueroa-Centeno et al. [8] proved that certain unicyclic graphs with special types of pendant trees (of arbitrary size) at each vertex of the cycle C_n, $n \equiv 0$ or $3 \pmod 4$, are graceful. In the unicyclic graphs they studied, the n pendant trees should have the same order p and should admit α-valuation f_i where for the critical value, we have $l_i = l_{n-i+1}$ when $1 \leq i \leq \lfloor n/2 \rfloor$. The unicyclic graph is obtained by identifying the vertex u_i of the cycle and the vertex v_i of the pendant path P_i, where $f_i(v_i) = l$, $i = 1, 2, \cdots, n$, and $l < p$. In figure 7, we present an example where $T_i = P_4$ with the α-valuation described in the figure, where the critical value is 1, $f(v_i) = l$, $i = 1, 2, 3, 4, 5$.



α-valuation of P_4 Graceful unicyclic graph

Figure 7: Graceful unicyclic graph with special types of pendant trees

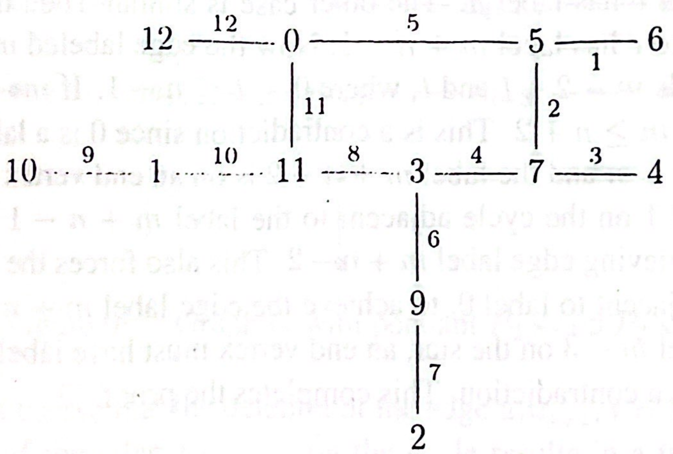


Figure 8: Graceful unicyclic graphs with pendant paths having the same length in consecutive pairs

Later, Chin-Mei Fu et al. [9] studied gracefulness of unicyclic graphs where a vertex of the cycle is identified with a leaf of a corresponding pendant path. The pendant paths have the same length in consecutive pairs. An example is displayed in figure 8, $P_{t_1} = P_2$, $P_{t_2} = P_{t_3} = P_3$, $P_{t_4} = P_{t_5} = P_2$.

4 Labels of central vertex of star in $K_{1,m-1} \oplus C_n$

In this section we present a proof of the following theorem that was stated in our paper [14] without proof. This result specifies the list of labels that the central vertex of the star can have in any graceful labeling of $K_{1,m-1} \oplus C_n$ (obtained by identifying an end vertex of the star $K_{1,m-1}$ and a vertex of the cycle C_n).

Theorem 1 *([14]) For $n \geq 4$, for $m \geq n+2$ and for any graceful labeling of $K_{1,m-1} \oplus C_n$, the central vertex of the star cannot have a label in the set $\{n, n+1, \cdots, m-1\}$.*

Proof. Suppose f is a graceful labeling of $K_{1,m-1} \oplus C_n$. Let v denote the central vertex of the star and let A be the set of labels $\{n, n+1, \cdots, m-1\}$. We prove the result by contradiction.

Suppose that v has a label in A. Consider an edge of label $m+x$, where $0 \leq x \leq n-1$. The end points of this edge must have labels $m+x+y$ and y, where $0 \leq x+y \leq n-1$. It follows that neither of the two end-points have labels in A. Thus the edges labeled $m, m+1, \cdots, m+n-1$ are all on the cycle, and in particular, 0 and $m+n-1$ are adjacent labels on the cycle. Furthermore, the edges on the star are labeled $1, 2, \cdots, m-1$. Now the end-points of the edge labeled $m-1$ must have labels $m-1+z$ and z, where $0 \leq z \leq n$. The only labels in A that satisfy these are n and $m-1$.

Assume that v has label n. The other case is similar Then the vertex on the cycle adjacent to v has label $m+n-1$. Now the edge labeled $m-2$ on the star must have labels $m-2+t$ and t, where $0 \leq t \leq n+1$. If $m-2+t=n$, we get $t=0$ since $m \geq n+2$. This is a contradiction since 0 is a label on the cycle. It follows that $t=n$ and the label $m+n-2$ is on an end vertex of the star. This forces the label 1 on the cycle adjacent to the label $m+n-1$ since this is the only way of achieving edge label $m+n-2$. This also forces the label $m+n-3$ on the cycle adjacent to label 0, to achieve the edge label $m+n-3$. Finally, to achieve the label $m-3$ on the star, an end vertex must have label $m+n-3$ or 1 (if $m=n+2$), a contradiction. This completes the proof. □

5 C_n-Unicyclic graphs with pendant paths P_2 and P_3

We study in this section the gracefulness of a class of unicyclic graphs where the cycle has pendant paths. For these unicyclic graphs, the deletion of any edge on the cycle does not result in a caterpillar. This class of unicyclic graphs has pendant paths P_3 and P_2 at the vertices of the cycle C_n. Let k be an even integer, $k \geq 2$, and $n = 4k-1$. Denote the vertices of C_n $u_1, u_2, u_3, \cdots, u_{4k-1}$, as depicted in figure 9.

We form the unicyclic graph G of this class by adding pendant paths to the vertices of C_n as follows. Add $\lfloor \frac{k}{2} \rfloor$ pendant P_2's at vertices u_{3k}, u_{3k-2}, u_{3k-4}, $\cdots$, $u_{3k-2\lfloor k\backslash 2\rfloor+2}$. Add pendant P_3's at all the other vertices of the cycle. Finally, add also $\lfloor \frac{k}{2} \rfloor$ pendant P_2's at u_{4k-1}, u_{4k-5}, u_{4k-9}, $\cdots$, $u_{4k+3-4\lfloor k/2 \rfloor}$.

