Generalized Nordhaus-Gaddum type results for general Randić index and geometric-arithmetic index of graphs *

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Abstract

Randić index and geometric-arithmetic index are two important chemical indices. In this paper, we give the generalized Nordhaus-Gaddum-type inequalities for the two kinds of chemical indices.

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1 Introduction

Let $G = (V(G), E(G))$ be a graph. The degree and the neighborhood of a vertex $u \in V(G)$ is denoted by $d_G(u)$ and $N_G(u)$ (or simply by $d(u)$ and $N(u)$), respectively. Given two adjacent vertices u and v of a graph G, the *Randić weight* of the edge uv is $R(uv) = (d(u)d(v))^{-\frac{1}{2}}$, and the *Randić index* of a graph G, $R(G)$, is the sum of the Randić weights of its edges. Randić [9] proposed the important topological index in his research on molecular structures, which is closely related with many chemical properties. Fixing $\alpha \in R - \{0\}$, the *general Randić index* is defined as

$$R_\alpha(G) = \sum_{uv \in E(G)} R_\alpha(uv) = \sum_{uv \in E(G)} (d(u)d(v))^\alpha.$$

Hence, $R_{-\frac{1}{2}}(G)$ is the ordinary Randić index of G. There are also a large number of other chemical indices of molecular graphs.

In [10], Vukičević and Furtula defined a new topological index "geometric-arithmetic index" of a graph G, denoted by $GA(G)$ and is defined by

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{deg_G(u)deg_G(v)}}{deg_G(u) + deg_G(v)}.$$

Let $f(G)$ be a graph invariant. The *Nordhaus–Gaddum Problem* is to determine sharp bounds for $f(G) + f(\overline{G})$ and $f(G) \cdot f(\overline{G})$, as G ranges over the class of all graphs of order n, and to characterize the extremal graphs, i.e., graphs that achieve the bounds. A further problem is to determine the set of all integer pairs (x, y) such that $f(G) = x$ and $f(\overline{G}) = y$ for some graph G of order n. We refer to this latter problem as the *Realizability problem.* In their paper, Nordhaus and Gaddum [8] determined bounds for $\chi(G) + \chi(\overline{G})$ and $\chi(G) \cdot \chi(\overline{G})$, where $\chi(G)$ denotes the chromatic number of graph G. The characterization of the corresponding extremal graphs and the realizability problem were resolved by Finck [6]. Nordhaus–Gaddum type relations have received wide attention; see the recent survey [1] by Aouchiche and Hansen. Let $k \geq 2$ be an integer.

A *k-decomposition* $(G_1, G_2, \ldots, G_k)$ of a graph G is a partition of its edge set to form k spanning subgraph $G_1, G_2, \ldots, G_k$. That is, each G_i has the same vertices as G, and every edge of G belongs to exactly one of $G_1, G_2, \ldots, G_k$. For a graph parameter f, a positive integer k, and a graph G, the *Generalized Nordhaus–Gaddum Problem* is to determine sharp bounds for $\left\{\sum_{i=1}^{k} f(G_i) : (G_1, G_2, \ldots, G_k) \text{ is a decomposition of } G\right\}$ and $\left\{\prod_{i=1}^{k} f(G_i) : (G_1, G_2, \ldots, G_k) \text{ is a decomposition of } G\right\}$.

Das [3] got the Nordhaus-Gaddum type results for geometric-arithmetic index. Zhang and Wu [12] obtained the Nordhaus-Gaddum type results for Randić index.

Theorem 1.1 [12] *Let* $G \in \mathcal{G}(n)$.
(1) *If* $\alpha > 0$, *then* $\binom{n}{2}\left(\frac{n-1}{2}\right)^{2\alpha} \leq R_\alpha(G) + R_\alpha(\overline{G}) \leq \binom{n}{2}(n-1)^{2\alpha}$.
(2) *If* $\alpha < 0$, *then* $\binom{n}{2}(n-1)^{2\alpha} \leq R_\alpha(G) + R_\alpha(\overline{G}) \leq \binom{n}{2}\left(\frac{n-1}{2}\right)^{2\alpha}$.