For a vertex u_i on the cycle, we denote by $P_s(u_i)$ the pendant path P_s at the vertex u_i, where $s = 2$ or $s = 3$. We present the special cases for $k = 2$ and $k = 4$ in figure 10.

Theorem 2 *Suppose that G is a C_{4k-1}-unicyclic graph as described above. Then G is graceful.*

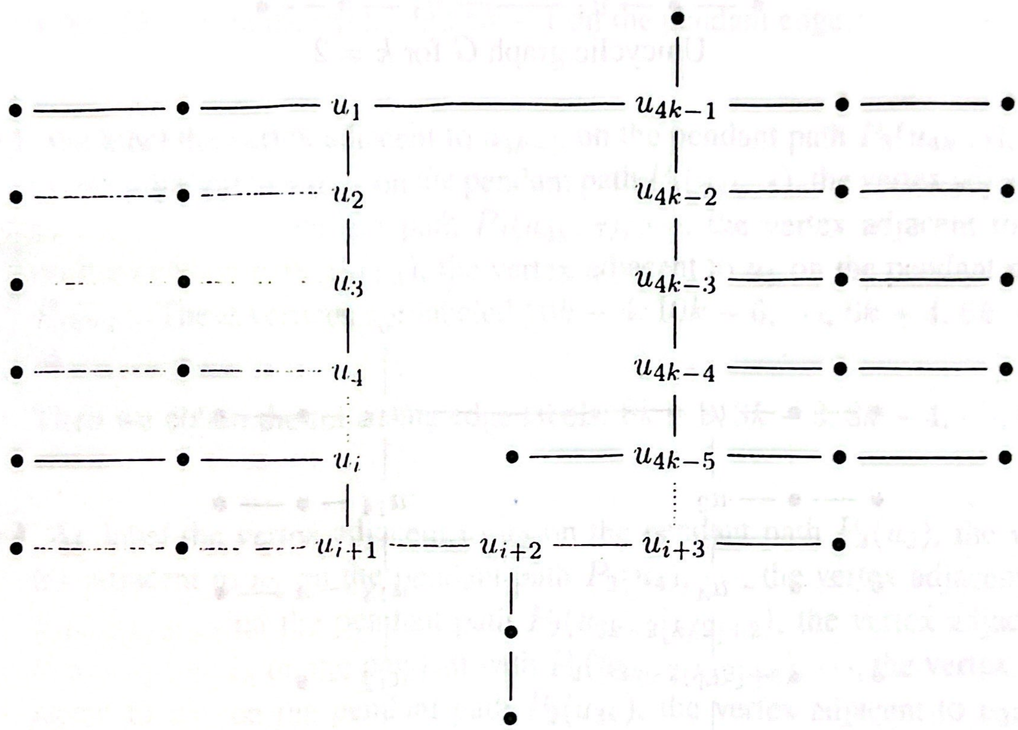
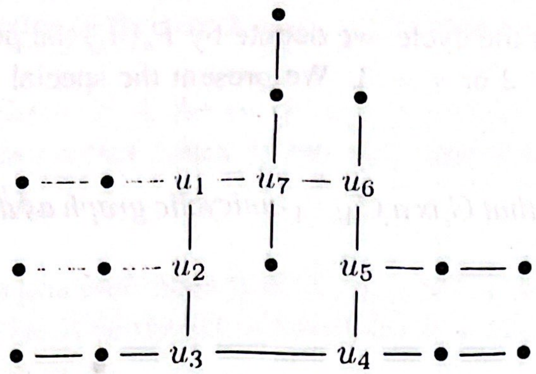


Figure 9: Graph G with pendant P_3's and P_2's

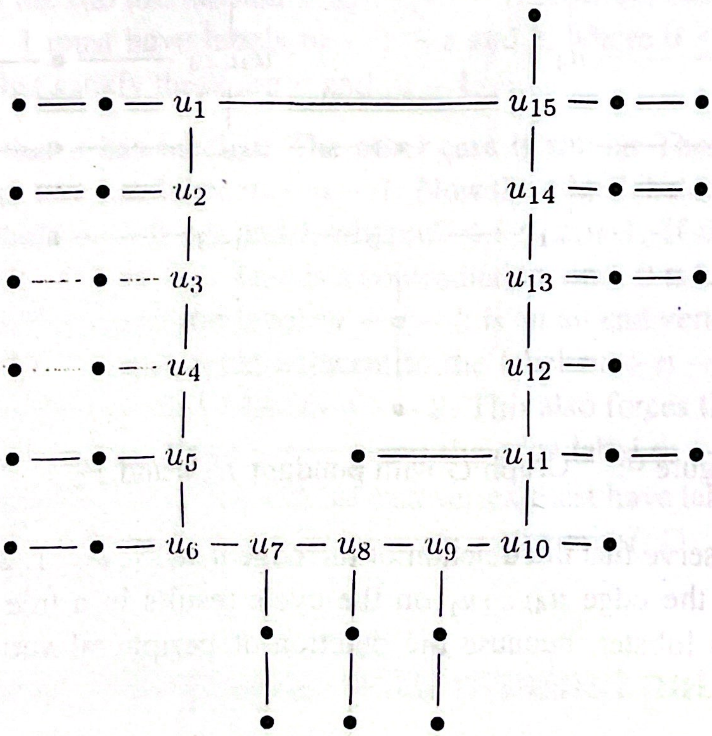
Proof. We first observe that the deletion of the edge u_iu_{i+1}, $i = 1, 2, \cdots, 4k-2$, or the deletion of the edge $u_{4k-1}u_1$ on the cycle results in a tree that is not a caterpillar. It is a lobster, because the deletion of peripheral vertices leaves a caterpillar.

We have $k \geq 2$, the cycle has $4k - 1$ edges, $2\lfloor \frac{k}{2} \rfloor$ pendant paths P_2's, and $4k - 1 - \lfloor \frac{k}{2} \rfloor$ pendant paths P_3's. We have

$$\begin{aligned} S &= 4k - 1 + 2\lfloor \frac{k}{2} \rfloor + 2(4k - 1 - \lfloor \frac{k}{2} \rfloor) \\ &= 12k - 3 \end{aligned}$$



Unicyclic graph G for $k = 2$



Unicyclic graph G for $k = 4$

Figure 10: Special cases of unicyclic graphs considered when $k = 4$ and $k = 2$

Thus G has $12k - 3$ edges. We first describe 7 steps used to label some vertices of G (step i must be completed before step $i + 1$, $i = 1 \cdots 6$). These steps are used to label the vertices of G, except the vertex adjacent to u_{4k-1} on pendant path $P_2(u_{4k-1})$, the vertex adjacent to u_{4k-5} on pendant path $P_2(u_{4k-5})$, the vertex adjacent to u_{4k-9} on pendant path $P_2(u_{4k-9})$, $\cdots$, the vertex adjacent to $u_{4k+3-4\lfloor k/2 \rfloor}$ on pendant path $P_2(u_{4k+3-4\lfloor k/2 \rfloor})$.

step 1 We do a canonical labeling of the vertices u_1, u_2, u_3, $\cdots$, u_{4k-1} of the cycle. So these vertices are labeled 0, $12k - 3$, 1, $12k - 4$, 2, $12k - 5$, $\cdots$, $2k - 3$, $10k$, $2k - 2$, $10k - 1$, $2k - 1$. We next label the vertex adjacent to u_{4k-1} on the pendant path $P_3(u_{4k-1})$ with the label $10k - 2$.

Then we obtain the following edge labels: $12k-3, 12k-4, 12k-5 \cdots, 8k+1, 8k, 2k \quad 1$ on the cycle, and $8k - 1$ on the pendant edge.

step 2 We label the vertex adjacent to u_{4k-3} on the pendant path $P_3(u_{4k-3})$, the vertex adjacent to u_{4k-5} on the pendant path $P_3(u_{4k-5})$, the vertex adjacent to u_{4k-7} on the pendant path $P_3(u_{4k-7})$, $\cdots$, the vertex adjacent to u_3 on the pendant path $P_3(u_3)$, the vertex adjacent to u_1 on the pendant path $P_3(u_1)$. These vertices are labeled $10k - 4$, $10k - 6$, $\cdots$, $6k + 4$, $6k + 2$, $6k$.

Then we obtain the following edge labels: $8k - 2$, $8k - 3$, $8k - 4$, $\cdots$, $6k$.

step 3 We label the vertex adjacent to u_2 on the pendant path $P_3(u_2)$, the vertex adjacent to u_4 on the pendant path $P_3(u_4)$, $\cdots$, the vertex adjacent to $u_{3k-2\lfloor k/2 \rfloor+2}$ on the pendant path $P_2(u_{3k-2\lfloor k/2 \rfloor+2})$, the vertex adjacent to $u_{3k-2\lfloor k/2 \rfloor+4}$ on the pendant path $P_2(u_{3k-2\lfloor k/2 \rfloor+4})$, $\cdots$, the vertex adjacent to u_{3k} on the pendant path $P_2(u_{3k})$, the vertex adjacent to u_{3k+2} on the pendant path $P_3(u_{3k+2})$, $\cdots$, the vertex adjacent to u_{4k-4} on the pendant path $P_2(u_{4k-4})$, the vertex adjacent to u_{4k-2} on the pendant path $P_2(u_{4k-2})$. These vertices are labeled $8k - 3$, $8k - 5$, $8k - 7$, $\cdots$, $4k + 3$, $4k + 1$. We next label the last vertex non labeled on $P_3(u_{4k-1})$ with the label $4k - 1$.

Then we obtain the following edge labels: $6k - 1$, $6k - 2$, $6k - 3$, $\cdots$, $4k$.

step 4 We label the last vertex non labeled on $P_3(u_{2k})$, the last vertex non labeled on $P_3(u_{2k-2})$, the last vertex non labeled on $P_3(u_{2k-4})$, $\cdots$, the last vertex non labeled on $P_3(u_4)$, the last vertex non labeled on $P_3(u_2)$. These vertices are labeled $2k$, $2k + 4$, $2k + 8$, $\cdots$, $6k - 8$, $6k - 4$.

Then we obtain the following edge labels: $4k-1, 4k-3, \cdots, 2k+3, 2k+1$.

step 5 We label the last vertex non labeled on $P_3(u_1)$, the last vertex non labeled on $P_3(u_3), \cdots$, the last vertex non labeled on $P_3(u_{2k-3})$, the last vertex non labeled on $P_3(u_{2k-1})$. These vertices are labeled $2k+2$, $2k+6$, $2k+10$, $\cdots$, $6k-6$, $6k-2$.

Then we obtain the following edge labels: $4k-2$, $4k-4$, $\cdots$, $2k+2$, $2k$.

step 6 We label the last vertex non labeled on $P_3(u_{4k-3})$, the last vertex non labeled on $P_3(u_{4k-5})$, the last vertex non labeled on $P_3(u_{4k-7}), \cdots$, the last vertex non labeled on $P_3(u_{2k+3})$, the last vertex non labeled on $P_3(u_{2k+1})$. These vertices are labeled $8k+1$, $8k+3$, $8k+5$, $\cdots$, $10k-7$, $10k-5$, $10k-3$.