In this paper, we study the generalized Nordhaus-Gaddum type results for general Randić index and geometric-arithmetic index.

2 General Randić index

In this section, we give the generalized Nordhaus-Gaddum type results for general Randić index.

Theorem 2.1 *Let* $k \geq 2$ *be an integer, and* $(G_1, G_1 \cdots, G_k)$ *be a* k-*decomposition of* K_n. *If* $\alpha > 0$, $\delta = \min\{\delta(G_1), \delta(G_2), \cdots, \delta(G_k)\}$, $\Delta = \max\{\Delta(G_1), \Delta(G_2), \cdots, \Delta(G_k)\}$, *then*

$$\binom{n}{2}\delta^{\alpha\delta k} \leq R_\alpha(G_1) + R_\alpha(G_2) + \cdots + R_\alpha(G_k) \leq k\binom{n}{2}\Delta^{2\alpha}.$$

Moreover, the lower and upper bounds are sharp.

Proof. For a graph $K_n = (V, E)$ of order n, let $\varepsilon(K_n) = |E(K_n)|$ and $N = \binom{n}{2}$. First we consider the upper bound. Since $\alpha > 0$, we have

$$\begin{aligned}
& R_\alpha(G_1) + R_\alpha(G_2) + \cdots + R_\alpha(G_k) \\
= & \sum_{uv \in E(G_1)} (d_{G_1}(u) d_{G_1}(v))^\alpha + \sum_{uv \in E(G_2)} (d_{G_2}(u) d_{G_2}(v))^\alpha + \cdots \\
+ & \sum_{uv \in E(G_k)} (d_{G_k}(u) d_{G_k}(v))^\alpha \\
\leq & \varepsilon(G_1)(\Delta(G_1))^{2\alpha} + \varepsilon(G_2)(\Delta(G_2))^{2\alpha} + \cdots + \varepsilon(G_k)(\Delta(G_k))^{2\alpha} \\
\leq & \varepsilon(G)[(\Delta(G_1))^{2\alpha} + (\Delta(G_2))^{2\alpha} + \cdots + (\Delta(G_k))^{2\alpha}] \\
\leq & \varepsilon(G)[\Delta^{2\alpha} + \Delta^{2\alpha} + \cdots + \Delta^{2\alpha}] \\
= & k\binom{n}{2}\Delta^{2\alpha}.
\end{aligned}$$

Now we aim to the lower bound. It is clear that

$$\begin{aligned}
& R_\alpha(G_1) + R_\alpha(G_2) + \cdots + R_\alpha(G_k) \\
= & \sum_{uv \in E(G_1)} (d_{G_1}(u) d_{G_1}(v))^\alpha + \sum_{uv \in E(G_2)} (d_{G_2}(u) d_{G_2}(v))^\alpha + \cdots \\
+ & \sum_{uv \in E(G_k)} (d_{G_k}(u) d_{G_k}(v))^\alpha \\
\geq & N \sqrt[N]{\prod_{uv \in E(G_i)} (d_{G_i}(u) d_{G_i}(v))^\alpha} = N \sqrt[N]{\prod_{uv \in E(G_i)} (d_{G_i}(u))^{d_{G_i}(v)\alpha}} \\
\geq & N \sqrt[N]{\prod_{uv \in E(G_i)} (\delta)^{\delta\alpha}} = N \sqrt[N]{(\delta)^{\delta\alpha N k}} \\
= & N(\delta)^{\delta\alpha k} = \binom{n}{2}\delta^{\alpha\delta k}.
\end{aligned}$$

The lower and upper bound are sharp when G_i is a $\frac{n-1}{k}$-regular graph. ∎

Similarly, we can get the bounds for $\alpha < 0$.

Theorem 2.2 *Let $k \geq 2$ be an integer, and $(G_1, G_1 \cdots, G_k)$ be a k-decomposition of K_n. If $\alpha < 0$, $\delta = \min\{\delta(G_1), \delta(G_2), \cdots, \delta(G_k)\}$, $\Delta = \max\{\Delta(G_1), \Delta(G_2), \cdots, \Delta(G_k)\}$, then*

$$k\binom{n}{2}\Delta^{2\alpha} \leq R_\alpha(G_1) + R_\alpha(G_2) + \cdots + R_\alpha(G_k) \leq \binom{n}{2}\delta^{\alpha\delta k}.$$

Moreover, the upper and lower bounds are sharp.

3 Geometric-arithmetic index

Let $G = (V(G), E(G))$. If $V(G)$ is the disjoint union of two nonempty sets $V_1(G)$ and $V_2(G)$ such that every vertex in $V_1(G)$ has degree r and every vertex in $V_2(G)$ has degree s, then G is a (r, s)-*semiregular graph*. When $r = s$, G is called a *regular graph*.

Das [3] first got the following lower bound of $GA(G)$ for a connected graph G.

Lemma 3.1 [3] *Let G be a simple connected graph of m edges with maximum vertex degree Δ and minimum vertex degree δ. Then*

$$GA(G) \geq \frac{2m\sqrt{\Delta\delta}}{\Delta + \delta}, \tag{3.1}$$

with equality holding in (3.1) if and only if G is isomorphic to a regular graph or G is isomorphic to a bipartite semiregular graph.

In this section, we give the generalized Nordhaus-Gaddum type results for geometric-arithmetic index.

Theorem 3.1 *Let $k \geq 2$ be an integer, and $(G_1, G_1 \cdots, G_k)$ be a k-decomposition of K_n. $\delta = \min\{\delta(G_1), \delta(G_2), \cdots, \delta(G_k)\}$, $\delta_i = \delta(G_i)$ $(1 \leq i \leq k)$, $\Delta = \max\{\Delta(G_1), \Delta(G_2), \cdots, \Delta(G_k)\}$, $\Delta_i = \Delta(G_i)$ $(1 \leq i \leq k)$. Let p_i is the number of maximum degree in G_i, and $p = \min\{p_1, p_2, \cdots, p_k\}$. Let $l_i = \max\left\{\sqrt{\frac{\Delta_i}{\delta_i}}, \sqrt{\frac{n-1-\delta_i}{n-1-\Delta_i}}\right\}$, and $l = \max\{l_1, l_2, \cdots, l_k\}$. Then*

$$\frac{2l}{l^2+1}\binom{n}{2} \leq GA(G_1) + GA(G_2) + \cdots + GA(G_k) \leq \binom{n}{2} - \frac{1}{2}pk\delta + \frac{pk\Delta\sqrt{\Delta\delta}}{\delta + \Delta}.$$

Moreover, the upper and lower bounds are sharp.

Proof. For a graph $K_n = (V, E)$ of order n, let $\varepsilon(K_n) = |E(K_n)|$ and $N = \binom{n}{2}$. From the definition of geometric-arithmetic index, we have

$$GA(G) = \sum_{v_iv_j \in E(G)} \frac{2\sqrt{d_id_j}}{d_i + d_j} \le \sum_{v_iv_j \in E(G)} \frac{2\sqrt{\delta\Delta}}{\delta + \Delta} + \left(m - \frac{p\Delta}{2}\right)$$

$$\le \frac{p\Delta}{2}\frac{2\sqrt{\delta\Delta}}{\delta+\Delta} + \left(m - \frac{p\Delta}{2}\right) = \frac{p\Delta\sqrt{\delta\Delta}}{\delta+\Delta} + \left(m - \frac{p\Delta}{2}\right),$$