Then we obtain the following edge labels: $1, 3, 5, \cdots, 2k-5, 2k-3$.

step 7 We label the vertices non labeled on pendant P_3s: the last vertex non labeled on pendant path $P_3(u_{4k-2})$, the last vertex non labeled on pendant path $P_3(u_{4k-4}), \cdots$, the last vertex non labeled on pendant path $P_3(u_{4k-2\lfloor(k-1)/2\rfloor})$. These vertices are labeled $4k-3, 4k-5, 4k-7, \cdots, 4k-1-2\lfloor(k-1)/2\rfloor$. Then we obtain the following edge labels: $4, 8, 12, \cdots, 4\lfloor(k-1)/2\rfloor$.

Table 1 shows the edge labels produced after these 7 steps. After these 7 steps, the peripheral vertices of the $\lfloor\frac{k}{2}\rfloor$ pendant P_2s at $u_{4k-1}, u_{4k-5}, u_{4k-9}, \cdots, u_{4k+3-4\lfloor k/2\rfloor}$ are not yet labeled. Now we label these remaining vertices: the vertex adjacent to u_{4k-1} on pendant path $P_2(u_{4k-1})$, the vertex adjacent to u_{4k-5} on pendant path $P_2(u_{4k-5})$, the vertex adjacent to u_{4k-9} on pendant path $P_2(u_{4k-9})$, $\cdots$, the vertex adjacent to $u_{4k+3-4\lfloor k/2\rfloor}$ on pendant path $P_2(u_{4k+3-4\lfloor k/2\rfloor})$. These vertices are labeled $2k+1$, $2k+3$, $2k+5, \cdots$, $2k-1+2\lfloor k/2\rfloor$.

Then we obtain the following edge labels: $2, 6, \cdots, 4\lfloor k/2\rfloor-2$.

All the vertices are labeled now. The missing vertex label is $8k-1$. We have the edge labels from $12k-3$ to 1. Now we must check that the vertex labels and the edge labels are distinct. Let V_i be the set of vertex labels used in step i, and E_j the set of edge labels obtained in step j. We have

$$\begin{aligned}
V_1 &= \{0, 1, 2, \cdots, 2k-1\} \cup \{10k-2, 10k-1, 10k, \cdots, 12k-3\} \\
V_2 &= \{6k, 6k+2, 6k+4, \cdots, 10k-6, 10k-4\} \\
V_3 &= \{4k-1, 4k+1, 4k+3, \cdots, 8k-7, 8k-5, 8k-3\} \\
V_4 &= \{2k, 2k+4, 2k+8, \cdots, 6k-4\}
\end{aligned}$$

$$V_5 = \{2k+2, 2k+6, 2k+10, \cdots, 6k-2\}$$
$$V_6 = \{8k+1, 8k+3, \cdots, 9k-5, 9k-3, \cdots, 10k-5, 10k-3\}$$
$$V_7 = \{4k-1-2\lfloor(k-1)/2\rfloor, 4k+1-2\lfloor(k-1)/2\rfloor, \cdots, 4k-7, 4k-5, 4k-3\}$$
$$V_8 = \{2k+1, 2k+3, 2k+5, \cdots, 2k-1+2\lfloor k/2\rfloor\}$$

Step	Vertex labels assigned	Edge labels produced
1	$0, 12k-3, 1, 12k-4, 2, 12k-5, \cdots, 2k-3, 10k, 2k-2, 10k-1, 2k-1, 10k-2$	$12k-3, 12k-4, 12k-5 \cdots, 8k+1, 8k, 8k-1$ and $2k-1$
2	$10k-4, 10k-6, 10k-8, \cdots, 6k$	$8k-2, 8k-3, 8k-4 \cdots, 6k$
3	$8k-3, 8k-5, 8k-7, \cdots, 4k-1$	$6k-1, 6k-2, \cdots, 4k$
4	$2k, 2k+4, 2k+8, \cdots, 6k-4$	$4k-1, 4k-3, \cdots, 2k+3, 2k+1$
5	$2k+2, 2k+6, 2k+10, \cdots, 6k-2$	$4k-2, 4k-4, \cdots, 2k+2, 2k$
6	$10k-3, 10k-5, \cdots, 9k-3, 9k-5, \cdots, 8k+3, 8k+1$	$1, 3, 5, \cdots, 2k-5, 2k-3$
7	$4k-3, 4k-5, 4k-7, \cdots, 4k-1-2\lfloor(k-1)/2\rfloor$	$4, 8, 12, \cdots, 4\lfloor(k-1)/2\rfloor$

Table 1: Details of labeling steps

It is clear that $V_1 \cap V_2 = \emptyset$, $(V_1 \cup V_2) \cap V_3 = \emptyset$, $(V_1 \cup V_2) \cap V_4 = \emptyset$, $(V_1 \cup V_2) \cap V_5 = \emptyset$, $V_6 \cap (V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5) = \emptyset$, $V_7 \cap (V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6) = \emptyset$. It is also evident that $(V_3 \cap V_4) = \emptyset$, $(V_3 \cap V_5) = \emptyset$, because V_4 and V_5 are two sets of even numbers, V_3 is a set of odd numbers. Indeed

$$V_3 = \{4k+2j-1 \mid j = 0, 1, 2, \cdots, 2k-1\}$$
$$V_4 = \{2k+4i \mid i = 0, 1, 2, \cdots, k-1\}$$