and hence

$$GA(G_1) + GA(G_2) + \cdots + GA(G_k)$$

$$= \sum_{v_iv_j \in E(G_1)} \frac{2\sqrt{d_id_j}}{d_i + d_j} + \sum_{v_iv_j \in E(G_2)} \frac{2\sqrt{d_id_j}}{d_i + d_j} + \cdots + \sum_{v_iv_j \in E(G_k)} \frac{2\sqrt{d_id_j}}{d_i + d_j}$$

$$\le \frac{p_1\Delta_1\sqrt{\delta_1\Delta_1}}{\delta_1+\Delta_1} + \left(m_1 - \frac{p_1\Delta_1}{2}\right) + \frac{p_2\Delta_2\sqrt{\delta_2\Delta_2}}{\delta_2+\Delta_2} + \left(m_2 - \frac{p_2\Delta_2}{2}\right)$$

$$+ \cdots + \frac{p_k\Delta_k\sqrt{\delta_k\Delta_k}}{\delta_k+\Delta_k} + \left(m_k - \frac{p_k\Delta_k}{2}\right)$$

$$= (m_1 + m_2 + \cdots + m_k) - \frac{1}{2}(p_1\Delta_1 + p_2\Delta_2 \cdots + p_k\Delta_k)$$

$$+ \frac{p_1\Delta_1\sqrt{\delta_1\Delta_1}}{\delta_1+\Delta_1} + \frac{p_2\Delta_2\sqrt{\delta_2\Delta_2}}{\delta_2+\Delta_2} + \cdots + \frac{p_k\Delta_k\sqrt{\delta_k\Delta_k}}{\delta_k+\Delta_k}$$

$$\le \binom{n}{2} - \frac{1}{2}pk\delta + \frac{pk\Delta\sqrt{\Delta\delta}}{\delta+\Delta}.$$

We now consider the lower bound. From Lemma 3.1, we have

$$GA(G) = \sum_{v_iv_j \in E(G)} \frac{2\sqrt{d_id_j}}{d_i + d_j} \ge \frac{2m\sqrt{\Delta\delta}}{\delta+\Delta}.$$

Since $l_i \ge \sqrt{\frac{\Delta}{\delta}} \ge 1$ and $1 - \frac{\sqrt{\delta}}{l_i\sqrt{\Delta}} \ge 0$, we have $(l_i - \sqrt{\frac{\Delta}{\delta}})(1 - \frac{\sqrt{\delta}}{l_i\sqrt{\Delta}}) \ge 0$, and hence

$$\frac{\sqrt{\Delta_i\delta_i}}{\Delta_i + \delta_i} \ge \frac{l_i}{l_i^2 + 1}.$$

Therefore, we have

$$\begin{aligned}
& GA(G_1)+GA(G_2)+\cdots+GA(G_k) \\
= & \sum_{v_iv_j\in E(G_1)} \frac{2\sqrt{d_id_j}}{d_i+d_j}+\sum_{v_iv_j\in E(G_2)} \frac{2\sqrt{d_id_j}}{d_i+d_j}+\cdots+\sum_{v_iv_j\in E(G_k)} \frac{2\sqrt{d_id_j}}{d_i+d_j} \\
\geq & \frac{2m_1\sqrt{\Delta_1\delta_1}}{\delta_1+\Delta_1}+\frac{2m_2\sqrt{\Delta_2\delta_2}}{\delta_2+\Delta_2}+\cdots+\frac{2m_k\sqrt{\Delta_k\delta_k}}{\delta_k+\Delta_k} \\
\geq & \frac{2m_1l_1}{l_1^2+1}+\frac{2m_2l_2}{l_2^2+1}+\cdots+\frac{2m_kl_k}{l_k^2+1} \\
\geq & \frac{l}{l^2+1}(2m_1+2m_2+\cdots+2m_k) \\
= & \frac{2l}{l^2+1}\binom{n}{2}.
\end{aligned}$$

The upper and upper bound are sharp when G_i is a $\frac{n-1}{k}$-regular graph.

■

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