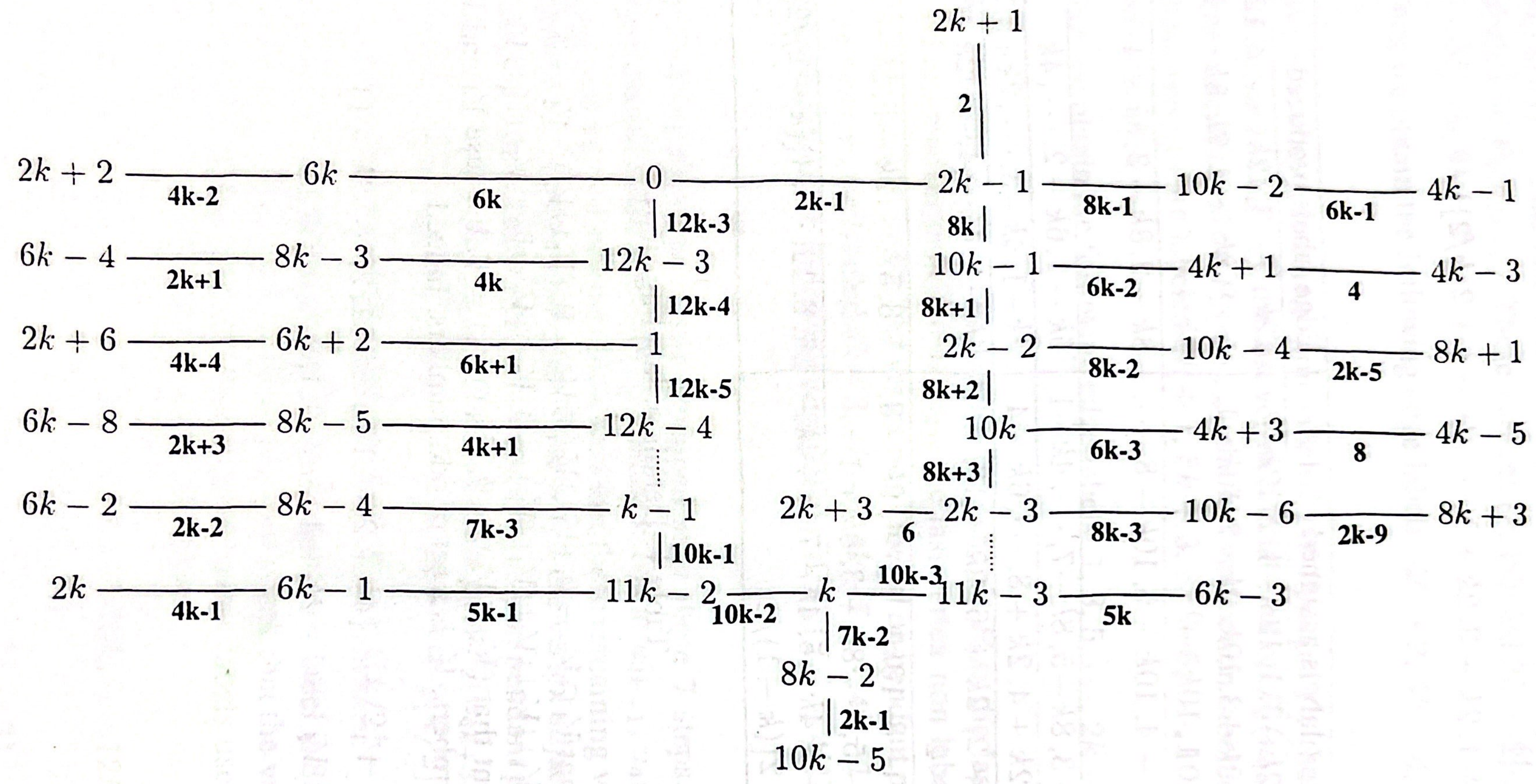


Figure 11: A graceful labeling of G

Suppose there exists $i_0 \in \{0, 1, 2, \cdots, k-1\}$ and $j_0 \in \{0, 1, 2, \cdots, 2k-1\}$ such that

$$\begin{aligned} 2k + 4i_0 &= 4k + 2j_0 - 1 \\ \Rightarrow \quad 2(2i_0 - j_0) &= 2k - 1 \quad \text{Impossible because } k > 1 \end{aligned}$$

Then $V_3 \cap V_4 = \emptyset$. Now we should verify $V_4 \cap V_5$. $V_5 = \{2k + 2 + 4\alpha \mid \alpha = 0, 1, 2, \cdots, k-1\}$. Suppose there exists $i_0 \in \{0, 1, 2, \cdots, k-1\}$ and $\alpha \in \{0, 1, 2, \cdots, k-1\}$ such that

$$\begin{aligned} 2k + 4i_0 &= 2k + 2 + 4\alpha \\ 4i_0 - 4\alpha &= 2 \\ \Rightarrow \quad 2(i_0 - \alpha) &= 1. \quad \text{This is impossible because } i_0 \text{ and } \alpha \text{ are integers.} \end{aligned}$$

Then $V_5 \cap V_4 = \emptyset$. We have also

$$V_8 \cap (V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6 \cup V_7) = \emptyset$$

Concerning the sets of edge labels, we have

$$E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \cap E_6 \cap E_7 \cap E_8 = \emptyset.$$

The vertex labels are distinct, the edge labels are also distinct from 1 up to $12k-3$, then we obtain a graceful labeling of the graph G when k is even. We present the graceful labeling in figure 11. □

We now present two examples. We begin with the unicyclic graph G where $k = 4$. The results of the steps are in the table 2 and the figure 12 presents the graceful labeling obtained. For our second example, figure 13 presents the graceful unicyclic graph G where $k = 8$ and table 3 shows results of the different steps.

Step	Vertex labels assigned	Edge labels produced
1	0, 45, 1, 44, 2, 43, 3, 42, 4, 41, 5, 40, 6, 39, 7, 38	45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 34, 33, 32, 31, 7
2	36, 34, 32, 30, 28, 26, 24	30, 29, 28, 27, 26, 25, 24
3	29, 27, 25, 23, 21, 19, 17, 15	23, 22, 21, 20, 19, 18, 17, 16
4	8, 12, 16, 20	15, 13, 11, 9
5	10, 14, 18, 22	14, 12, 10, 8
6	33, 35, 37	1, 3, 5
7	13	4

Then we use the vertex labels 9, 11 to obtain the edge labels 2 and 6.

Table 2: Steps of labeling unicyclic graph G when $k = 4$

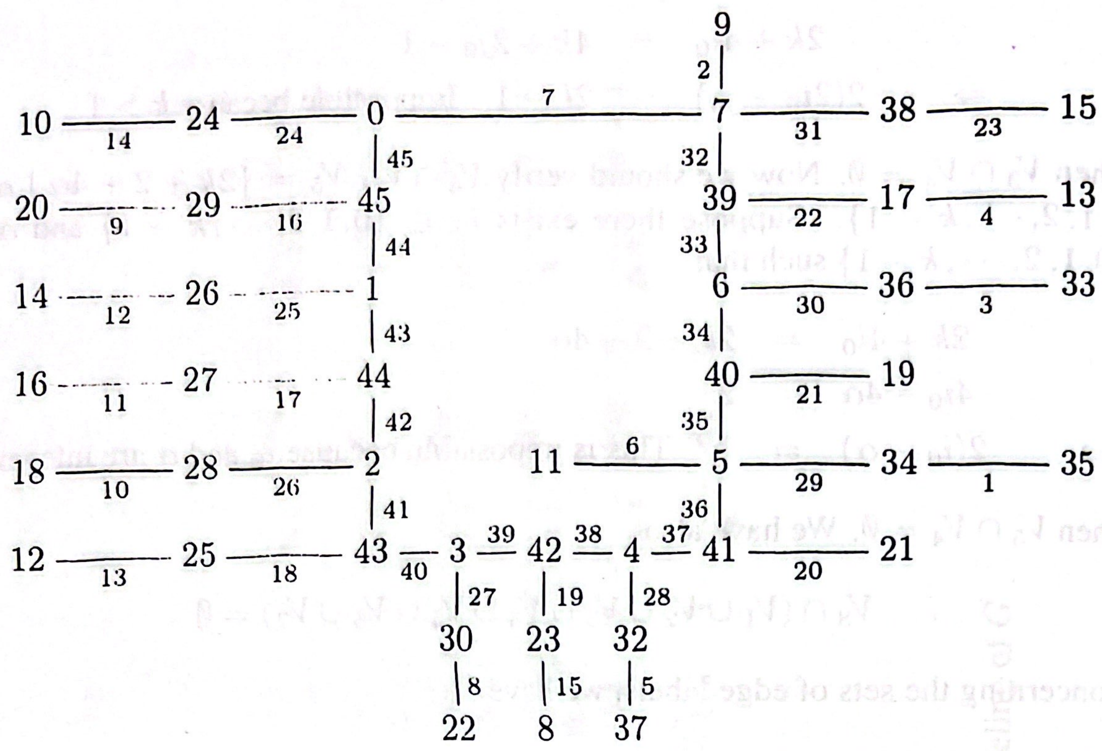


Figure 12: A Graceful labeling of unicyclic graph G when $k = 4$

Step	Vertex labels assigned	Edge labels produced
1	0, 93, 1, 92, ···, 80, 14, 79, 15, 78	93, 92, 91, ···, 63, 15
2	76, 74, 72, ···, 48	62, 61, 60, ···, 48
3	31, 33, ···, 61	47, 46, 45, ···, 32
4	16, 20, 24, ···, 44	31, 29, 27, ···, 19, 17
5	18, 22, 26, ···, 46	30, 28, 26, ···, 18, 16
6	75, 73, 71, 69, 67, 65	1, 3, 5, 7, 9, 11, 13
7	29, 27, 25	4, 8, 12

Then we use the vertex labels 17, 19, 21 and 23 to obtain the edge labels 2, 6, 10, 14.

Table 3: Steps of labeling when $k = 8$

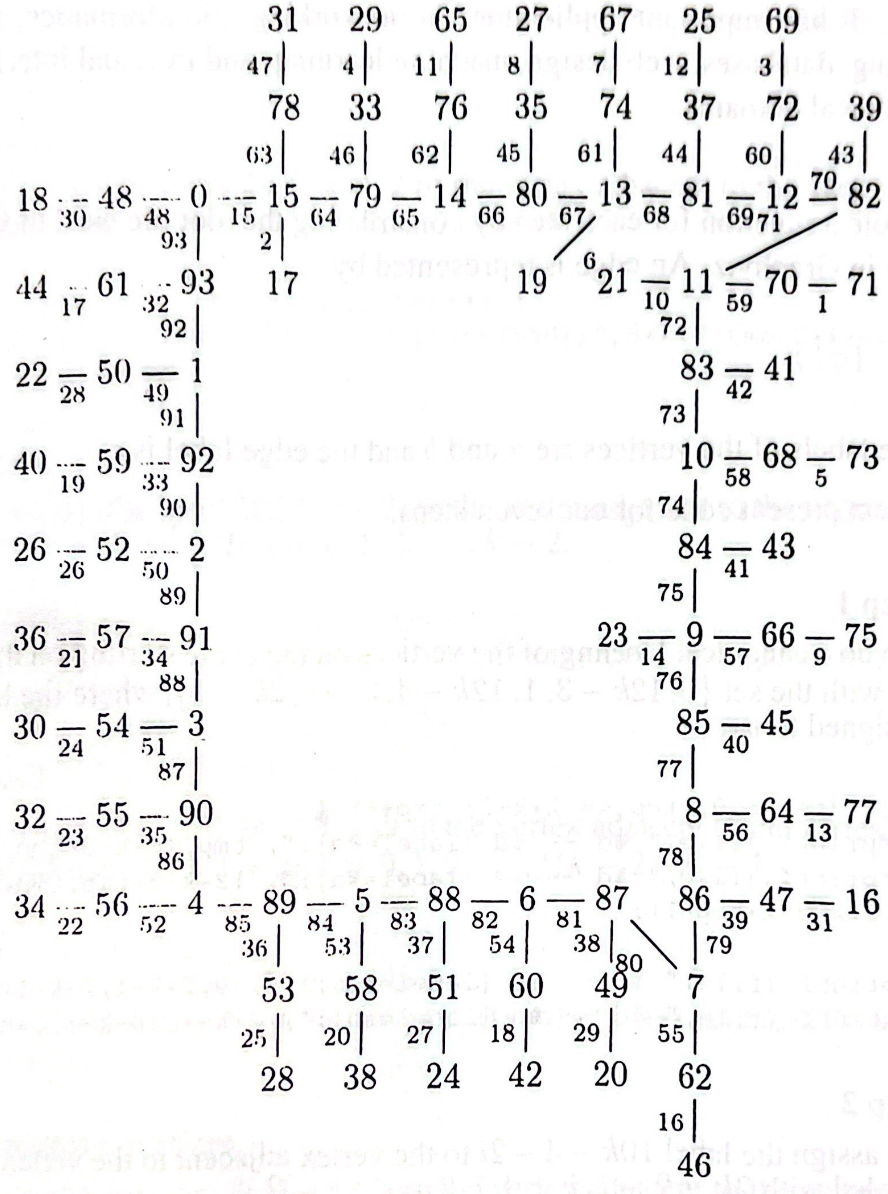


Figure 13: A Graceful labeling of unicyclic graph G when $k = 8$

6 Visualization of our graceful unicyclic graphs

In this section, we build software code to visualize our graceful unicyclic graphs. We design C programs constructing dot files with the language DOT using the generator neato [11, 15] in the tool Graphviz (Graph Visualization Software [12]). Graphviz is an open source graph visualization software. Graph visualization is a way of representing structural information as diagrams of abstract graphs and networks. It has important applications in networking, bioinformatics, software engineering, databases, web design, machine learning, and in visual interfaces for other technical domains.

We wrote a function for each step by constructing the .dot file used to visualize the graph in Graphviz. An edge is represented by

```
a -- b [c];
```

where the labels of the vertices are a and b and the edge label is c.

We next present code for our seven steps:

- **Step 1**

  We do a canonical labeling of the vertices on the cycle starting on the vertex u_1 with the set $\{0, 12k-3, 1, 12k-4, 2, \cdots, 2k-1\}$, where the label 0 is assigned to u_1.

```
for (tmp = 0; tmp <= 2*k-2; tmp++) {
 fprintf (file," %d -- %d [label=%d];", tmp,12*k-3-tmp,12*k-3-2*tm
 fprintf (file," %d -- %d [label=%d];", 12*k-3-tmp,tmp+1,
 12*k-3-2*tmp-1);
}
fprintf (file," %d -- %d [label=%d];\t", 0,2*k-1,2*k-1);
fprintf (file," %d -- %d [label=%d];", 2*k-1,10*k-2,8*k-1);
```

- **Step 2**

  We assign the label $10k-4-2i$ to the vertex adjacent to the vertex already labeled with $2k-2-i, i = 0, 1, 2, \cdots, 2k-2$.

```
for (tmp = 0; tmp <= 2*k-2; tmp++)
  fprintf (file," %d -- %d [label=%d];", 2*k-2-tmp,10*k-4-2*tmp,
  8*k-2-tmp);
```

- **Step 3**

  We assign the label $8k-3-2i$ to the vertex adjacent to the vertex already labeled with $12k-3-i, i = 0, 1, 2, \cdots, 2k-1$.

```
for (tmp = 0; tmp <= 2*k-1; tmp++)
  fprintf (file," %d -- %d [label=%d];", 12*k-3-tmp,8*k-3-2*tmp,
  6*k-1+tmp-(2*k-1));
```

- **Step 4**

  We assign the label $2k + 4i$ to the vertex adjacent to the vertex already labeled with $6k + 2i$, $i = 0, 1, 2, \cdots, k - 1$.

```
for (tmp = 0; tmp <= k-1; tmp++) {
  fprintf (file," %d -- %d [label=%d];",6*k+2*tmp-1,2*k+4*tmp,
  4*k-1-2*tmp);
```

- **Step 5**

  We assign the label $2k + 4i + 2$ to the vertex adjacent to the vertex already labeled with $6k + 2i$, $i = 0, 1, 2, \cdots, k - 1$.

```
for (tmp = 0; tmp <= k-1; tmp++) {
  fprintf (file," %d -- %d [label=%d];",6*k+2*tmp,2*k+4*tmp+2,
  4*k-2-2*tmp);
```

- **Step 6**

  We assign the label $10k - 3 - 2i$ to the vertex adjacent to the vertex already labeled with $8k + 2i$, $i = 0, 1, 2, \cdots, k - 2$.

```
for (tmp = 0; tmp <= k-2; tmp++) {
  fprintf (file," %d -- %d [label=%d];",8*k+2*tmp,10*k-3-2*tmp,
  abs(2*k-4*tmp-3));
```

- **Step 7**

  We assign the label $4k - 3 - 2i$ to the vertex adjacent to the vertex already labeled with $4k + 1 + 2i$, $i = 0, 1, 2, \cdots, \lfloor (k-2)/2 \rfloor - 1$.

```
max = (k-1)/2-1;
for (tmp = 0; tmp <= max; tmp++) {
  fprintf (file," %d -- %d [label=%d];", 4*k+1+2*tmp,4*k-3-2*tmp,
  4+4*tmp);
```

- **Remaining vertices**

  We assign the label $2k + 1 + 2i$ to the vertex adjacent to the vertex already labeled with $2k - 1 - 2i$, $i = 0, 1, 2, \cdots, \lfloor k/2 \rfloor - 1$.

```
max = k/2-1;
for (tmp = 0; tmp <= max; tmp++) {
  fprintf (file," %d -- %d [label=%d];",2*k-1-2*tmp,2*k+1+2*tmp,
  4*tmp+2);
```

```
/*
graphpaper3.dot -- k=2
-----
By Pambe, for the graphical representation of graceful labelings of
some uniyclic graphs Goal: This file is used by neato in Graphviz
to generate the labeled graph
*/
graph {
  graph [overlap=false,splines=true]; node [shape=circle,fontsize=12];
  edge [labelfontsize=10,fontcolor=blue];
  /* STEP 1 -----------------------------------*/
  0 -- 21 [label=21]; 21 -- 1 [label=20];
  1 -- 20 [label=19]; 20 -- 2 [label=18];
  2 -- 19 [label=17]; 19 -- 3 [label=16];
  0 -- 3 [label=3]; 3 -- 18 [label=15];
  /* STEP 2 -----------------------------------*/
  2 -- 16 [label=14]; 1 -- 14 [label=13]; 0 -- 12 [label=12];
  /* STEP 3 -----------------------------------*/
  21 -- 13 [label=8]; 20 -- 11 [label=9]; 19 -- 9 [label=10];
  18 -- 7 [label=11];
  /* STEP 4 -----------------------------------*/
  11 -- 4 [label=7]; 13 -- 8 [label=5];
  /* STEP 5 -----------------------------------*/
  12 -- 6 [label=6]; 14 -- 10 [label=4];
  /* STEP 6 -----------------------------------*/
  16 -- 17 [label=1];
  /* STEP 7 -----------------------------------*/
  /* REMAINING VERTICES --------------------*/
  3 -- 5 [label=2];
}
```

Figure 14: Dot file obtained when $k = 2$ with DOT

At the end of the labeling, we obtain the dot file. For $k = 2$, figure 14 presents the dot file. We then used the generator `neato` to display the graph. Neato [15] is a program that makes layouts of undirected graphs following the filter model of DOT [11]. The command is

```
neato -Tps graphpaper3.dot -o graphdot.ps
```

The `graphdot.ps` containing the labeling of the unicyclic graph G obtained when $k = 2$ is in the figure 15.

7 Conclusion

In this paper we study gracefulness of certain unicyclic graphs. We have presented the proof of the generalization of the labels for the central vertex of the star in the unicyclic graph $K_{1,m-1} \oplus C_n$. This theorem appeared without proof in [14]. We have shown the gracefulness of certain C_{4k-1}-unicyclic graphs, where the cycle vertices have pendant paths P_3 or P_2, and the deletion of any edge on the cycle does not result in a caterpillar, but in a lobster. The results of Doma[7], Barrientos[5], S.K.Vaidya et al.[19] together with our results add more evidence to Truszczyński's conjecture. The authors thank the referee for useful suggestions and corrections.

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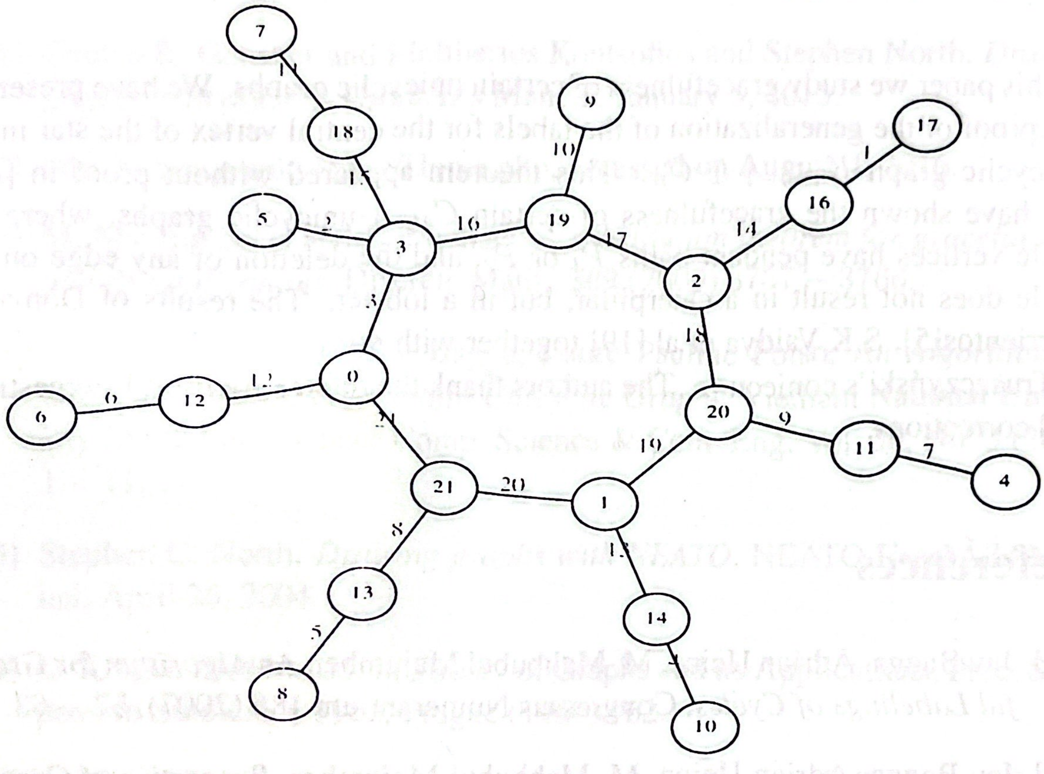

Figure 15: Image in the Ps file obtained with neato for the unicyclic graph G when $k = 2$

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